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A TREATISE

ON

CRYSTALLOGRAPHY

BY

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PREFACE.

MY purpose in this text-book is to set before the student the views held at the present time as to the classification of crystals and the principles of symmetry on which the classification is based; to describe the 'forms' which are a consequence of the symmetry; to determine the geometrical relations of the forms; and, finally, to explain the methods by which the crystals are drawn and their forms represented graphically. The treatment has been as far as possible geometrical, and, with the exception of the formulæ of plane and spherical trigonometry necessary for the solution of triangles, very little analysis has been introduced into the main discussion. Seeing that good figures are important aids to the understanding of the geometrical relations of crystals, and that practice in drawing crystals develops the student's power of solving crystallographic problems, I have throughout given prominence to the methods used in making diagrams. The principles of projection followed in the construction of such figures are therefore explained at an early stage. Hence it is necessary in certain parts of Chapters VI and VII to presuppose some knowledge of the crystals of the different systems: and, accordingly, those articles which refer to a given system should

be read with the corresponding chapter, and with the examples which it contains. A brief sketch of the methods depending on analytical geometry suitable for the treatment of crystallographic problems has been given in Chapter XIX.

It is now generally held that each of the thirty-two possible classes of crystals is a definite group, the forms in which are a direct consequence of the 'elements of symmetry' (p. 21) present in it. Some of these classes have certain geometrical and physical relations in common, and form larger groups called systems. In discussing the classes seriatim, I have taken first the crystals which have no symmetry; and, in passing from one class to another, I have, in general, proceeded to that which involves the least addition to the symmetry of its predecessor, or follows from it on the addition of a centre of symmetry. When no further classes can be obtained by this process, a fresh start is made with a class which has the least symmetry of a new kind. Thus, for instance, the acleistous tetragonal class, which has only a single tetrad axis, comes third in Chapter XIV, whilst the two preceding classes of the tetragonal system have been derived from particular combinations of dyad axes. The geometrical characters of rhombohedral and hexagonal crystals require special treatment, and, accordingly, their systems have been placed last. The general order of discussion has involved some repetition; and, unfortunately, in almost every system it entails discussing at the outset a class in which the geometrical relations do not lend themselves readily to elementary treatment. The beginner will therefore do well to read through the general discussion of all the classes of a system before endeavouring to master the geometrical relations between the face-symbols and the interfacial angles.

It is not easy to find distinctive and appropriate names

for the several classes. Those which I have adopted are, with one or two exceptions, intended to indicate the shape of the 'general form' of the class. Names based on physical characters fail to bring clearly before the mind the shape of the crystal, or to indicate the elements of symmetry present; and those based on particular substances may, with greater knowledge, have to be changed, if the name-substance has to be transferred from one class to another.

The simple notation for the crystal-forms, and the elegant method of geometrical treatment by the stereographic projection and the anharmonic ratio of four tautozonal faces, with which Miller's name is indissolubly associated, have been adopted throughout. In the scalenohedral class of the rhombohedral system Naumann's symbols are so expressive of the geometrical relations of the various rhombohedra and scalenohedra, that I have used his notation as well as Miller's in the representation of these forms.

Free use has been made of the works of previous writers—Naumann, Haidinger, Miller, Story-Maskelyne, Groth, and others. From the late Professor W. H. Miller and Professor M. H. N. Story-Maskelyne I received my training as a crystallographer, so that my debt to them is of a very special and personal nature. I have drawn largely on the stores of information in Professor E. S. Dana's *System of Mineralogy*, 1892; and I am indebted to him and to his publishers, Messrs. Wiley of New York, for clichés of several of the figures in that work. I have to thank Professor Groth and his publisher, Herr Engelmann, for clichés of several of the figures in the *Physikalische Krystallographie*; Mr. Hilary Bauerman for the loan of the blocks of several of the figures in Miller's works; and Dr. J. H. Pratt, of the Geological Survey of North Carolina, for several figures, the clichés of which were kindly supplied by Professor

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CAMBRIDGE,
28th September, 1899.

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CRYSTALLOGRAPHY.

CHAPTER I.

CRYSTALS AND THEIR FORMATION.

1. CRYSTALS are homogeneous solid bodies bounded by plane surfaces arranged according to definite laws of symmetry: their physical properties, such as cohesion, elasticity, optical and thermal characters, are intimately connected with the symmetry of the external form. The plane surfaces are called the *faces* of the crystal.

2. Examination of a few crystals of common substances, such as calcite ($CaCO_3$), quartz (SiO_2), gypsum ($CaSO_4 . 2H_2O$), and soda ($Na_2CO_3 . 10H_2O$), brings to our notice two important characteristics of the external form. They are: (1) the parallelism of the planes in pairs, and (2) the arrangement of them in *zones*, *i.e.* in sets, the planes of each of which intersect one another in parallel edges. A straight line drawn through some fixed point, called the *origin*, parallel to the edges of a set is called a *zone-axis*. In Fig. 1, the planes marked b, m, $m_,$ and those parallel to them form a zone with a vertical zone-axis; the faces marked b, l and l' form a second zone.

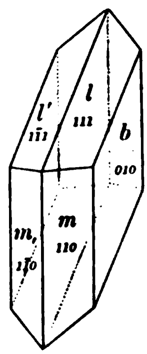

FIG. 1.

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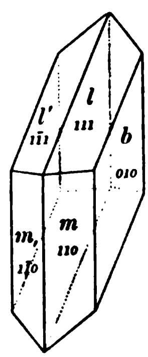

FIG. 1.

3. The connection of the physical properties with external form is strikingly manifested in the case of the cleavage, *i.e.* the property, characteristic of many crystals, of splitting along plane surfaces having certain directions. It is well shown in calcite, fluor and

gypsum. Thus in the latter there is a good cleavage parallel to the plane marked b in Fig. 1. The cleavage-planes, shortly called cleavages, are always parallel to actual or *possible*[1] (Chap. IV. Art. 9) faces of the crystal. They often enable us to get fragments, completely bounded by cleavages, which differ from the original crystals in no respect except in their mode of formation. Such cleavage-fragments as, for instance, those of calcite and rock-salt are sometimes hard to distinguish from true crystals. Experience in the criticism of the character of the crystal-faces will in most cases enable us to distinguish between them and those resulting from cleavage. For the natural faces often show coarse or fine markings—striæ, pittings, &c.—which accord with the symmetry of the face, and are an important characteristic of it. Thus the cubes of pyrites, Fig. 2, are frequently striated in the manner shown in the diagram. A perfect cleavage should be smooth; and markings, perceived on a cleavage-face of a simple crystal, arise from the accidental interruptions in the continuity of the cleavage.

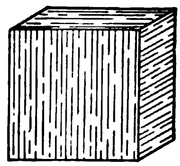

FIG. 2.

Imitations of crystals in glass, or by the cutting of artificial faces on fragments of crystals or metal, such as gold, can generally be easily distinguished from true crystals. False faces, however, sometimes occur in Nature, and may be due to several causes:—as, for instance, to the growth of the crystal being hampered by the surface of some other body, just as that of a crystal growing in a vessel is hindered by the sides and bottom; or to accidental cleavage followed by corrosion, as may have occurred in some crystals of calcite.

True crystal-forms are also found in which the internal structure is not correlated with the external form. In such cases the originally formed crystal has undergone a change in its internal structure with, or without, change of substance. Such altered bodies, in which the substance, or the internal structure, or both, no longer correspond with the external form, are called *pseudomorphs*; and are not to be included amongst crystals: as instances of such transformations we may mention the brownish translucent crystals of sulphur

[1] By a possible face or edge of a crystal is meant one which satisfies the law of development of the crystal. A face possible on one crystal may actually exist on another crystal of the same substance.

formed by fusion, which after the lapse of a day or two change into aggregates of yellow sulphur, and the brown pseudomorphs after pyrites (FeS_2) resulting from the conversion of the iron disulphide into a hydroxide.

4. Crystallization is a property common to almost every substance of which the structure is not the result of organic growth. Instances of crystallization are familiar to everyone: *e.g.* salt and sugar. The crystalline form is characteristic of the substance, and conversely, each chemical compound has a distinctive and characteristic crystalline form, subject however to the modifying principles of isomorphism and polymorphism.

Crystallography is the science which treats of the distribution of the faces on a crystal, of the geometrical relations to which their positions are subject, and of the classification of crystals in groups depending on the distribution of the faces. It applies equally to the natural bodies known as minerals and to the crystals manufactured in the laboratory.

Formation of Crystals.

5. Crystals are formed during the passage of their substance, under favourable conditions, from a fluid to a solid state. Thus, snow is the result of the rapid cooling of aqueous vapour, and consists of numerous minute crystals of ice aggregated together. Crystals of phosphorus have been formed by volatilizing the substance in a hermetically closed tube heated at one end and cooled at the other. Liquid bismuth, on cooling, gives six-faced crystals of bismuth. In the larger number of cases crystals are formed from solutions, in which the liquid, which serves as solvent, contains one or several substances. The quantity of a substance which a given quantity of liquid can, under ordinary circumstances, hold in solution at a definite temperature is limited. At this limit the solution is said to be saturated. The higher the temperature the greater the quantity which can as a rule be held in solution; consequently a solution saturated at a high temperature will deposit crystals as the temperature falls. Thus a solution of nitre (KNO_3) saturated at 30° C. deposits crystals when it is allowed to cool freely. Crystals are also obtained by the evaporation of a solution. They are likewise formed by the mixture of two non-saturated solutions of different substances which react on one another chemically; thus,

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found that the more slowly the substance is deposited, and therefore as a rule the more slowly evaporation takes place, the more perfect are the crystals formed. It has not been found possible to form in the laboratory crystals of all substances, and those of the minerals common in Nature have in many cases not been obtained artificially, and very little is known of the conditions under which the latter have been, and are being, produced.

7. Crystals have, in many cases, the same form at every stage of their growth from the smallest to the largest. The growth here consists simply in the uniform deposition of the substance which passes from the fluid to the solid state. This gives rise to a structure in which the layers parallel to a particular face are completely similar and similarly orientated to each other, for the larger crystal has its faces and edges parallel to those of the nucleus. There seems in this case to be no essential difference in the growth of the crystal at different periods.

In certain cases there is a tendency to grow more rapidly in particular directions or places, as is evidenced by the fact that some crystals are flat tablets, others long and columnar, &c. This tendency affects the general aspect of the crystal and is described by the word *habit*. The crystals of some substances have a fairly constant habit:—as, for instance, sulphur. The habit in other substances is often very variable:—as in quartz, barytes and calcite; and this variableness of habit is one of the greatest difficulties a beginner has to encounter. It may depend, as is generally the case in quartz, on the exceptional development of particular faces at the expense of others; the faces present being the same, or very nearly so, in all the crystals. Or, as is often the case in calcite, it may

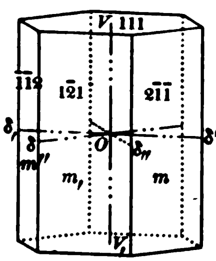

FIG. 3.

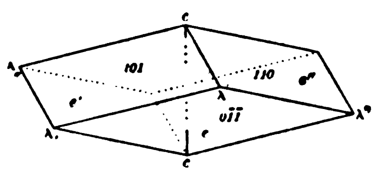

FIG. 4.

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which result in a completely new form. Crystals of alum (Fig. 5) under certain circumstances develop new faces, replacing the edges and corners. The six corners may be cut off and replaced by squares as shown in Fig. 7. In some exceptional cases the change may be so extensive as to completely obliterate certain faces. When, however, new faces appear during the growth of a crystal, they always come (allowance being made for accidental irregularities) in such a way as to modify all similar corners and edges in the same manner. Further, in the disappearance of faces, like ones disappear together. The growth of crystals consists, therefore, partly in the deposition of layers of uniform thickness on each face so as to retain the parallelism of the earlier and later planes; and partly in the development of new faces, or in the obliteration of old ones, in such a way that the symmetry is retained.

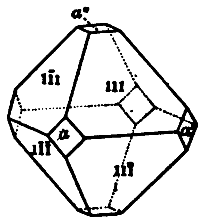

Fig. 7.

CHAPTER II.

THE LAW OF CONSTANCY OF ANGLE.

1. OBSERVATIONS made on the growth of crystals show that the faces when once formed can undergo considerable modifications in their development, but that their relative positions remain the same. Hence, many crystals, which seem strikingly dissimilar, are found on close inspection and measurement of the angles to be bounded by similar faces similarly orientated with respect to one another. Thus Figs. 8 and 9 represent two crystals of garnet. The only difference between the two figures arises from the fact that in the latter the vertical planes meet in edges whilst in the former they meet in points where two other faces inclined to the vertical also meet.

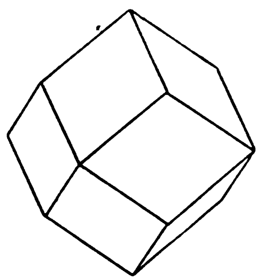

FIG. 8.

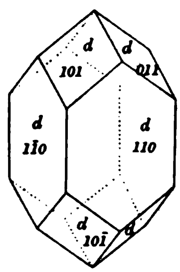

FIG. 9.

2. The relative position of two faces is determined by the *dihedral* (*i.e.* between two faces) angle which they make with one another. This angle is that between the two lines in which a plane perpendicular to the edge of intersection meets the faces—the angle ABC in Fig. 10. We may however take the angle to be N_1ON_2, the angle between the two perpendiculars ON_1, ON_2, drawn from a point within the crystal to meet the faces. For the angles N_1ON_2 and ABC are supplementary, as can be seen from the fact that the lines ON_1, ON_2, AB and BC all lie in a plane perpendicular to the edge, which is supposed to be perpendicular to

the paper. The figure N_1ON_2B can be divided into two triangles N_1OB, N_2OB. But the interior angles of a triangle make up two right angles. The angles of the figure N_1ON_2B are therefore together equal to four right angles, and those at N_1 and N_2 are each 90°. Hence $N_1ON_2 + ABC = 180°$. And AB being produced to E, $ABC + EBC = 180°$. Therefore $N_1ON_2 = EBC$. The lines ON_1, ON_2, drawn through an arbitrary point within the crystal are called the *normals* to the faces. Similarly the inclination of BC to a third plane CD in the zone is given by BCD or by N_2ON_3. The normal-angle between the faces AB and CD is then clearly $N_3ON_1 = N_3ON_2 + N_2ON_1$.

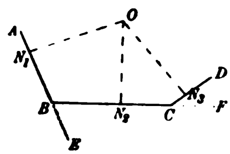

FIG. 10.

The angles given in most modern works on Mineralogy and Crystallography are those between the normals, and normal-angles will be given in this book when the contrary is not stated. When precision of statement is needed to avoid ambiguity, the phrase *normal-angle* will be used in contradistinction to that of *face-angle* used to denote the Euclidean angle such as ABC. French mineralogists still employ the face-angle, and also Lévy's modification of Haüy's notation to represent the faces.

3. The first investigation into the values of the dihedral angles of crystals was made by Nicolaus Steno, whose work[1] was published in Florence in 1669. He cut a series of crystals of quartz of different habits one of which is given in Fig. 11, in two directions at right angles to each other, (i) perpendicular to the edge $[mm_{,}]$, and (ii) perpendicular to one or other of the edges $[mr]$, $[m_{,}z']$, $[m_{,,}r_{,,}]$ of the figure. By placing the sections on paper and drawing lines carefully parallel to the edges he was able to measure the Euclidean angle. In the first set of sections he found that all the angles between the faces m on the same crystal were equal to 120°, and that this was also the angle between similarly placed faces on other crystals. The normal-angle $mm_{,}$ is therefore 60°. In the second series of sections he got figures which were not equiangular and of

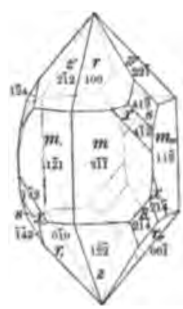
FIG. 11.

[1] An English translation of his work was published in London in 1671. It is hardly necessary to state that the figures and the values of β and γ, given in the text, are not taken from Steno's work.

which Fig. 12 may be taken to be a type. Two of the angles at the vertices had the same value $\beta = 76^\circ\, 26'$ in all cases; the four others had the same value $\gamma = 141^\circ\, 47'$. He, therefore, inferred that in all crystals of quartz corresponding pairs of faces are inclined to one another at constant angles.

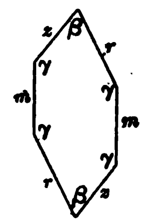

FIG. 12.

4. To Romé de l'Isle, whose 'Essai de Cristallographie' was published in Paris in 1772, is due the credit of establishing, by an extensive series of measurements, the fundamental law of crystallography, viz. that the angles between corresponding pairs of faces of similarly orientated crystals of one and the same substance are always equal. This experimental law we shall denote as that of *the constancy of angles* between corresponding pairs of faces.

He showed that the numerous suites of forms on such minerals as fluor (CaF_2) and calcite may be developed by the modification of edges and corners of a *primitive form*, which he assumed in the former case to be a cube, and in the latter a rhombohedron one of whose face-angles is $105^\circ\, 5'$. When an edge or coign of the primitive form is modified by the introduction of one or more faces it is necessary to repeat the same process on each like edge or coign; in fact to retain the symmetry characteristic of the primitive form adopted. It was seen, however, that the same suites of forms could be derived from different primitive forms. Thus, for instance, those of fluor may be derived with equal ease from either of the common forms, the cube or the regular octahedron. There is nothing to indicate the nature of the primitive form, except frequency of occurrence, or magnitude of development, or the simplicity of character of the form.

5. In the year 1784 the celebrated 'Essai d'une théorie sur la structure des crystaux' of the Abbé Haüy was published, which fully established the correctness of Romé de l'Isle's law and placed the science on a firm basis, although his mode of interpreting the facts has long since been abandoned. Haüy's attention was attracted to the cleavage of calcite by an accident which happened to one of his specimens, which led him to try a series of experiments on the cleavage of crystals of calcite of totally different shapes. He found that cleavage-rhombohedra could be obtained which had identically

the same angles, however different might be the shapes of the original crystals. This suggested to him the idea that the crystals were built up of small nuclei, or 'constituent molecules,' which had the form of regular rhombohedra having the same angles as the cleavage-rhombohedron. A cleavage-slice from a hexagonal prism of calcite, Fig. 3, can be again cleaved in directions transverse to the original cleavage, so that the fragments from the middle of the slice are rhombohedra, whilst the fragments at the edges are not rhombohedra, for they have one face of the original prism as part of their boundary. It is necessary, therefore, to assume either that the nuclei are very small and that the secondary face is of the nature of a slope produced by a series of extremely small uniform steps, or else that the secondary planes on crystals are due to a difference in their nuclei. Haüy saw that the former assumption gave a simple solution of the derivation of secondary faces.

Thus Fig. 13 shows a crystal of calcite of a shape frequent in Derbyshire, and Fig. 14 represents a model (drawn on a larger

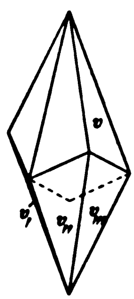

Fig. 13.

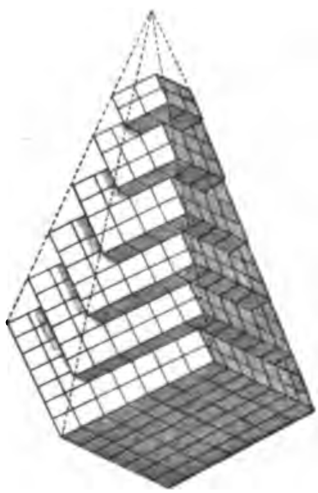

Fig. 14.

scale) by which such a crystal is by Haüy derived from the cleavage-rhombohedron. In Fig. 14 a large rhombohedron, whose middle edges are the same as those of Fig. 13, is built up of layers of

regularly disposed nuclei, each nucleus being similar and similarly placed to the large rhombohedron. A cap is then formed of three plates, one nucleus thick, and placed on the upper faces of the rhombohedron. Each plate is a rhombus similar to the face of the rhombohedron on which it rests, but it has one nucleus less in the edge than the face, and consequently two rows of nuclei in the face are left uncovered. Such a cap can easily be imitated by three equal rhombuses of thick card-board cut with angles of 101° 55′ and 78° 5′, and attached to one another along the edges so that the obtuse angles are at the apex. On this cap a similar cap of three plates is superposed in like manner so as to leave two rows of nuclei of the first uncovered. Each of the plates in this cap is one nucleus thick and has one nucleus less in the edge than the first cap. The same process is continued until the apex is reached. The plates in the last cap must clearly consist of four nuclei, as shown in Fig. 14.

If now a series of exactly similar caps is placed in succession on the lower faces of the large rhombohedron, the model would present a similar aspect at the two ends.

Suppose now the nuclei to be very small, as the ultimate molecules of bodies must be, then the discontinuity in the lines joining the corners of the caps, which is so conspicuous in the dotted lines of Fig. 14, becomes imperceptible and we have the crystal shown in Fig. 13, known as a scalenohedron of the rhombohedral system.

Had the number of nuclei in the edges of the first cap placed on the large rhombohedron been the same as in its edges, the lines joining the projecting corners would not have converged, but would have been parallel. The same will clearly be true of any number of successive caps, for in this case they have the same dimensions. In this way a hexagonal prism $\{10\bar{1}\}$ terminated by the primitive rhombohedron can be easily obtained.

6. Crystals of galena (PbS) and rock-salt (NaCl) cleave along planes parallel to the faces of a cube. We can in these cases derive the secondary faces from a square plate formed in one of two ways. (i) The sides of the cubic nuclei may be parallel to the edges of the plate in chessboard fashion, or (ii) the sides of the nuclei may be diagonally placed to the edges of the plate. In the latter the edges of the plate will resemble a staircase in which the height of the step

is equal to the tread. If now to a plate of this kind similar plates are attached on both sides with a *decrement* of one nucleus on each side of the plate, and then to the outer surfaces of these plates similar plates with a like decrement of one nucleus are attached, and the process repeated to the limit, we obtain a solid like that in Fig. 15, which, when the nuclei are very small, is the regular octahedron common on crystals of galena. Now the height of each layer is the edge of the cube, a, and the breadth of the decrement on each edge is clearly one-half the diagonal of the cubic face $= a \div \sqrt{2}$. Hence, the inclination of the face to the plane of the original plate, θ, is given by $\tan \theta = \text{height} \div \text{breadth} = \sqrt{2}$, $\therefore \theta = 54° 44'$. The Euclidean angle between the faces of the octahedron meeting in an edge is $2 \times 54° 44' = 109° 28'$.

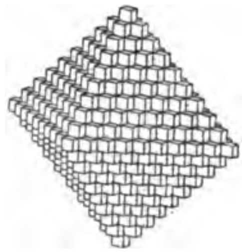

FIG. 15.

The result of a decrement of one nucleus on each edge of a plate in the arrangement (i) is shown in the rhombic-dodecahedron, Fig. 16. In this the plates are shown undivided into their constituent cubes, and the figure is supposed to start with a large cube, and not with a plate of single nuclei. The angle each face of the secondary form makes with that of the cube on which it is raised is 45°, for the height of each layer is equal to the breadth of the decrement. Hence, the faces meeting at an edge of the large cube coalesce in one plane, and form a rhombus. Had we, however, taken a decrement on each edge whose breadth was double the height of the layer, each face would be inclined at 26° 34′ to the cubic face, for $\tan 26° 34' = 1 \div 2$. We then get a tetragonal pyramid on each face of the cube, and the form is that fully described in Chap. XIV. Both these secondary forms are found on crystals of rock-salt.

FIG. 16.

7. In the manner illustrated by the preceding examples Haüy gave an explanation of the constancy of the angles between corresponding pairs of faces, and established the further fact that the angles, which the secondary faces make with those of the primitive

form are not accidental and arbitrary, but depend on the ratio of whole numbers involved in the decrements. This relation of whole numbers is now expressed by the phrase 'the law of rational indices' (see Chap. IV. Art. 9). It is not essential to Haüy's theory that the nuclei should be solid polyhedra exactly filling the space. It suffices to suppose that his nuclei are cells each of which contains a similarly placed molecule.

These are none of them "difficulties"

8. The difficulties in accepting Haüy's theory are serious. (1) It does not follow that, because a body breaks up by cleavage into definite polyhedra, it has been built up by layers of these polyhedra. (2) It has not been possible to obtain cleavage polyhedra of all crystals. Some crystals cleave only in one, or two, directions, and some not at all. (3) It seems reasonable to expect that fairly compact, and therefore stable, bodies can be built up when the cleavage polyhedra are cubes as in galena, or rhombohedra as in calcite, or are four-sided prisms terminated by a pair of parallel faces as in barytes. For in all these cases the constituent polyhedra can be arranged so as completely to fill a definite portion of space. But in fluor the cleavage form is the regular octahedron, and the nuclei, when arranged in like orientation, will only touch one another at their edges and will leave a considerable portion of space unoccupied. Such a structure would probably be very unstable and be easily deformed. Furthermore, the facts, (1) that we have a series of different forms all connected together, and (2) that the secondary faces depend on whole numbers, would still be explained if any other simple form of the suite were taken as primitive.

CHAPTER III.

SYMMETRY.

1. ATTENTION has been already called to the fact that the faces on crystals usually occur as pairs of parallel planes. When the two faces of each pair are coexistent and have the same physical character, *i.e.* the same lustre, striation or other marks, the crystal is said to be centro-symmetrical, or to be symmetrical with respect to a point within it, which is called the *centre of symmetry*. Figs. 1 and 2 are diagrams of centro-symmetrical crystals. Professor Maskelyne expresses the relation by saying that the crystal is *diplohedral*, *i.e.* that every face-normal through the centre terminates on opposite sides in a pair of faces perpendicular to it.

2. There are crystals, however, like the tetrahedron, Fig. 17, in which the faces do not occur as pairs of parallel planes. From such figures it is easy to derive figures having parallel faces. Thus, by drawing planes as indicated by the small triangles, the octahedron, Fig. 18, can be obtained. Conversely, from the parallel-faced figure, such as Fig. 5, it is easy to derive a non-central figure like Fig. 17, by the omission of one of each pair of parallel planes. In more complex crystals there may be more than one way of selecting the faces to be omitted, so that we may get different non-central figures. Up to very recent times it has been usual to regard the parallel-faced crystal as the normal one and the non-central figure as caused by some deficiency in the symmetry. Other cases of a geometrical connection between different forms are also common, such that the form

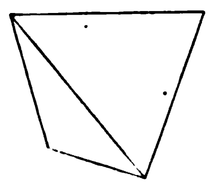

FIG. 17.

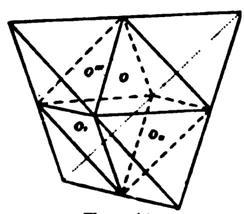

FIG. 18.

having fewest faces manifests one-half, or one-quarter, the number observed on the most symmetrical cognate form; the faces present being arranged according to definite rules and not being due to accident in growth. The most highly developed form was denoted as *holohedral*[1], whilst the partial form was, and still is, called *hemihedral*[2] when it contains half the number of faces in the holohehral form. When the number of faces in the lower form is only one-fourth of that in the holohedral form, the partial form is said to be *tetartohedral*[3]. When it is not desired to express the exact numerical relation, the partial form is said to be *merohedral*[4]. These words do not accord with modern views on the symmetry of crystals, and are only retained for the sake of convenience.

3. From what has been said in Chapter I. as to the growth of crystals we see that, if the particles be themselves centro-symmetrical, and likewise the forces which cause them to cohere, a crystal growing freely in a liquid will grow equally well in opposite directions. If, however, there be a lack of symmetry, either in the particles or in the forces, the facility of growth may be on one side very different from that on the other side.

It has been already pointed out that accidents of various kinds may modify the regularity of growth of a crystal. Hence the opposite and parallel faces of centro-symmetrical crystals need not be of the same dimensions. All that is needed is that the physical characters, as indicated by the lustre, striation, &c., and also the resistance to corrosion, should be of an exactly like kind. Furthermore, it is clear that the deposition of matter on the opposite faces must under like conditions take place with equal facility.

4. Again the geometric law connecting crystal faces, known as that of rational indices (Chap. IV. Art. 9), shows that parallel faces are always possible, but does not require coexistence or identity of character. In non-central crystals parallel faces, which are not parallel to an axis of two-, four-, or six-fold symmetry (Arts. 7 and 8), have to be regarded as quite distinct faces, and often differ in lustre, striations and other physical properties. The minerals fahlerz and boracite give good instances of non-central crystals of shape like that given in Fig. 17. Crystals of quartz often appear to be

[1] From ὅλος whole, and ἕδρα base. [2] From ἡμι- half, and ἕδρα base.
[3] From τέταρτος one-fourth, and ἕδρα base.
[4] From μέρος part, and ἕδρα base.

centro-symmetrical, for they are bounded by pairs of parallel faces, *m*, *r* and *z*, but the physical characters of all untwinned crystals of the substance and also the development of certain subordinate faces on some crystals, such as the faces *s* and *x*, in Fig. 11, have conclusively proved them to have no centre of symmetry.

5. In simple crystals, such as are represented in the preceding diagrams, certain faces are observed to have the same physical characters, and, when the crystals are very regularly developed as in Figs. 1, 2, 5, 7 and 8 the similar faces have the same shape and magnitude. On quartz, Fig. 11, the faces *m* manifest this similarity in a most striking manner, for they are almost invariably all striated in directions perpendicular to their mutual intersections. Such like faces are found to be symmetrically arranged with respect to certain planes, or axes, or both, which are parallel to actual, or possible, faces or edges of the crystal. In complicated and irregularly developed crystals, such as Figs. 6 and 9, the similarity of shape of the faces is often lost, and it frequently happens that the physical characters are not sufficiently marked to enable us to discriminate those which belong to the same set of planes. The symmetry becomes obscured, and can only be established by measurement of the angles and careful criticism of the development of several crystals. Dodecahedra of garnet are frequent in which the symmetry is concealed by the unequal development of the faces. Fig. 9 shows one type of such irregular growth. The nature of the figure can, however, be generally perceived in such crystals by careful inspection of the physical characters of the faces and by comparison with a regularly grown dodecahedron like Fig. 8, and is at once established by measurement of the angles between adjacent faces which are either 60° or 90°.

6. By symmetry with respect to a plane is meant the similarity of position of like coigns[1], edges and faces on opposite sides of the plane. A plane of symmetry divides a crystal into two like halves, which are to one another as an object is to its image seen in a mirror occupying the position of the plane. It thus bisects the angle between corresponding faces and edges. We shall often express the relation of two planes, or two lines, to a plane of

[1] By *coign* is meant the solid angle or corner at which three or more faces meet.

symmetry bisecting the angle between them by the statement that they are *reciprocal reflexions* in the plane. We shall also use the phrase to express that two crystals are related to one another as an object is to its image in a mirror in those classes of crystals in which such a relation exists. Thus the right and left hands are, when held out palm to palm, reciprocal reflexions of one another. Symmetry does not refer in crystals to the actual position, but to the relative angular position of corresponding faces and edges; *i.e.* they are equally inclined to a plane of symmetry on opposite sides of it. In a regular figure, such as that of a crystal of barytes given in Fig. 19, corresponding faces, indicated by the same letters, will (if produced) meet in straight lines lying in the planes of symmetry. In the figure two planes of symmetry are perpendicular to the paper: the one bisecting the faces *o*, *c*, *o′*; the other at right angles to the former and bisecting the faces *u*, *d*, *l*, *c* etc. The figure illustrates the repetition, in pairs, of faces inclined to the planes of symmetry, and also the fact that when the face is perpendicular to the plane of symmetry there is but one face, the two halves of which are reciprocal reflexions. It is extremely rare to find crystals so regularly developed as is indicated in diagrams, and the presence of a plane of symmetry, Σ, is proved by the equality of angles between faces having similar physical characters and lying on opposite sides of Σ.

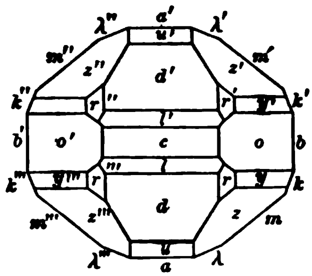

FIG. 19.

7. An axis of symmetry is a straight line about which the faces, edges and coigns are regularly disposed, so that if the crystal be turned about it through a definite aliquot part of a complete revolution similar faces, edges and coigns are interchanged; that is, any particular face or edge in the new position occupies exactly the place of a similar face or edge in the original position. We shall generally express the relation by saying that the like faces, edges and coigns disposed about an axis of symmetry are interchangeable or *metastrophic*[1].

[1] μεταστρέφειν to turn round, to change.

8. If the least angle of rotation about an axis be 180°, the faces, edges and coigns are associated in pairs, and it is clear that a second rotation brings the crystal back into its original position. Such an axis of symmetry will be called a *dyad axis*. The line through the centre perpendicular to the paper in Fig. 19 is a dyad axis; also the lines joining the middle points of the opposite edges of the regular tetrahedron, Fig. 17.

If the least angle of rotation giving interchangeability of like faces etc. be 120°, then the faces, edges and coigns, occur in sets of three, *i.e.* in triplets or triads. The axis will be called a *triad axis*. The vertical line through the two apices of Fig. 11, and parallel to the faces *m*, is such an axis. The faces *r*, *z*, *s* and *x*, occur at each end in sets of three which are interchangeable on rotation about the vertical axis through 120°.

When the least angle which gives interchangeability is 90°, the rotation can be effected four times before the crystal returns to its original position. The axis is in this case a *tetrad axis*. The vertical lines through the apices in Figs. 8 and 9 are tetrad axes.

The only remaining axis of symmetry occurring in crystals is one of six-fold symmetry—the *hexad axis*. The least angle of rotation is 60°, and a crystal face or edge can occupy in succession six different positions. Such an axis is observed in well developed crystals of apatite and is shown in Fig. 20, where the hexad axis is parallel to the lines [*ma*] and perpendicular to the face *c*.

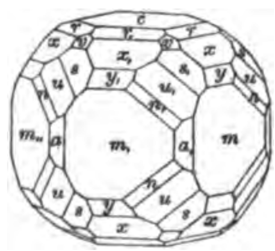

FIG. 20.

An axis of pentad (five-fold) symmetry, so common in flowers, is not possible in crystals; that is, it would give a set of associated crystal-faces inconsistent with the laws which have been found to regulate the positions of faces.

Several of these axes of symmetry of the same, or of different, degrees[1] may occur together in one and the same crystal; and, in fact, an axis of degree *n* is often associated with a set of *n* dyad axes all perpendicular to it.

[1] The degree of an axis of symmetry indicates the number of like things disposed around it, or the number of times rotation giving interchangeability can be effected before the original position is regained. Thus a triad axis is of degree 3.

9. For the purpose of illustration the following table of angles on the crystal of barytes, Fig. 19, is given; we shall find the angles useful at a later stage.

(i)	$a\lambda$	22°	10′
	λm	17	1
	mk	28	34
	kb	22	15
	bk'	22	15
	$k'm'$	28	34
	$m'\lambda'$	17	1
	$\lambda'a'$	22	10

(ii)	au	31°	50′
	ud	19	19
	dl	16	55
	lc	21	56
	cl'	21	56
	$l'd'$	16	55
	$d'u'$	19	19
	$u'a'$	31	50

(iii)	mz	25°	41′
	zr	18	13
	rc	46	6
	cr''	46	6
	$r''z''$	18	13
	$z''m''$	25	41

(iv)	az	45°	41′
	zy	18	18
	yo	26	1
	oy'	26	1
	$y'z'$	18	18
	$z'a'$	45	41

(v)	bz	55°	18′
	zu	34	42
	uz'''	34	42
	$z'''b'$	55	18

The angles in the columns connected by a straight bond are all measured in one zone and this method of giving the angles between a number of tautozonal faces will be frequently used throughout the book. For the sake of brevity[1], the equal angles between the pairs of faces parallel to those shown have been omitted.

The symmetry manifested in the drawing, and inferred from inspection of the crystal with respect to the central planes parallel to a and b, is now confirmed by the equality of the angles on opposite sides of these planes. Measurement of one zone, such as (i), is not enough to establish this symmetry, for in a centro-symmetrical crystal this zone will only show that there is at least one plane of symmetry parallel to a or to b, and one dyad axis perpendicular to the plane. But the equality of angles between corresponding pairs of faces in zones (i), (ii) and (iv) is only consistent with a plane of symmetry parallel to face a. Again the equality of angles in zones (i) and (v) entails a plane of symmetry parallel to b and at right angles to the former one. We shall see later on that (1) the lines of intersection of these two planes and (2) the perpendiculars

[1] In actual measurement of a zone the student should always measure every angle and go completely round until he reaches the face from which he started. The second reading on this face should be recorded, for it will serve to control the accuracy of the angles. Any accidental displacement of the crystal during the observations will probably be indicated by a defect in adjustment, and also by a considerable divergence from the first reading.

on them from the centre are axes of dyad symmetry. That the line through the centre perpendicular to the paper is an axis of dyad symmetry is clear from the figure, and is confirmed by the equality of angles on opposite sides of it in zones (ii) and (iii). As the crystal has parallel faces the plane of the paper occupies the position of a plane of symmetry.

The student should bear in mind that the two halves into which a crystal is divided by a plane of symmetry cannot, as a rule, be brought into the same position. They are related to one another in a manner similar to that of the right and left hands, and may be said to be *antistrophic*[1]. Similarly the corresponding faces and edges may be said to be antistrophic.

10. Planes and axes of symmetry have been found to satisfy the following conditions: (1) They must be parallel to possible faces, or zone-axes, of the crystal. (2) A plane of symmetry must be perpendicular to a possible zone-axis; and an axis of symmetry perpendicular to a possible face of the crystal. (3) When there are several planes of symmetry in a crystal the angles between any pair of them must be one of the four—90°, 60°, 45°, or 30°. These angles owing to their importance in determining the *system*, or type of symmetry, of a crystal have been called the *crystallometric angles.* (4) The angles of rotation possible about an axis of symmetry are 180°, 120°, 90°, or 60°. This may be stated otherwise: that axes of symmetry can only have the degrees, 2, 3, 4, or 6. The above and other relations between *the elements of symmetry*—using this phrase to denote generally planes, axes and centre of symmetry—form the subject of Chapter IX.

The first relation has been found to be true in all the crystals which have been observed; but in the case of a triad axis it is not capable of deduction from the purely geometric relations resulting from the rationality of indices.

11. The set of like faces, edges and coigns, which are connected together by the symmetry of the crystal and which have like physical characters, will be called *homologous* faces, edges and coigns. The simple figure, whether closed or not, which consists of one set of homologous faces will be called a *form*. Many of the figures so far given, such as Figs. 2, 8, 17, represent single forms, whilst

[1] ἀντιστρέφειν to turn to the other side.

that of quartz, Fig. 11, represents five separate forms, whose faces are denoted by the letters *m*, *r*, *z*, *s* and *x*. The same letter will be used to denote the several faces of a form, and, as a rule, dashes or numerals will be used to distinguish the one from the other. The dashes of parallel faces will be the same but placed at the top and the bottom respectively. Occasionally some of the faces are left unlabelled (as has been done in Fig. 13) when no confusion is likely to arise from doing so, and this is more especially the case where very many like faces occur together. If the faces of forms on a crystal are approximately equidistant from a point within it, the faces belonging to the same form are similar in size and shape; and each form is said to be *equably developed.* Any displacement, outwards or inwards, of one or more of the faces will clearly disturb this regularity of development. In the pairs of figures, Figs. 5 and 6 and Figs. 8 and 9, we see that the irregular figures are deduced from the regular ones by the displacement of some of the faces whilst they retain the constancy of their inclinations to the adjacent planes and therefore the parallelism of the edges. The new positions of these displaced faces are respectively parallel to those they would occupy in equably developed crystals. In theoretic discussions we consider the forms to be equably developed. The crystal-figures given in books, as also crystal-models, are, for the most part, such ideal representations of natural crystals.

This intensely stupid representation of an extremely simple matter is better treated in Chap XIX p 557; though not too well even there

CHAPTER IV.

AXIAL REPRESENTATION.

1. THE relative positions of faces on a crystal can be given by the axial method of geometry. For this purpose three planes—parallel to three faces of the crystal which are inclined to one another at any finite angles, but are not in one and the same zone—are taken as *axial planes*. The lines in which they intersect are the *axes of reference*, and are sometimes called the *crystallographic axes* or, shortly, the *axes*. The axes are supposed to meet at a point within the crystal, and this point is called *the origin*. It is generally denoted by the letter O, and the axes by the letters OX, OY and OZ. The projection on the plane of the paper, or perspective view, of such a set of axes is shown in Fig. 21. From what precedes, the axes are necessarily parallel to three edges, or zone-axes, which do not lie in one and the same plane. When they are shown in diagrams they will usually be given by interrupted lines consisting of strokes and dots.

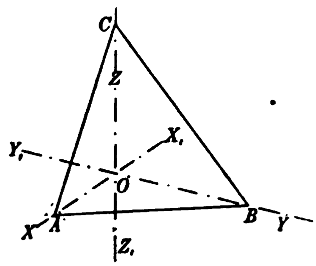

FIG. 21.

2. A fourth face of the crystal, taken at any arbitrary distance from the origin and meeting all three axes at finite distances, cuts off from them respectively lengths, OA, OB and OC, which depend on the actual position of the face. But the ratios of these lengths for all planes parallel to the face are constant, and depend only on the angular relations of the fourth and the axial planes to one another. Furthermore the angles between the similarly placed faces on crystals of the same substance are constant (Chap. II. Art. 4). Hence the angles between the axes and the ratios $OA : OB : OC$ are constant for all crystals of the same substance,

provided the faces selected for this purpose be similarly placed on the several crystals, or be (as we shall, for the sake of brevity, say) the same faces. In theoretical and general discussions the three lengths OA, OB and OC on the axes are denoted by a, b and c respectively, and are called *the parameters* of the crystal. In systematic descriptive treatises on minerals and chemical compounds, one parameter, usually b, is given as unity, and the other two are then the numbers which give the fixed ratios $a : b$ and $c : b$. These ratios are determined by trigonometrical formulæ, which will be given under the different systems, from the angular relations between the fourth (*parametral*) plane and the three axial planes to one another. The angles between these several planes are in many cases determined by direct measurement, or are calculated from angles measured between other faces by methods depending on the laws of crystallography.

3. In crystals which manifest no symmetry, or only centrosymmetry, planes parallel to any three faces of the crystal, which are not in a zone, serve equally well as axial planes. Such crystals are said to belong to the anorthic, or triclinic, system. In other classes of crystals the faces of the crystals are arranged in sets of homologous faces (forms) in a manner dependent on the elements of symmetry present. It is, therefore, convenient to select the axial planes in a manner which accords with the symmetry, and which enables us to represent the set of faces constituting a form in the simplest manner.

4. In order, however, to be able to represent planes on all sides of the crystal, and lying therefore in any one of the eight segments—commonly called *octants*—into which space is divided by the axial planes, we must adopt the further convention of analytical geometry, that lengths measured along OX are positive, and those along $OX_{,}$ are negative. Similarly lengths along OY and OZ are positive, whilst those along $OY_{,}$ and $OZ_{,}$ in the opposite directions are negative. The combinations of these various signs of the intercepts enable us to represent a plane lying in any one of the eight octants. Thus a plane in the upper left-hand octant to the front would in Fig. 21 have the intercepts on OX and OZ both positive, that on $OY_{,}$ negative.

5. If, as is frequently the case, several planes be so situated that they meet the axes on the same side of the origin, one of

them, as ABC in Fig. 22 can be selected to give the three parameters a, b and c. And any other, such as HKL in the same figure, can be represented as intercepting lengths, $OH = a \div h$, $OK = b \div k$ and $OL = c \div l$, on the axes OX, OY and OZ respectively. The numbers h, k, l depend on the relative position of the new face HKL to the axes and the parametral plane ABC. They are called the *indices* of the face, and clearly fix its position when the positions of the axes and the ratios $a : b$, $c : b$, are all known.

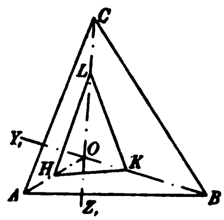

Fig. 22.

6. If the intercepts $(a \div h,\ b \div k,\ c \div l)$ on the axes are multiplied by any common factor r, *i.e.* are increased or diminished in any common ratio, the plane is simply displaced parallel to itself.

Let HKL, Fig. 23, represent a face making intercepts, $a \div h$, $b \div k$, $c \div l$, on the axes; and let $OH' = ra \div h$, $OK' = rb \div k$, $OL' = rc \div l$.

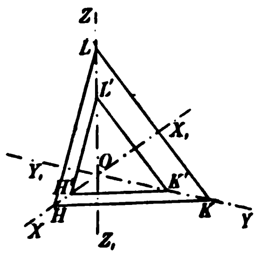

Fig. 23.

Then $OH' : OH = OK' : OK$; and by Euclid vi. Prop. 4 the line $H'K'$ is parallel to the line HK.

Similarly, $OK' : OK = OL' : OL$; and the line $K'L'$ is parallel to KL.

Hence, the two planes $H'K'L'$ and HKL have two lines of the one parallel to two lines of the other. They are therefore parallel planes (Euclid xi. Prop. 15).

The face can be equally well represented by $a \div h$, $b \div k$, $c \div l$; or by $ra \div h$, $rb \div k$, $rc \div l$; since we are not concerned with its actual distance from the origin. Hence parallel faces on the crystal, which are, therefore, on opposite sides of the origin, can be deduced the one from the other by multiplying the intercepts, or the indices, by -1, *i.e.* by a simple change of signs of all the intercepts or indices. This is directly obvious if a model of the axes, similar to Fig. 24, be made in cane or wire, and the positions of the two planes be indicated by lines of Berlin wool. For

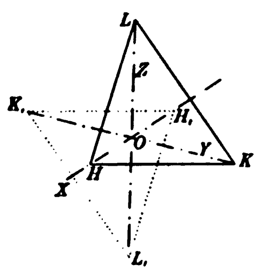

Fig. 24.

$OH_, = -OH = -a \div h$, $OK_, = -OK = -b \div k$, $OL_, = -OL = -c \div l$.

7. Suppose a face to be parallel to one of the axes, OZ (say), and therefore to meet this axis at an infinite distance. Such a plane can be represented as meeting the axes at ma, nb, ∞c, or at $a \div h$, $b \div k$, $c \div 0$, according as we take multiples, or submultiples, of the parameters to give the intercepts. The symbol ∞ represents an infinitely large number, which can also be expressed as $1 \div 0$.

8. We have already pointed out that, in the case of planes meeting the axes in octants other than the first, the signs of the intercepts must correspond with the directions of the axes forming the edges of the octant. Thus a face passing through $HK_{,}L$ of Fig. 24 makes on the axes the intercepts, $a \div h$, $-b \div k$, $c \div l$. A face passing through $HK_{,}L_{,}$ has the intercepts $a \div h$, $-b \div k$, $-c \div l$. The face through $H_{,}KL$ the intercepts $-a \div h$, $b \div k$, $c \div l$, and the face $H_{,}KL_{,}$ the intercepts, $-a \div h$, $b \div k$, $-c \div l$. In the two remaining octants we can have planes $H_{,}K_{,}L$ given by $-a \div h$, $-b \div k$, $c \div l$; and $H_{,}K_{,}L_{,}$ given by $-a \div h$, $-b \div k$, $-c \div l$.

The law of rational indices.

9. We are now in a position to enunciate this law, which is the same as that of decrements discovered by Haüy.

If a crystal be referred to three axes parallel to three edges of the crystal, which do not all lie in one plane; and if a face be taken which meets all the axes at finite distances a, b and c, respectively; then the position of all other faces possible on the crystal can be given by intercepts, $a \div h$, $b \div k$, $c \div l$, where h, k and l, are commensurable (rational) numbers, positive or negative. Amongst these commensurable numbers zero has to be included. The three numbers are called *the indices of the face.*

10. A plane's position, given by the ratios $3a \div 4 : b \div 2 : 3c$, can equally well be given by $a \div 4 : b \div 6 : c \div 1$; the latter being obtained by dividing each term of the first set by 3. The latter representation is convenient for the following reason. Whewell proposed, and Miller brought into use, the convention that the parameters a, b, c should always occupy *the same order*, and that the ratios giving the intercepts should be so arranged that they consist of these parameters divided by integers which may be positive or negative, or one or two of them but not all three may be zero. The notation can then be simplified by omitting the parameters and writing only the integers which serve as divisors. Care must be taken in writing these latter that they are given in the

order in which they refer to the axes of X, Y and Z, respectively. The sign must also be placed over the number when the intercept is negative. This position of the minus sign is not essential, but it has the advantage of condensation and avoids all risk of ambiguity.

When the intercepts have been arranged in the manner required by the convention, so that for the intercepts $a \div h$, $b \div k$, $c \div l$, the numbers h, k and l are integers, the face can be represented by *the symbol* (hkl). A simple curved bracket is used to inclose the numbers if there is any likelihood of their being confused with other sets of integers. When letters are used to indicate numbers in general, they may be essentially negative although this is not shown. If, however, the same letters apply to planes meeting the axes on opposite sides of the origin then the sign must be expressed. Thus $(\bar{h}\bar{k}\bar{l})$ is the symbol of the face parallel to (hkl). When particular numbers such as 1, 2, 3 &c. are involved, the sign must, if negative, be always indicated, as is done in $(1\bar{2}1)$.

Thus on a crystal referred to definite axes and a definite parametral plane, the parameters a, b and c, being therefore known, the following are possible faces:

(112)	with intercepts	$a, b, c \div 2$
$(1\bar{2}1)$	,, ,,	$a, -b \div 2, c$
(231)	,, ,,	$a \div 2, b \div 3, c$
(201)	,, ,,	$a \div 2, b \div 0, c$
(100)	,, ,,	$a, b \div 0, c \div 0.$

11. Haüy's law excludes from crystal faces such planes as:— $a \div \sqrt{2}$, b, $c \div 3$. For the intercept on the axis of X has the incommensurable surd $\sqrt{2}$ for the divisor of the parameter. No surd can be the index (representing the submultiple of the parameter) of a crystal-face. The parameters themselves may have to one another any ratios commensurable or incommensurable; and it is easy to show that the lengths cut off by definite crystal faces on lines having *different directions* are in some systems to one another as the surds $\sqrt{2}$ and $\sqrt{3}$.

12. We shall see later on that, by a judicious selection of the axial system and parametral plane, the indices of all the faces of a form are easily determined when those of one of its faces are known. Except in the case of certain classes of crystals having a single triad or hexad axis, all the remaining faces have the same indices as the given one, but either the signs, or the order, or both

signs and order, must be changed. When the rule governing this change is known it is clearly sufficient to give the symbol of one face. In order to make it clear that all the connected faces are included we shall use complex brackets, {}, to denote the form. Thus the *symbol* $\{hkl\}$ denotes the whole set of faces constituting the form of which (hkl) is one face. In the exceptional cases above mentioned some of the faces have the same indices in which sign and order may both be changed, whilst other faces have a second set of indices connected with the first by a simple rule. When we discuss these classes we shall use symbols such as $\{hkl, pqr\}$ to denote the form.

When the faces are labelled in the manner described in Chap. III. Art. 11, we shall often refer to a particular face or form by the letter serving as label or by its symbol; and we shall often join the two together. Thus $P(hkl)$ denotes the face labelled P whose symbol is (hkl).

13. The parameters a, b, c are lengths, which can be expressed in terms of the units, inches or millimetres. The *parametral ratios* are numbers which are constant for the same crystal, provided they are deduced from the same axial planes combined with a *definite* face of the crystal as parametral plane. In most cases this latter face is selected arbitrarily with a view to assign the simplest indices to the faces, but any other face of the crystal meeting all three axes at finite distances may be taken. It follows, however, from the law of rational indices that the parameters obtained by taking a new parametral plane are simple commensurable multiples, fractional or integral, of the parameters given by the first plane. So far no limitation has been placed on the possible values of the parametral ratios, which will be best discussed when we consider the several classes of crystals.

14. By the aid of the law of rational indices and the principle of symmetry the faces of a crystal can be represented in a simple manner. For this purpose, the nature of the symmetry, or, what amounts to the same thing, the *system* and its subdivision must be first determined. This is often easily done by inspection, but the inference needs confirmation by goniometric measurement. The angles between the chosen axial planes, when these are not already given by the symmetry, must then be determined; and finally the parametral plane must be selected and the angles it makes with the axial planes determined. Five constants are thus involved in the

complete determination of the character of a crystal as above indicated; viz. the three angles between the axial planes, or between the axes—the latter being deducible from the former, or vice versâ—and the two ratios $a \div b$ and $c \div b$ deduced from the angular relations of the axial and parametral planes to one another. These five constants, or five angles from which they can be determined, constitute *the elements* of the crystal. They involve measurement of five of the angles between the four planes. When the five elements are known, every face of the crystal can be represented by a symbol such as (hkl), and the corresponding form by the symbol $\{hkl\}$.

Furthermore, the angles which any face (hkl) makes with the axial, or any other, planes of the crystal can be determined by comparatively simple formulæ for all possible values of h, k and l. The converse problem is that of finding h, k and l from measurement of the angles between the faces actually present on the crystal. The solution of these problems involves some knowledge of geometry and of plane and spherical trigonometry. If, however, the zones on the crystal be well marked and easily recognised, the determination of the symbols of the faces becomes easy without the knowledge of any angles. In such determinations, which will be illustrated by examples in the following chapters, the angles between the axes and the values of the parameters are supposed to be known; and it often happens that even their determination involves only the measurement of two or three angles.

The equations of the normal.

15. The connection of the indices of faces with the angles between them will be the subject of much of our future discussion, and it will be well to give at once the fundamental relations which connect the normal on the face (hkl) with the axes.

Let, in Fig. 25, OX, OY and OZ be the three axes, the parameters on which are a, b, c respectively. Let HKL be the face (hkl) meeting the axes at the points H, K and L respectively; and let OP be the normal on the face from the origin: and let P be the point where the normal meets the face. Join HP, KP and LP. These lines, shown by interrupted strokes, lie in the

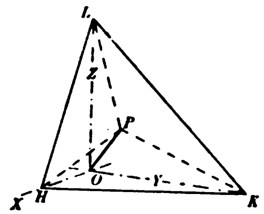

Fig. 25.

face and are at right angles to OP; for the perpendicular on a plane is at right angles to every line in it. Hence in the right angled triangles POH, POK and POL we have:

$$\frac{OP}{OH} = \cos XOP,\ \frac{OP}{OK} = \cos YOP,\ \frac{OP}{OL} = \cos ZOP.$$

But $$OH = a \div h,\ OK = b \div k,\ OL = c \div l.$$

$$\therefore\ OP = \frac{a \cos XOP}{h} = \frac{b \cos YOP}{k} = \frac{c \cos ZOP}{l} \ \ldots\ldots\ldots(1).$$

The length OP is a positive length, hence each of the terms of equations (1) is also positive. But OP is otherwise quite an arbitrary length and depends on the accidents of the freedom of growth on particular portions of the crystal. Hence equations (1) give us only two definite equations which regulate: (a) the angles, when the parameters and indices are known, or (b) the parameters, when the angles and indices are known, or (c) the indices, when the angles and parameters are given.

16. The trigonometrical ratios, $\cos XOP$, $\cos YOP$, $\cos ZOP$ are called the direction-cosines of the line OP. They are connected together by a constant relation, independent of the parameters and indices, which is to be found in all treatises on Analytical Solid Geometry (*e.g.* Frost and Wolstenholme, Ed. I. p. 19). In the systems in which rectangular axes are employed (viz. the cubic, tetragonal, and prismatic) the relation reduces to the very simple one:

$$\cos^2 XOP + \cos^2 YOP + \cos^2 ZOP = 1 \ \ldots\ldots\ldots\ldots(2).$$

Equation (2), or its equivalent when the axes are not rectangular, and the two definite equations obtained from (1) enable us to completely determine the angles and therefore the direction of the normal when a, b, c and h, k, l are known.

Thus, if P be the face (111) of cassiterite (SnO_2) in which the axes are rectangular and $a = b$:

$$a \cos^* XP = a \cos YP = c \cos ZP = \frac{^{\dagger}\sqrt{\cos^2 XP + \cos^2 YP + \cos^2 ZP}}{\sqrt{\frac{2}{a^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{2}{a^2} + \frac{1}{c^2}}},$$

since $\cos^2 XP + \cos^2 YP + \cos^2 ZP = 1$.

* When, as is usually the case, angles are measured at the origin we need not insert the letter O. XP may be taken to be the arc of a circle subtending the angle at the origin.

† This term and the similar one in p. 31 are obtained by adding the squares of the numerators and denominators of the preceding terms after they have undergone a slight modification and then extracting the square root of the sums.

Whence $\cos XP = \cos YP = \dfrac{c}{\sqrt{2c^2+a^2}}$; and $\cos ZP = \dfrac{a}{\sqrt{2c^2+a^2}}$.

Similarly for z (321) of the same mineral

$$\frac{a\cos Xz}{3} = \frac{a\cos Yz}{2} = \frac{c\cos Zz}{1} = \frac{+\sqrt{\cos^2 Xz + \cos^2 Yz + \cos^2 Zz}}{\sqrt{\frac{13}{a^2}+\frac{1}{c^2}}} = \frac{ac}{\sqrt{13c^2+a^2}};$$

$$\therefore \cos Xz = \frac{3c}{\sqrt{13c^2+a^2}}, \quad \cos Yz = \frac{2c}{\sqrt{13c^2+a^2}}, \quad \cos Zz = \frac{a}{\sqrt{13c^2+a^2}}.$$

It might seem, at first sight, as if in extracting the square root we might equally well take the negative sign. But in the above instances OP and each term of equations (1) are positive and therefore the angles must be all less than 90° and their cosines positive.

17. So far equations (1) have only been shown to hold for positive values of h, k and l. They are, however, true generally whatever be the values of the indices. Suppose the face in Fig. 26 to be parallel to one of the axes, OZ (say), and to pass through the same points H and K as before. Then the intercept $c \div l$ on the axis of Z is infinite, and l is zero. The denominator in the last term of (1) becomes zero. Now each term of equations (1) is equal to OP, a finite positive length. But if the denominator of a fraction becomes zero, the term can only remain finite when the numerator is also zero. Hence, $\cos ZOP = 0$, and $ZOP = 90°$. This is seen to be true from the fact that any number of lines can be drawn in the face parallel to OZ. But every line in the face is at right angles to the normal OP. Hence, OPL_∞ is 90°. But the interior angles $ZOP + OPL_\infty$ are together equal to two right angles (Euclid I. 29). Hence $ZOP = 90°$, and $\cos ZOP = 0$. When therefore the face is ($hk0$) the equations of the normal are given by

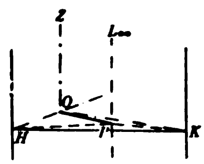

FIG. 26.

$$OP = \frac{a\cos XOP}{h} = \frac{b\cos YOP}{k} \quad \text{..................(3).}$$

If the face is parallel to a second axis, OY (say), then k is also zero and $YOP = 90°$. Then both the numerator and denominator of the term in (1) involving the axis of Y vanish together, and the term has an indeterminate form. The normal is then given by

$$OP = \frac{a\cos XOP}{h} \quad \text{........................(4).}$$

18. It is clear that as the inclination of the plane to the axis of Z varies, the distance at which OZ is met by it varies also, and that in the case just discussed we have come to the limit possible on the positive side of this axis. If the plane be tilted beyond this limit it must cut the axis on the negative side as in Fig. 27. But when the face was parallel to OZ the angle ZOP had exactly reached 90° and the additional tilt has made it exceed 90°. But the cosine of an angle greater than 90° is negative though equal in magnitude to the cosine of the supplementary angle. Hence both numerator and denominator have become negative at the same time and the ratio remains positive.

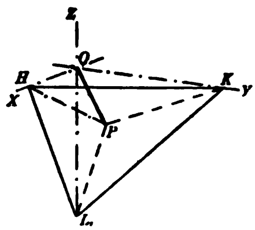

FIG. 27.

The proof can be easily given by the elementary geometry of Fig. 27. For in the right-angled triangle $L_{,}PO$, $OP = OL_{,} \cos L_{,}OP$. Here $OL_{,}$ is a length $c \div l$, without taking into consideration the direction in which it is measured, and $L_{,}OP$ is less than 90°. But $L_{,}OP + ZOP = 180°$, and $\cos L_{,}OP = \cos(180° - ZOP) = -\cos ZOP$.

Hence $$OP = -OL_{,} \cos ZOP = \frac{c \cos ZOP}{-l}.$$

The index in the denominator must therefore be negative when $ZOP > 90°$.

The same holds for the other axes, and therefore equations (1) give the position of all possible normals.

19. Equation (4), which is all that is left of (1), when the normal is perpendicular to the axial plane YOZ, *i.e.* to the face (100), serves only to indicate that the normal to (100) makes an angle less than 90° with OX. This normal we shall denote by OA. The normal $OA_{,}$ to $(\bar{1}00)$ must make an angle greater than 90° with OX: for OA and $OA_{,}$ are in the same straight line, being both perpendicular to the plane YOZ. The exact value of XOA is obtained from the general expression mentioned in Art. 16 when YOA and ZOA are made 90° and therefore $\cos AY = \cos AZ = 0$.

The same reasoning applies to the normals OB (010) and OC (001).

The normals OA, OB, OC do not coincide with the axes OX, OY and OZ, except when the axes are at right angles to one another, and the student should therefore be careful to avoid confusing the two sets of lines.

CHAPTER V.

ZONE-INDICES AND RELATIONS OF ZONES.

1. In the preceding Chapters, we have remarked that crystal-faces are arranged in zones; it is clear that the direction of an edge belonging to a zone, and therefore of the zone-axis, is known if the positions of the two faces meeting in the edge be given. It is also clear that a limitation is placed on the position of a plane when it has to pass through a known line, or, as it may be expressed crystallographically, has to lie in a known zone. This limitation gives, as we shall see in Art. 8, a simple relation between the indices of a face and the indices of a zone.

2. As a simple case of the intersection of two faces, let, in Fig. 28, the face ABC (111) intersect the face BLM (012) in the line BM. The two points B and M, common to both faces, are easily found, and the line is then known. But the line of intersection of two faces cannot always be so easily found. The general method of finding the direction of a zone-axis, OT, is to construct with edges along the axes a parallelepiped of which OT is the diagonal. This involves the determination of the lengths of the edges of the parallelepiped. We shall illustrate the method by finding the edges of the parallelepipeds $OGMLB$ and $OCANT$, of which BM and the zone-axis OT are the diagonals.

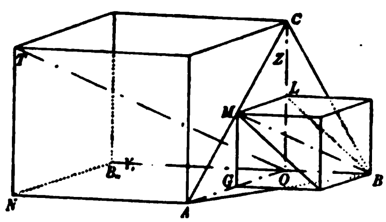

Fig. 28.

The first face meets the axes at A, B and C, where $OA = a$, $OB = b$, $OC = c$. The second meets OY at B, OZ at L, where $OL = c \div 2$; it also meets the plane XOZ, or AOZ, in the line LM parallel to OA. But the line AC is also in the plane XOZ. Hence LM and AC meet at M. The edges of the parallelepiped $OGMLB$,

of which BM is the diagonal, can be now found. The edge $OG = ML$, is obtained by Euclid VI. 4 from the similar triangles MLC, AOC. For $ML : AO = LC : OC = 1 : 2$; $\therefore OG = ML = a \div 2$. The edges of the parallelepiped are therefore $a \div 2$, b, $c \div 2$.

The same direction is given if the edges of the parallelepiped are all doubled. The line BM is then also doubled in length but not altered in direction. The *direction* is also given by any similar parallelepiped having its edges parallel to and in the same ratios as those of the original one, provided the diagonal joins corresponding corners. The required zone-axis OT, parallel to the edge BM, is the diagonal through O of the parallelepiped having edges: $OA = a$, $OB_{,,} = -2OB = -2b$, $OC = c$. The numbers *multiplying* a, b, and c, are called *the indices of the zone-axis*, or shortly *the zone-indices*, and OT is represented by the symbol $[1\bar{2}1]$.

3. Let, in Fig. 29, the two faces $(h_1k_1l_1)$, $(h_2k_2l_2)$ meet the axes at the points H_1, K_1, L_1 and H_2, K_2, L_2, respectively. It is required

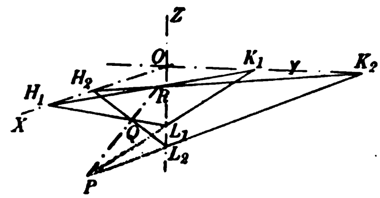

FIG. 29.

to find the zone-axis through the origin parallel to the line of intersection PQR.

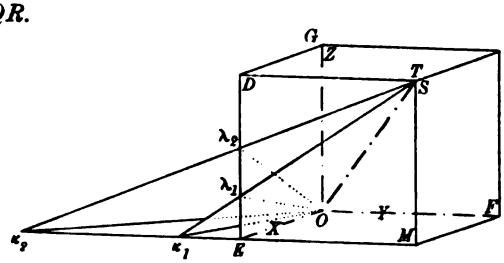

FIG. 30.

The actual position of the face, on which its dimensions depend, is immaterial, provided the angles it makes with the axial and other planes remain constant. If then a face be shifted, remaining parallel to its original position, the line of intersection with an axial plane remains parallel to the original line in which the planes met.

Suppose the face $(h_1k_1l_1)$ to be shifted parallel to itself until it passes through the origin, and let its position be then given by the plane $O\kappa_1\lambda_1$ of Fig. 30. The face in its new position may be denoted as an *origin-plane* of the crystal. The lines $O\kappa_1$, $O\lambda_1$ are parallel to H_1K_1 and H_1L_1 of Fig. 29; for, by Euclid XI. 16, parallel planes intersect a third plane in parallel lines. In Fig. 30 construct, with lengths OE, OF, OG on the axes as edges, the parallelepiped $OEMFT$; and let $OE = au = a(k_1l_2 - l_1k_2)$, $OF = bv = b(l_1h_2 - h_1l_2)$, $OG = cw = c(h_1k_2 - k_1h_2)$. If the symbols of the two faces are known, the differences of the products of the indices given above are easily calculated, and the lengths OE, OF and OG are then known.

Let now the lines $O\kappa_1$, $O\lambda_1$ meet the edges EM and ED in κ_1, λ_1 respectively. Join $\kappa_1\lambda_1$, and produce it to meet MT. The line $\kappa_1\lambda_1$ lies in the plane $DEMT$ of the parallelepiped, and is the intersection of this plane and the origin-plane parallel to $H_1K_1L_1$. If then $\kappa_1\lambda_1$ can be shown to pass through T, the origin-plane $O\kappa_1\lambda_1$ passes through the diagonal OT.

Now the triangle $OE\lambda_1$ in Fig. 30 is similar to OH_1L_1 in Fig. 29, and $OE\kappa_1$ to H_1OK_1. Hence, by Euclid VI. 4,

$$\frac{E\lambda_1}{EO} = \frac{OL_1}{OH_1}, \quad \therefore E\lambda_1 = -au\frac{c}{l_1}\frac{h_1}{a} = -cu\frac{h_1}{l_1} \quad \text{.........} (1),$$

$$\frac{E\kappa_1}{EO} = \frac{OK_1}{OH_1}, \quad \therefore E\kappa_1 = -au\frac{b}{k_1}\frac{h_1}{a} = -bu\frac{h_1}{k_1} \quad \text{.........} (2).$$

When the values of OH_1, OK_1, OL_1 are introduced, attention must be paid to the directions in which the lines are measured. From the figures, it is clear that $E\lambda_1$ and OL_1 are measured in opposite directions; as are also $E\kappa_1$ and OK_1. Hence the introduction of the negative signs in equations (1) and (2). If we denote the directions of measurement by the order in which the letters are written, we can regard $E\kappa_1$ as negative, and κ_1E as the same length measured from κ_1 in the positive direction. We express this shortly by the equation $\kappa_1E = -E\kappa_1$.

Hence $$\kappa_1E = bu\frac{h_1}{k_1} \quad \text{.........................} (2^*).$$

Suppose the line $\kappa_1\lambda_1$ to meet the edge MT in a point S_1, the point S of Fig. 30. Then from the similar triangles $\kappa_1E\lambda_1$, κ_1MS_1 we have

$$\frac{S_1M}{\kappa_1M} = \frac{\lambda_1E}{\kappa_1E} \quad \text{..........................} (3).$$

Here the lengths on parallel lines are measured in the same direction and the two ratios have the same sign.

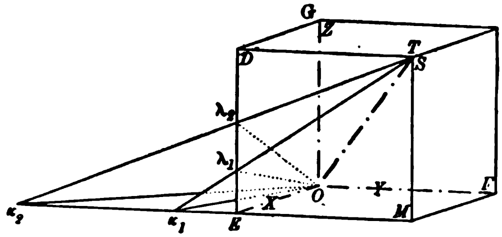

FIG. 30.

Now $\dfrac{\lambda_1 E}{\kappa_1 E} = \dfrac{OL_1}{OK_1} = \dfrac{ck_1}{bl_1}$, and no change of sign is needed ... (4).

$$\therefore \frac{S_1 M}{\kappa_1 M} = \frac{ck_1}{bl_1} \quad \text{.......} \quad (5).$$

$$\text{But } \kappa_1 M = \kappa_1 E + EM = bu\frac{h_1}{k_1} + bv \quad \text{.......} \quad (6).$$

$$\therefore S_1 M = \frac{ck_1}{bl_1}\left(bu\frac{h_1}{k_1} + bv\right) = \frac{c}{l_1}(h_1 u + k_1 v) \quad \text{.......} \quad (7).$$

$$\text{But } h_1 u + k_1 v = h_1 (k_1 l_2 - l_1 k_2) + k_1 (l_1 h_2 - h_1 l_2) = -l_1 (h_1 k_2 - k_1 h_2) = -l_1 w.$$

$$\therefore S_1 M = -\frac{c}{l_1} l_1 w = -cw \quad \text{.......} \quad (8).$$

$$\therefore MS_1 = -S_1 M = cw = MT \quad \text{.......} \quad (9).$$

For the edge MT was made equal to cw. Hence S_1 coincides with T, and the plane $O\kappa_1\lambda_1$ contains the diagonal OT[1].

To prove that the origin-plane $O\kappa_2\lambda_2$, parallel to the face $(h_2 k_2 l_2)$, contains the diagonal OT, we proceed in exactly the same way, and

[1] The student will find it advantageous to get a tinman to join together, at right angles to one another, three pieces of tube, and to fit tightly into these tubes sticks of bamboo. These will represent the three axes, and the jointed tubes may be called axial brackets. Eight sets of them will be needed to form a parallelepiped. By fastening strings of differently coloured wool or thread to the sticks, models can be made to represent the diagrams in this and the following Chapters. Or a set of axes may be made by sticking rusty knitting-needles into corks. If the needles are smooth the threads slip. The axes in the text are supposed to make any arbitrary angles with one another, but it is difficult to get the sets made accurately at angles which are not right angles.

obtain exactly similar equations. These equations can be deduced from those given for the first face by changing the suffix 1 and replacing it by 2. Lengths of lines and ratios exactly similar to those referring to $(h_1k_1l_1)$ are obtained. The sole difference is that h_2, k_2, l_2 take the place of h_1, k_1, l_1 respectively. Employing letters with suffixes 2 to denote the corresponding points, and starred reference-numbers to denote corresponding equations, the final relations are:

$$\frac{S_2M}{\kappa_2M} = \frac{\lambda_2E}{\kappa_2E} = \frac{ck_2}{bl_2} \quad \dots\dots\dots\dots\dots\dots\dots (5^*);$$

and
$$\kappa_2M = \kappa_2E + EM = bu\frac{h_2}{k_2} + bv \quad \dots\dots\dots\dots\dots\dots (6^*);$$

$$\therefore\ S_2M = \frac{c}{l_2}(h_2u + k_2v) \quad \dots\dots\dots\dots\dots\dots (7^*).$$

But
$$\begin{aligned} h_2u + k_2v &= h_2(k_1l_2 - l_1k_2) + k_2(l_1h_2 - h_1l_2) \\ &= -l_2(h_1k_2 - k_1h_2) = -l_2w. \end{aligned}$$

$$\therefore\ MS_2 = \frac{c}{l_2}l_2w = cw = MT \quad \dots\dots\dots\dots\dots\dots (9^*).$$

Hence the origin-plane $O\kappa_2\lambda_2$, parallel to the face $(h_2k_2l_2)$, contains the diagonal OT.

4. The direction of the zone-axis of the two faces $(h_1k_1l_1)$, $(h_2k_2l_2)$, is therefore given by the following construction. With the origin as vertex, and lengths au, bv, cw along the axes as edges, construct a parallelepiped. Then the diagonal through the origin is the direction required. The numbers u, v, w are called the *zone-indices*, and are obtained by the following rule:

$$\begin{array}{c|ccccccc|c} & & u & & v & & w & & \\ h_1 & k_1 & & l_1 & & h_1 & & k_1 & l_1 \\ & & \times & & \times & & \times & & \\ h_2 & k_2 & & l_2 & & h_2 & & k_2 & l_2 \end{array} \quad \dots\dots\dots\dots\dots\dots (10).$$

Each of the numbers u, v, w, is the difference of the products of the face-indices joined by the arms of the cross below it. It is only necessary to subtract in each case the product of the pair joined by the faint arm from that of the pair joined by the thick arm to secure that the signs are correct.

The values of the zone-indices may be either positive or negative, and one or two, but not all three, may in particular cases be zero. Thus in the example given in Art. 2 the second index, referring to the axis of Y, was negative.

5. Again, if in (10) the order of the two faces be altered the signs of all the zone-indices are changed; thus

$$\begin{array}{c|ccccc|c} & & u' & v' & w' & & \\ h_2 & k_2 & l_2 & h_2 & & k_2 & l_2 \\ & & \times & \times & \times & & \\ h_1 & k_1 & l_1 & h_1 & & k_1 & l_1 \end{array} \quad \cdots\cdots\cdots\cdots \quad (10^*),$$

$$u' = k_2l_1 - k_1l_2 = -(k_1l_2 - l_1k_2) = -u,$$
$$v' = l_2h_1 - h_2l_1 = -(l_1h_2 - h_1l_2) = -v,$$
$$w' = h_2k_1 - h_1k_2 = -(h_1k_2 - k_1h_2) = -w.$$

The parallelepiped is clearly constructed in the opposite octant, but with equal edges. The diagonal is the same straight line produced through the origin. It follows that the direction of the line of intersection of the faces is independent of the order in which the faces are taken when the parallelepiped is constructed; as is also obvious from the geometry.

6. The numbers u, v, w, are necessarily integers, for the indices of the faces are integral. They are employed as *multipliers* of a, b and c respectively; and as the parameters are supposed to be known, it suffices to give u, v, w to enable us to construct the parallelepiped. They must be carefully distinguished from face-indices, which are *divisors* of a, b, and c. For this purpose it is usual to enclose them in *crotchets* [ ]. Hence the above zone-axis is denoted by the symbol $[uvw]$. When the zone-indices have not been determined, the zone is often denoted by enclosing the letters denoting two or more of the faces in crotchets, and sometimes their symbols are introduced. Thus if P_1 be $(h_1k_1l_1)$ and P_2 be $(h_2k_2l_2)$, the *zone-symbol* may be given by $[P_1P_2]$, or by $[h_1k_1l_1,\ h_2k_2l_2]$, or by $[uvw]$. The zone-symbol of the example in Art. 2 is $[1\bar{2}1]$.

The distinction between face-indices and zone-indices must be carefully kept in mind. The points on the axes given by the former are points in the face, and the three lines joining them are three lines in the face. The points on the axes given by the zone-indices fix the edges of the parallelepiped, the diagonal of which from the origin is the direction of the edges of intersection of the *tautozonal*[1] faces. Thus (321) indicates the face meeting the axes respectively at distances $a \div 3$, $b \div 2$, and c, from the origin. A parallel plane is obtained by taking points at six times the distance from the origin, *i.e.* with intercepts $2a$, $3b$, and $6c$. The lengths

[1] From ταὐτό "the same."

given by [321], are $3a$, $2b$, c respectively, which form the edges of a parallelepiped, the diagonal of which from the origin is the zone-axis. And although, in consequence of the equations established in Chap. IV. Art. 15, we shall also use the face-indices to denote the face-normal, equations (1) of that article are very different from the relations of the diagonal to the edges of a parallelepiped. In the Chapter in which the relations of faces and zone-axes are treated analytically, it will be shown that, except in special cases and in the cubic system, a zone-axis cannot be perpendicular to a possible face.

7. If the zone-axis lies in one of the axial planes, XOY say, then clearly $MT = cw = 0$. Hence $w = 0$, and the symbol of such a zone-axis is $[uv0]$. The axis is the diagonal OT of the parallelogram $OEFT$, Fig. 31, the edges of which are $OE = au$, $OF = bv$.

If v becomes zero at the same time as w, then the length bv of the side of the parallelogram measured along OY becomes zero, and ET also becomes zero. The point T then coincides with E, and the zone-axis $[u00]$ is the line OE, *i.e.* the axis OX. In this case it is immaterial what length is taken on OX, and the zone-axis may equally well be given by the symbol $[100]$.

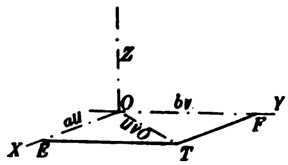

FIG. 31.

If, however, the second index to become zero is u, then OE and the parallel side TF become zero, and the zone-axis $[0v0]$ or $[010]$ is the axis OY.

Similarly a zone-axis lying in the plane XOZ is $[u0w]$, and the axis OZ is $[00w]$ or $[001]$. If the zone-axis lies in YOZ, its symbol is $[0vw]$.

Weiss's zone-law.

8. Suppose a third face (hkl)—not parallel to either of the first pair—to lie in the same zone with $(h_1k_1l_1)$, $(h_2k_2l_2)$. When it is shifted parallel to itself so as to pass through O, it must contain the zone-axis OT. It is required to find the relation between the symbols which corresponds to this geometrical fact.

We may, as before, represent the line in which the transposed face meets the plane TME by $T\lambda\kappa$ (using no suffixes). We shall clearly obtain ratios similar to those already given in (5), (5*), &c., for $(h_1k_1l_1)$, $(h_2k_2l_2)$ in which the suffixes are now omitted.

Thus $$\frac{TM}{\kappa M} = \frac{\lambda E}{\kappa E} = \frac{ck}{bl} \quad\text{.....................}\quad (5^{**}),$$

also $$\kappa M = \kappa E + EM = bu\frac{h}{k} + bv \quad\text{..............}\quad (6^{**}),$$

$$\therefore\ TM = \frac{ck}{bl}\left(bu\frac{h}{k} + bv\right) = \frac{c}{l}(hu + kv) \quad\text{.........}\quad (7^{**}).$$

But $TM = -MT = -cw$ by construction,

$$\therefore\ \frac{c}{l}(hu + kv) = -cw;$$

hence $$hu + kv + lw = 0 \quad\text{......................}\quad (11).$$

This important relation which connects the indices of any face in a zone with the zone-indices, as determined from any known pair of faces, we shall call *Weiss's zone-law*. For the distinguished crystallographer, C. S. Weiss, was the first to call attention to the relations discussed in this Chapter. This he did in his translation into German (1804–6) of Haüy's *Minéralogie*. Weiss used the intercepts as the symbol of a face—e.g. $\frac{a}{3} : \frac{b}{2} : c$—so that his expression for the law was not in the form given in (11). His analysis of the relations is given in a memoir published in the *Abh. d. Berlin Akad.*, 1820–1.

The equation just given may be written out in full as follows:

$$h(k_1l_2 - l_1k_2) + k(l_1h_2 - h_1l_2) + l(h_1k_2 - k_1h_2) = 0 \quad\text{......}\quad (11^{*}).$$

This may also be rearranged, so as to make the indices of any one of the faces the factors multiplying expressions of the form of zone-indices. Thus (11^{*}) is equivalent to

$$h_2(kl_1 - lk_1) + k_2(lh_1 - hl_1) + l_2(hk_1 - kh_1) = 0 \quad\text{......}\quad (11^{**}).$$

It may also be given in the determinant form:—

$$\begin{vmatrix} h & k & l \\ h_1 & k_1 & l_1 \\ h_2 & k_2 & l_2 \end{vmatrix} = 0 \quad\text{........................}\quad (12).$$

To a student familiar with mathematics this latter form is far the most convenient one, for a glance is usually sufficient to show whether the determinant is zero, and whether therefore the three faces, the symbols of which have been introduced, are tautozonal. It also enables him to prove that the indices of any face (hkl) in the zone can be given by the following equations:

$$\left.\begin{aligned} h &= mh_1 + nh_2 \\ k &= mk_1 + nk_2 \\ l &= ml_1 + nl_2 \end{aligned}\right\} \quad\text{.....................}\quad (13);$$

where m and n are whole numbers. The student can easily see that the values given by (13) satisfy the condition by introducing them into (11**).

9. It is clear from the analysis that any two *non-parallel* faces in the zone can be employed to determine the zone-symbol; and likewise that, when the zone-symbol is once known, any other face whatever, which lies in the zone, must have indices which satisfy the equations given in (11) or (12). Thus we can show that the following faces are tautozonal: (321), (112), (433), $(21\bar{1})$, $(10\bar{3})$. From any pair (say the first) we find[1] $u_{12} = 3$, $v_{12} = -5$, $w_{12} = 1$. The equation, which a face (hkl) in this zone must satisfy, is

$$3h - 5k + l = 0 \qquad \text{(14).}$$

It is easily seen that the indices of the faces (433), $(21\bar{1})$, $(10\bar{3})$ will, when substituted for h, k, l in equation (14), reduce the left side to zero. The faces are therefore tautozonal.

But attention should be directed to the fact that different pairs of faces will not give exactly the same values for u, v, w, but only values in the same ratio. Thus, if the third and fourth faces (433) and $(21\bar{1})$ are taken, we obtain

$$u_{34} = -6, \quad v_{34} = 10, \quad w_{34} = -2.$$

But
$$\frac{u_{34}}{u_{12}} = \frac{v_{34}}{v_{12}} = \frac{w_{34}}{w_{12}} = -2 \qquad \text{(15).}$$

The variation in the values of u, v, w, means no more than that the size of the parallelepiped is altered, and that the length of OT given by one pair of faces is a multiple, or submultiple, of the length given by another pair; and likewise that it may be taken, as in the example just given, on the other side of the origin. We have already seen in Art. 5 that the signs of the indices, and the direction of OT as viewed from the origin, are changed by reversing the order in which the two faces are taken in the table.

10. Let $[u_{12}\ v_{12}\ w_{12}]$, and $[u_{34}\ v_{34}\ w_{34}]$ be the zone-symbols of the same zone deduced from two different pairs of non-parallel faces in the zone, it is required to prove that

$$\frac{u_{12}}{u_{34}} = \frac{v_{12}}{v_{34}} = \frac{w_{12}}{w_{34}} \qquad \text{(16).}$$

[1] When there are several faces in a zone, it is convenient to indicate the faces taken to give the zone-indices by attaching suffixes to the zone-indices. Thus u_{12}, &c., indicate that the indices of the first and second face are introduced in this order in the table (10) of Art. 4.

Now the pairs of faces in a zone must meet in lines all parallel to the zone-axis. Hence the *direction* of the diagonal of the parallelepiped which gives the zone-axis cannot depend on the pair of faces selected. But the only variables are the lengths of the edges of the parallelepipeds, and these remain proportional to one another if the diagonals lie in the same straight line. The directions in which the edges are measured from the origin may, in some cases, be on opposite sides of it, but it will suffice to prove the proposition for the case in which they lie, as in Fig. 32, on the same side of it; for by taking the second pair in the reverse order, we reverse the directions of the edges of the parallelepiped (Art. 5). Let $OETF$ be the parallelepiped obtained by means of $(h_1k_1l_1)$, $(h_2k_2l_2)$, and $OE'T'F'$ that obtained by another pair of faces in the same zone $(h_3k_3l_3)$, $(h_4k_4l_4)$. Then OT and OT' are co-linear. Also $OE=au_{12}$, $OF=bv_{12}$, $MT=OG=cw_{12}$; $OE'=au_{34}$, $OF'=bv_{34}$, $M'T'=OG'=cw_{34}$.

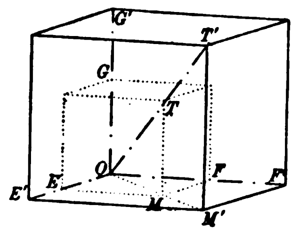

FIG. 32.

But if a plane be drawn through OG and OTT', it will contain the edges MT, $M'T'$ parallel to OG. The points M and M' must therefore lie on a straight line through O. Hence we have the similar triangles OTM, $OT'M'$, and by Euclid VI. 4,

$$\frac{MT}{M'T'}=\frac{OM}{OM'}.$$

Again, from the similar triangles OEM, $OE'M'$ we have

$$\frac{OE}{OE'}=\frac{EM}{E'M'}=\frac{OM}{OM'}=\frac{MT}{M'T'}.$$

Hence, introducing the values of the edges, we have

$$\frac{au_{12}}{au_{34}}=\frac{bv_{12}}{bv_{34}}=\frac{cw_{12}}{cw_{34}}.$$

And
$$\frac{u_{12}}{u_{34}}=\frac{v_{12}}{v_{34}}=\frac{w_{12}}{w_{34}}\quad\text{.............................(16).}$$

The three ratios given in (16) are each equal to

$$\frac{hu_{12}+kv_{12}+lw_{12}}{hu_{34}+kv_{34}+lw_{34}}\quad\text{............................(17),}$$

where h, k, l, are any three numbers whatever. If they be commensurable they may be the indices of a face. If this face do *not* lie in the zone $[u_{12}v_{12}w_{12}]$, the ratio has some definite commensurable value. It is clearly useless to take for h, k, l, the indices of a face in the zone, for the numerator and denominator then both become zero by Weiss's law, Art. 8. That the terms in (16) are equal to that given in (17) is proved as follows.

Let each term in (16) $= r$;

$$\therefore \quad u_{12} = ru_{34}, \ v_{12} = rv_{34}, \ w_{12} = rw_{34}.$$

$$\therefore \quad hu_{12} = rhu_{34}, \ kv_{12} = rkv_{34}, \ lw_{12} = rlw_{34}.$$

Adding together the left sides, as also the right sides, we have

$$hu_{12} + kv_{12} + lw_{12} = r(hu_{34} + kv_{34} + lw_{34}).$$

Hence r has the value given in (17).

Face common to two zones.

11. We can now show that a plane which contains two zone-axes is parallel to a possible face of the crystal, and that the intercepts this face makes on the axes are commensurable sub-multiples of the parameters. The face is clearly common to the two zones of which the indices are given.

Let the two zones have the symbols $[u_1v_1w_1]$, $[u_2v_2w_2]$; then, from equation (11), the face (hkl) lies in $[u_1v_1w_1]$ when

$$hu_1 + kv_1 + lw_1 = 0 \qquad (18),$$

and in order that it should also lie in $[u_2v_2w_2]$ we must have

$$hu_2 + kv_2 + lw_2 = 0 \qquad (19).$$

If (18) be multiplied throughout by w_2 and (19) throughout by w_1, and if the one equation be then subtracted from the other, we have

$$h(u_1w_2 - w_1u_2) + k(v_1w_2 - w_1v_2) = 0 \qquad (20).$$

Similarly, by multiplying (18) and (19) by v_2 and v_1 respectively, and then subtracting, we have

$$h(u_1v_2 - u_2v_1) + l(w_1v_2 - w_2v_1) = 0 \qquad (21).$$

Hence, transforming the equations (20) and (21), we find

$$\frac{h}{v_1w_2 - w_1v_2} = \frac{k}{w_1u_2 - u_1w_2} = \frac{l}{u_1v_2 - v_1u_2} \qquad (22).$$

Now u_1, v_1, &c., are all commensurable numbers and, if derived by table (10) from face-indices, they are integers. Hence h, k, l, are also commensurable numbers and may be expressed by integers. The face (hkl), having commensurable indices, is therefore a possible face of the crystal.

The relation between the indices h, k, l of the face common to

two zones and their zone-indices, given by equations (22), can be expressed by a table (23) similar in all respects to table (10). Thus

$$\begin{array}{c|cccccc|c} & & h & & k & & l & \\ u_1 & v_1 & & w_1 & u_1 & & v_1 & w_1 \\ u_2 & v_2 & & w_2 & u_2 & & v_2 & w_2 \end{array} \quad \ldots\ldots\ldots\ldots\ldots\ldots(23).$$

The indices of a face common to two zones, viz. $h = v_1w_2 - w_1v_2$, $k = w_1u_2 - u_1w_2$, $l = u_1v_2 - v_1u_2$, are obtained in a manner exactly similar to that by which zone-indices were by (10) obtained from the indices of two faces in the zone. The same precaution must be taken in both cases, and the product of the indices joined by a faint arm must be subtracted from the product of those joined by the thick arm.

As an example let us find the face (hkl) common to $[3\bar{5}1]$ and to $[102]$.

By table (23)

$$\begin{array}{c|cccc|c} & h & k & l & & \\ 3 & \bar{5} & 1 & 3 & \bar{5} & 1 \\ 1 & 0 & 2 & 1 & 0 & 2, \end{array}$$

$\therefore$ $h = -10$, $k = 1 - 6 = -5$, $l = 5$, and rejecting the common measure 5, the face has the symbol $(\bar{2}\bar{1}1)$. Had we however taken the zone [102] first, we should have found $h = 10$, $k = 5$ and $l = -5$, and the face to be $(21\bar{1})$. These two faces are parallel, and the first is necessarily common to every zone to which the second belongs.

12. The equation $hu + kv + lw = 0$, involving as it does the face-indices h, k, l, and the zone-indices u, v, w, in precisely the same manner, serves equally to connect: (1) all the faces lying in the zone $[uvw]$, and (2) all the zones to which a particular face (hkl) is common.

13. By the aid of Fig. 33, similar in some respects to Fig. 30, and the expressions given in Art. 3 with reference to the lines in the latter figure, we can give a direct proof of the proposition—that a plane parallel to two zone-axes is a possible face.

Let OT be the zone-axis $[uvw]$, and $OST_{,}$ the zone-axis $[u_1v_1w_1]$, and let $TO\kappa\lambda T_{,}$ be the origin-plane determined by them. We can find the ratios $E\kappa : EO$, and $E\lambda : EO$; and can then, by a comparison with Fig. 29, find the intercepts on the axes of a parallel plane which will give the indices.

From the similar triangles $E\kappa\lambda$, κMT we have

$$\frac{E\kappa}{E\lambda} = \frac{M\kappa}{MT} = -\frac{\kappa M}{MT} = -\frac{\kappa E + bv}{cw} \quad \ldots\ldots\ldots\ldots\ldots\ldots (24).$$

Let the second zone-axis OS—the diagonal of the parallelepiped $OFNS$ with edges au_1, bv_1, cw_1—be produced to meet the line κT in $T_{,}$.

Draw $T_{,}M_{,}$ and $ONM_{,}$. The second zone-axis is equally well given as the diagonal of the parallelepiped $OEM_{,}T_{,}$. But from the two pairs of similar triangles, $OT_{,}M_{,}$, ONS, and $OM_{,}E$, ONF, we have

$$\frac{M_{,}T_{,}}{NS}=\frac{OM_{,}}{ON}=\frac{OE}{OF}=\frac{EM_{,}}{FN}.$$

$$\therefore \quad M_{,}T_{,}=\frac{OE}{OF}NS=cw_1\frac{u}{u_1}; \; EM_{,}=\frac{OE}{OF}FN=bv_1\frac{u}{u_1} \quad \text{......(25).}$$

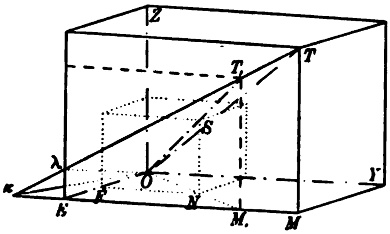

Fig. 33.

Again, from the similar triangles $E\kappa\lambda$, $T_{,}\kappa M_{,}$ we have

$$\frac{E\kappa}{E\lambda}=\frac{M_{,}\kappa}{M_{,}T_{,}}=-\frac{\kappa M_{,}}{M_{,}T_{,}}=-\frac{\kappa E+EM_{,}}{M_{,}T_{,}}=-\frac{\kappa E+bu\frac{v_1}{u_1}}{cu\frac{w_1}{u_1}} \quad \text{......(26).}$$

Equating the right sides of (24) and (26) we have

$$\frac{\kappa E+bv}{cw}=\frac{\kappa E+bu\frac{v_1}{u_1}}{cu\frac{w_1}{u_1}}.$$

$$\therefore \quad \kappa E=bu\frac{v_1w-vw_1}{uw_1-wu_1} \quad \text{......................... (27).}$$

Substituting this value of κE in the right side of (24), we have

$$-\frac{E\kappa}{E\lambda}=\frac{bv+bu\frac{v_1w-vw_1}{uw_1-wu_1}}{cw}=\frac{b}{c}\,\frac{v(uw_1-wu_1)+u(v_1w-vw_1)}{w(uw_1-wu_1)}$$

$$=\frac{b}{c}\,\frac{uv_1-vu_1}{uw_1-wu_1} \quad \text{.................(28).}$$

But, in a manner similar to that given in Art. 3, $\frac{OK}{OL}$ (of Fig. 29) $=\frac{E\kappa}{E\lambda}$ (of Fig. 33), if the two planes HKL and $O\kappa\lambda T$ are parallel.

$$\therefore \quad \frac{OK}{OL}=\frac{b}{c}\,\frac{vu_1-uv_1}{uw_1-wu_1} \quad \text{......................... (29).}$$

And similarly, $\frac{OK}{OH}$ (of Fig. 29) $= \frac{\kappa E}{OE}$ (of Fig. 33),

and (from 27), $$\frac{OK}{OH} = \frac{b}{a} \frac{v_1 w - v w_1}{u w_1 - w u_1} \text{ .... (30).}$$

Hence from (29) and (30)

$$\frac{\dfrac{OH}{a}}{w v_1 - v w_1} = \frac{\dfrac{OK}{b}}{u w_1 - w u_1} = \frac{\dfrac{OL}{c}}{v u_1 - u v_1} \text{ .... (31).}$$

The intercepts on the axes made by the plane HKL parallel to the two zone-axes are integral submultiples of the parameters, and the plane is therefore a possible face of the crystal.

14. The geometrical relations of a crystal may hence be given in a new way as follows. Take axes parallel to three zone-axes, not all in one plane, and a fourth zone-axis not in a plane with any pair of the first three zone-axes. Form a parallelepiped with edges along the axes and having the fourth zone-axis for diagonal. Denote the lengths of the edges by a, b and c respectively. Then a, b, and c, are the parameters of the crystal; and every other zone-axis, or edge, possible on the crystal is given by the diagonal of a parallelepiped, the edges of which are in the axes and have the lengths au, bv, cw, where u, v and w, are integers. Furthermore, every face common to two zones is given by $\frac{a}{h}$, $\frac{b}{k}$, $\frac{c}{l}$, where h, k and l are integers deduced from the zone-indices by the rule given in table (23).

15. *Example.* We may now apply the propositions in Chaps. IV. and V. to the solution, as far as the principles already established permit, of the crystal of barytes shown in Fig. 34.

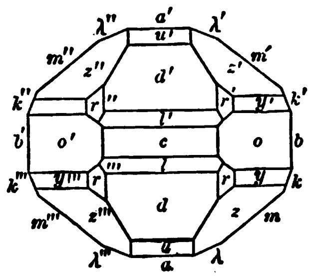

FIG. 34.

The lines of intersection of planes, parallel to a, b and c in the figure, drawn through a central point are taken as axes, so that face a is parallel to OY and OZ, face b to OZ and OX, and face c to OX and OY. These latter axes are shown in Fig. 37 which gives the faces m and c only. The indices of these faces are therefore at once known—a is (100), a' ($\bar{1}00$); b (010), b' ($0\bar{1}0$); c (001) and the parallel face below the paper will be c' ($00\bar{1}$).

The face z is taken as parametral face, and its symbol is (111). The homologous faces are symmetrically placed with respect to the axial planes and axes. Hence z and z' meet in a line in the plane YOZ, and their intercepts on the axes OY and OZ must be the same. The axis OX is met by z at a distance a on the positive side, by z' at an equal distance on the negative side. Hence z' meets the axes at distances, $-a$, b, and

c, respectively, and its symbol is $(\bar{1}11)$. Similarly, the symbol of z''' can be shown to be $(1\bar{1}1)$, and that of z'' to be $(\bar{1}\bar{1}1)$.

Again, if we note that the axis OZ is a dyad axis, and that therefore the pair of faces z and z'', and also the pair z', z''', interchange places when the crystal is turned through 180° about this axis, we perceive the fact that the symbols of pairs of faces, interchangeable by rotation about a dyad axis perpendicular to a pair of axes of reference, differ in having the signs of the indices referring to these axes both changed. This is clearly necessary, for the rotation about OZ interchanges, on each line perpendicular to it, equal lengths measured on opposite sides of the origin.

We can now determine the symbols of the homologous faces of the forms denoted by the letters m, u and o. The faces m lie in the zone containing the faces a and b which may be represented by $[ab]=[100, 010]=[001]$—the latter symbol being determined by table (10). Hence, by (11) of Art. 8, the last index of each face m is zero. This is obvious from the geometry, for the faces m, m', &c., are all parallel to OZ. But m and m'' lie in the zone $[cz]=[001, 111]=$ (by table (10)) $[\bar{1}10]$. Hence, if m be $(hk0)$, we have from equation (11) $-h+k=0$. This equation is satisfied by making $h=1$ and $k=1$; or by making $h=\bar{1}$ and $k=\bar{1}$. The former value gives for m the symbol (110), for this face meets both the axes of X and Y on the positive sides of the origin. The second set of values gives $(\bar{1}\bar{1}0)$ for the parallel face m'', which meets the axes on the negative sides of the origin. Similarly, m' and m''' lie in $[cz']=[110]$, and their indices $(hk0)$ satisfy the equation, $h+k=0$. Hence m' is $(\bar{1}10)$, and m''' $(1\bar{1}0)$.

The face u is determined in a similar manner from the two zones $[ac]=[010]$, and $[bz]=[\bar{1}01]$. From the first it follows that the middle index is zero, and the face can be represented by $(h0l)$. The second zone proves that, $-h+l=0$, which is satisfied by making $h=l=1$, and the symbol (101). Or we may use table (23), thus

$$\begin{array}{c|cccc|c} 0 & 1 & 0 & 0 & 1 & 0 \\ & & \times & \times & \times & \\ \bar{1} & 0 & 1 & \bar{1} & 0 & 1. \end{array}$$

Hence u is (101). Similarly, from $[ca]$ and $[bz']$, u' may be found to be $(\bar{1}01)$.

From the zones $[bc]=[100]$, and $[az]=[01\bar{1}]$, the face o is fixed and its symbol (011) determined. The symbol of o' is found, from $[bc]$ and $[az'']=[011]$, to be $(0\bar{1}1)$.

The symbols of the remaining faces will be determined in a later Chapter.

CHAPTER VI.

CRYSTAL-DRAWINGS.

1. CRYSTALS can be represented by models or by crystal-drawings, which give with sufficient accuracy the relative positions of the faces, those of the same form being usually equably developed. Models and drawings of crystals serve mainly to indicate their general habit, and aid us to distinguish the crystals of one substance from those of another: although, as already stated, the habit often varies greatly in crystals of the same substance; and, more especially, when they come from different localities, or are produced in the laboratory under different conditions. In models and drawings, irregularities in the size and shape of faces of a form are only admitted when they are needed to render manifest some peculiarity, either of general habit, or of a particular crystal. Drawings and models fail, however, to give that aid in the determination of symmetry which is afforded by the physical characters of the faces, for homologous faces on a crystal possess the same physical characters and show the same marks, such as striæ, pittings, &c., which can only imperfectly be represented in drawings.

2. The crystal-drawings to be found in scientific works are not true perspective drawings, for the parallelism of the edges of intersecting tautozonal faces would be thereby obscured. A crystal is drawn as if all the rays proceeding from the coigns to the eye were parallel; that is, the eye and crystal are supposed to be infinitely distant from one another. The crystal is, therefore, drawn in much the same way as it would appear, if viewed through a telescope adjusted for a very distant object. The crystal is held in front of the paper, supposed to be placed vertically, so that one of its zone-axes—usually called the *vertical axis*—lies in the vertical plane through the eye at right angles

to the paper. The lines, or *rays*, which join the eye to the coigns are all parallel, and either (*a*) perpendicular to the paper, or (*b*) inclined to it at a fixed angle not differing much from 90°. In the former case the figures are *orthogonal projections*, in the latter they will be called *clinographic drawings*. The points, in which the rays meet the paper, are joined by straight lines in a manner corresponding with the edges of the crystal, and the net-work of lines forms a crystal-drawing.

Drawings, which are orthogonal projections, may be divided into two classes: *plans* and *elevations*, described in section (i); and *orthographic drawings*, treated of in section (ii). In the former an important zone-axis, which may, or may not, be itself perpendicular to an important face, is placed at right angles to the paper. Strictly speaking, in a plan the vertical axis should be perpendicular to the paper, and in an elevation it should be parallel to the paper. But we shall always speak of diagrams of both kinds as plans; and shall specify the face which is parallel to the paper, or the zone-axis and faces which are perpendicular to the paper. In orthographic drawings the important edges and faces of the crystal are inclined at some arbitrary angles to the paper.

In clinographic drawings, treated of in section (iii), the vertical axis is *in*, or *parallel to*, the paper, but the crystal has in other respects an arbitrarily selected position with respect to the paper.

In this Chapter some knowledge of the relations which hold in the several systems is presupposed, so that the student should read the Articles referring to each system together with the Chapter on that system. As a matter of fact, an accurate drawing of a crystal is only made after the crystal has been completely determined. For working purposes a freehand sketch, which will enable the student to identify each face, is all that is needed. The student should, however, find no difficulty in following the description of the plans given in the next Articles and the example worked out in Art. 8.

(i) *Plans and Elevations.*

3. Figures of this kind are largely used in Brooke and Miller's edition of Phillips's *Mineralogy*, 1852. The drawings are easily executed, and give a fair idea of the end of the crystal directed towards the observer. They, however, fail to render sensible the

inclination of the faces and edges to the line of sight, and they are deficient in the appearance of solidity.

If a very simple crystal, such as a cube, or a tetragonal crystal of apophyllite bounded by three pairs of rectangular faces, be

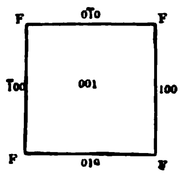

FIG. 35.

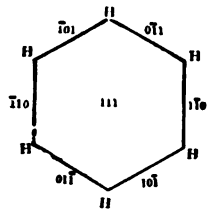

FIG. 36.

projected on a plane parallel to one of the faces, it will appear as a square, Fig. 35[1]. Similarly a hexagonal prism, like Fig. 3, will, if viewed endwise, be seen as the hexagon, Fig. 36.

4. A simple crystal of barytes, resembling a diamond-shaped lozenge, gives the plan, shown in Fig. 37. This crystal is bounded by four similar faces, perpendicular to the paper and denoted by the lines *m*. They are parallel to good cleavages. The rhombic face marked *c* is parallel to the paper, and is also parallel to a good cleavage. Two axes, OX and OY, are taken along the diagonals of the rhombus; the third, OZ, is perpendicular to the paper.

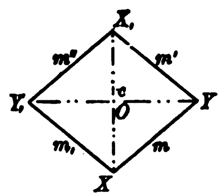

FIG. 37.

5. Fig. 38 shows a ditetragonal prism, terminated by planes parallel to the paper and at right angles to the prism-edges through the points F and K. Such a figure is readily made, if one of the face-angles F or K is known. If one be known the other is also, for $F+K=270°$. To make such a diagram, two lines, OX and OY, are drawn at right

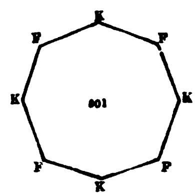

FIG. 38.

[1] Figs. 35, 36, 38–44, and some others used in later Chapters, are printed from Miller's original blocks which have been kindly placed at my disposal by Mr. H. Bauerman, F.G.S. Miller placed the axes X, Y, Z, in a different order to that adopted in this book, and his symbols, which have been left undisturbed, differ from those which are given in other diagrams.

angles, and equal lengths, OK, are marked off on them starting from the origin. Lines KF are then drawn making with the axes angles equal to $K \div 2$.

Fig. 39 represents a ditetragonal pyramid having eight similar faces meeting at an apex on a tetrad axis, which is taken to be OZ and is perpendicular to the paper. To construct this plan, the angles between the similar edges, labelled L, have to be found, and then the ditetragon is drawn as in Fig. 38. The lines joining the corners to the centre are then drawn. Those labelled K are similar edges, and are interchangeable by rotation through 90° about OZ. The edges F are also similar to one another, *i.e.* they are interchangeable, and the angles over them are all equal. The angles over the edges L are all equal to one another. But the angles of one of the above sets are never equal to those of either of the other sets.

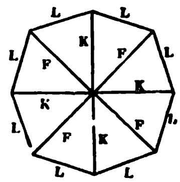

FIG. 39.

Two tetragonal pyramids, belonging to the same class of crystals, are shown in Figs. 40 and 41. In Fig. 40 the axes of X and Y are in the directions of the diagonals of the square formed by the edges L; in Fig. 41 they are parallel to the edges so marked.

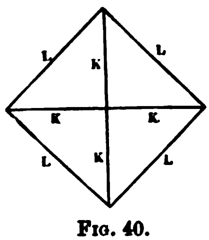

FIG. 40.

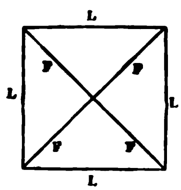

FIG. 41.

Each figure has two distinctly different sets of angles, those over the edges K and L in one, and those over F and L in the other. The angles L of the one pyramid differ from those of the other.

In the preceding diagrams the distribution of faces at the back of the crystal is supposed to be exactly like that shown in front, and the paper is supposed to be parallel to a plane of symmetry. It is clear also that the three last figures will be the same whatever be the distance at which the axis OZ is met, for this axis is projected in the central point. The figures fail, therefore, to show whether the pyramids are steeply inclined to the paper or not.

6. Figs. 42–44 show three simple rhombohedral crystals, in which the plane of the paper is perpendicular to the triad axis. The first represents a rhombohedron, and serves equally well

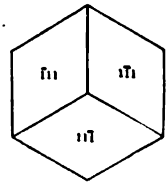

Fig. 42.

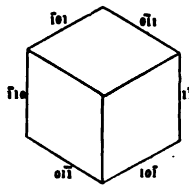

Fig. 43.

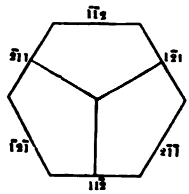

Fig. 44.

whether the rhombohedron is an acute one or an obtuse one, *i.e.* whether the apices are far apart or the reverse. If we suppose it, however, to represent a crystal of calcite, the symbols on the faces imply that the faces are steeply inclined to the paper. By placing on the faces the symbols on the upper faces of Fig. 4, the plan serves equally well for this obtuse rhombohedron. There is another objection to the plan. The hexagon bounding the figure represents edges which cross the paper and are not parallel to it. The parallel lines in Fig. 42 are, however, all inclined to the paper at the same angle; and so indeed are all the edges of the figure.

Fig. 43 represents a rhombohedron in a different azimuth. Its symbol may be $\{100\}$. From the fact that symbols are attached to the sides of the bounding hexagon, we are informed that prism-faces perpendicular to the paper are also present, opposite pairs of which are in a zone with two faces of the rhombohedron and with the faces below the paper parallel to them. All the edges of this figure are also equally inclined to the paper.

Fig. 44 may be taken to represent the same rhombohedron but associated with a different hexagonal prism, the faces of which truncate[1] the edges of the prism in Fig. 43. The alternate sides of the hexagon, labelled $2\bar{1}\bar{1}$, $\bar{1}2\bar{1}$, $\bar{1}\bar{1}2$, in which the faces of the prism meet each only one face of the rhombohedron, are edges parallel to the paper. Each of the remaining three sides represents two edges equally inclined to the paper.

Such diagrams serve to show the development at one end of a crystal fairly well, and the method will be employed in Chap. XVI. to illustrate the dissimilar ends of a crystal of tourmaline.

[1] A face situated in a zone with two faces and equally inclined to them is said to *truncate* their edge.

7. In the oblique system plans, or elevations, on the plane of symmetry are frequently employed by Miller. They are readily made when a few angles in the plane of symmetry and the symbols of the faces are known.

Occasionally a plan of an oblique crystal is made on a plane perpendicular to a zone-axis lying in the plane of symmetry. An instance of this kind is given in the plans used, in Chapter XVIII., in the discussion of twins of orthoclase. Similar plans are also occasionally employed for representing anorthic crystals; as, *e.g.* the plan of oligoclase given in Chapter XI. The paper is not in these cases parallel to a possible face of the crystal.

8. *Example.* Fig. 45 affords a good illustration of the use and method of construction of complicated plans. It gives the more important forms on barytes, which are not, however, often found associated together on the same crystal. The paper is parallel to the base $c\,(001)$. The faces a, λ, m, k, b are all perpendicular to the paper, and their projections are given by the lines so labelled.

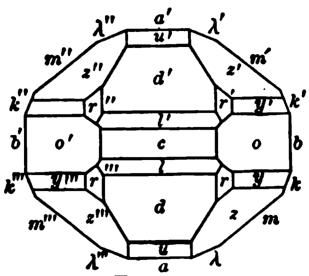

FIG. 45.

Two lines are drawn, as in Fig. 37, at right angles for the axes of X and Y. Points B and $B_{\prime}$ on OY are taken arbitrarily at any convenient distance, such that $OB = OB_{\prime} = b$. From B and $B_{\prime}$ lines m, m', &c. are drawn inclined to OY at angles of $39° \, 11' = 78° \, 22' \div 2$. (For this and other angles see Chap. III. Art. 9.) We thus find points A and $A_{\prime}$ on OX, where $OA = OA_{\prime} = a$. The corners at A, B, &c., are then truncated by lines a and b parallel to the axes, which represent faces parallel to them and to OZ. The corners between these lines and the intermediate m's are now modified by lines k and λ, where k makes with OX the angle $22° \, 15'$ and λ makes with OY the angle $22° \, 10'$. The length of these lines depends on their distance from the centre. They should be drawn so as fairly to represent the ratios of the breadths of the several faces to each other. The zones [yoz], [$y''' \, o' \, z'$] are then introduced by drawing lines parallel to OY and symmetrically situated on both sides of it. Their distances from one another should approximate to the impression of the width of the faces when the crystal is looked at endwise. The edges [ud], [dl], &c., are drawn in the same way parallel to OY. The edges [uz], [dr], [ry], [co], &c., are drawn parallel to OX; for [zuz'''], [yrd..], &c., are zones symmetrical with respect to the plane of symmetry parallel to the face b. The edges [rz], [$r'z'$], &c., are parallel to m, m', &c.; for m, z, r, c are tautozonal.

So far the symbols of the faces have not been needed. But to get the edges [zd], [ro], and [lr] the indices of the faces are required. In Chap. V. Art. 15, the symbols of the faces were determined as far as they could be by the relations of zones, and in Chap. VII. those of the remaining faces are found by the method explained in that chapter. Hence, we may suppose all the indices to be known. The symbols are: $z\{111\}$, $d(102)$, $l\{104\}$, $o\{011\}$, $y\{122\}$, and $r(112)$.

The zone-axis [zd] is $[111, 102] = [2\bar{1}1]$. Now in the diagram, the axis OZ and every length upon it are projected in the point O. Hence the parallelepiped with edges $2a$, $-b$, $-c$, is projected in the parallelogram having sides $2a$ and $-b$ on OX and OY. The diagonal through the origin of this parallelogram is the direction of the edge [zd]. The homologous edges are parallel to the diagonals of similar parallelograms. The remaining edges can be obtained in a similar manner, and the figure completed.

The student's attention is directed to the fact that the parameters a and b were obtained by simple construction from the measured angles. The length OB taken on OY was arbitrary, and the line m was then drawn making 39° 11′ with OY. The point where it met OX gave the point A at distance a on OX.

Hence, from Fig. 46, $OA \div OB = a \div b = \tan ABO = \tan AOM$, where OM is the normal to the face m. But $AOM = 39° \, 11'$, and, if OB be the unit of length, we have

$$OA = \tan 39° \, 11' = {\cdot}8151.$$

In a similar manner, if the angle bo, or co, be measured, the parameter c can be found. If OQ, Fig. 47, represents the normal to

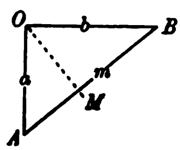

FIG. 46.

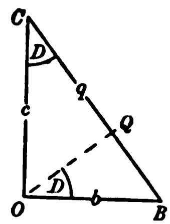

FIG. 47.

the face o, and if BC is the line in which the face meets the plane containing OY and OZ; then $\wedge COQ = \wedge OBC = co$. But

$$OC \div OB = c \div b = \tan OBC = \tan COQ.$$

By measurement, $co = 52° \, 43'$. Therefore, when $OB = 1$, $OC = 1{\cdot}3135$.

(ii) *Orthographic Drawings.*

9. In drawings of this kind the rays are not parallel to a zone-axis, and the paper is not parallel to a possible face. We first project a cube, as explained below. The three projected edges, meeting at any coign, give the directions and lengths of three equal lines at right angles to one another, which can serve as rectangular axes with equal parameters, *i.e.* for cubic crystals. For equal lengths on all parallel lines are projected as equal lengths; and unequal lengths on the same, or on parallel, lines in any given ratio to one another are projected as lengths having the same ratio. If the parameters are unequal, one of the edges is left unaltered, preferably the shortest in the projection; the other two are then lengthened, or shortened, in the ratios of the parameters. When the axes are not rectangular, new lines of unit length must be found making the required angles with one another by one or other of the methods given in Arts. 15—20.

We proceed to describe the principle by which Mohs obtained the projection of the edges of a cube. The cube seen with face parallel to the paper is projected as a square, Fig. 35. If it is then turned about a vertical edge until the face to the right has a width one-third (or one-nth) of that to the left, the cube is seen as two rectangles, Fig. 48. If the angle of rotation about the vertical edge be ρ, then

$$\tan\rho = \tan CAA^{II} = CA^{II} \div AA^{II} \quad \text{(Fig. 49)}$$
$$= DD^{II} \div CD^{II} = 1 \div 3;\ \text{or generally, } 1 \div n.$$

In the particular case when $n = 3$, $\rho = 18° 26'$.

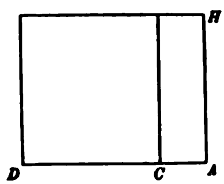

FIG. 48.

The horizontal faces are still foreshortened into straight lines. To see the upper face, the crystal has to be turned about a horizontal axis, which may be supposed to go through the nearest bottom corner of the cube. During this rotation the vertical edge is maintained in the vertical plane through the eye. Mohs and Haidinger adopted an angle of inclination to the horizon, such that the vertical distance between extreme points on the upper face, as seen on the paper, was one-eighth of the total breadth of the projected cube.

10. Figs. 49—51 are copied from Haidinger's memoir on Mohs' method of drawing crystals (*Mem. Wernerian Nat. Hist. Soc. Edin.*, 1824). The first represents a horizontal plane through

DCA of Fig. 48 after the first rotation. $ABDC$ is the base of the cube, the plane of projection passes through $D^{II}CA^{II}$, and the rays through the coigns A, B, D, are $A^{II}A$, $B^{II}B$, $D^{II}D$. Through the corners of the base draw the square $D^{III}A^{II}$, having two sides in the rays AA^{II}, DD^{II}; and through A, B, D, C, draw lines parallel to the sides of $D^{III}A^{II}$. The triangles CAA^{II}, CDD^{II}, are equal in all respects. Hence, $AA^{II} = CD^{II} = 3CA^{II}$, or, generally, nCA^{II}.

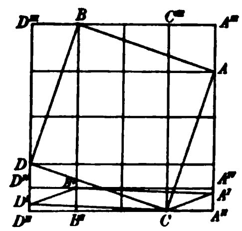

FIG. 49.

Let the cube be now tilted about $D^{II}A^{II}$ so as to expose the top to the observer, and suppose the square $D^{III}A^{II}$ to be carried with the cube. During the rotation the coigns A, B, D, will remain in the vertical planes through $A^{II}A$, $B^{II}B$, $D^{II}D$; and, if the rotation were continued through 90°, the square $D^{III}A^{II}$ and the base $ABDC$ would be brought into the plane of projection. If, however, the rotation is stopped when the coign B is projected in B^{I}, where $B^{II}B^{I} = D^{II}A^{II} \div 8 = B^{II}B \div 8$, then the square base of the cube is projected in the parallelogram $A^{I}B^{I}D^{I}C$ of Fig. 49.

To find the points A^{I} and D^{I}, suppose BA and BD produced to meet $D^{II}A^{II}$, prolonged both ways, in M and N, respectively. We thus obtain two similar right-angled triangles $BB^{II}M$ and $AA^{II}M$. On the other side of BB^{II} we have a second pair of similar right-angled triangles $BB^{II}N$ and $DD^{II}N$. Now during the rotation the straight lines BAM and BDN remain straight lines, and the pairs of similar triangles must remain similar. Therefore $B^{I}A^{I}$ passes through M, and $B^{I}D^{I}$ through N. Hence, the triangles $A^{I}A^{II}M$, $B^{I}B^{II}M$, are similar; and (Euclid VI. 4),

$$A^{I}A^{II} : B^{I}B^{II} = A^{II}M : B^{II}M.$$

Again, from the similar triangles $AA^{II}M$ and $BB^{II}M$,

$$AA^{II} : BB^{II} = A^{II}M : B^{II}M,$$

$$\therefore\ A^{I}A^{II} : B^{I}B^{II} = AA^{II} : BB^{II}.$$

But, $B^{I}B^{II} = BB^{II} \div 8$; $\therefore\ A^{I}A^{II} = AA^{II} \div 8 = 3CA^{II} \div 8$.

From the pairs of similar triangles $BB^{II}N$, $DD^{II}N$; and $B^{I}B^{II}N$, $D^{I}D^{II}N$, we get similar proportions and find

$$D^{I}D^{II} = DD^{II} \div 8 = CA^{II} \div 8.$$

11. The length of the projection of the nearly vertical edge of the cube through C has now to be determined. When the cube and square $D^{III}A^{II}$ have been rotated together through 90°, the lines in Fig. 49 are brought into the plane of projection and one edge of the cube through C is in the line of sight. Let CC^{III} and CH_I, in Fig. 50, represent the line CC^{III} of Fig. 49 and the edge of the cube when the latter is in the line of sight. If, now, the cube and Fig. 49 are turned back to the final position of the cube, BB^{II} and $C^{III}C$ are shortened to B^IB^{II} which is given by CL of Fig. 50. During this rotation the points C^{III} and H_I have traversed the circular arcs $C^{III}C^{IV}$ and H_IG. Draw the perpendiculars $C^{IV}L$ and HG on the vertical line HCC^{III}. Then $CL = B^IB^{II} = CC^{III} \div 8$, and CH is the projected length of the cubic edge CG.

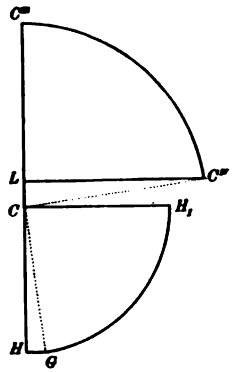

Fig. 50.

If the angle of tilt, $GCH = CC^{IV}L$, be denoted by σ, we have

$$\sin\sigma = CL \div CC^{IV} = 1 \div 8\text{; and } \sigma = 7°\ 11'.$$

For any other angle of tilt, $\sin\sigma = B^IB^{II} \div BB^{II} = 1 \div m$.

It is now easy to find the value of CH of Fig. 50 in terms of the unit CA^{II} of Fig. 49; for CG is an edge of the cube $= CA$ of Fig. 49 $= CA^{II}\sqrt{10}$, and $CH = CG\cos\sigma = CG\sqrt{1-\sin^2\sigma}$.

$$\therefore\ CH = CA^{II}\sqrt{10}\sqrt{1-\tfrac{1}{64}} = CA^{II}\sqrt{630} \div 8 = 3{\cdot}137\,CA^{II}.$$

If H', Fig. 51, be the middle point of AA^{III} of Fig. 49, then

$$A^IH' = A^{II}H' - A^{II}A^I = 3{\cdot}125\,CA^{II}.$$

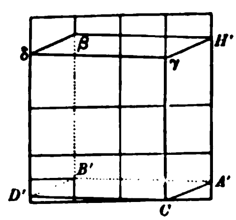

Fig. 51.

Hence, for practical purposes, the point H' may be substituted for H, and the cube projected, as in Fig. 51, by drawing through H', lines $H'\gamma$, $H'\beta$, parallel to CA' and $A'B'$, respectively. The remaining edges of the cube are then drawn through β and γ. Or lengths $C\gamma$, $B'\beta$, $D'\delta$, each equal to $A'H'$, may be measured off on the verticals from C, B' and D' by means of a pair of compasses.

The lines CA', CD' and $C\gamma$, produced in both directions give the projections of a rectangular system of axes. Furthermore, the length CA' gives the unit of length on OX or on any parallel line, CD' gives the same unit of length on OY or on any parallel line, and $C\gamma$—or, more correctly, CH, when theoretical values are involved—gives the same unit of length on all vertical lines.

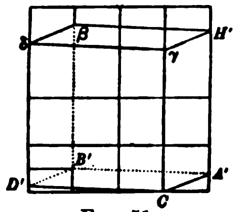

FIG. 51.

The student will find it convenient to have a set of rectangular axes of equal length, shortly called cubic axes, projected in the manner just described. They should be drawn on a large scale on card-board or stout paper. He should notice that CA' is the negative direction of the axis of X and CD' the negative direction on OY.

12. The angles $\gamma CD'$ and $\gamma CA'$ can be easily calculated from the data of Figs. 49–51, and also the actual lengths of CA', CD', and CH in terms of any scale.

For $\tan D^{I}CD^{II} = \dfrac{D^{I}D^{II}}{D^{II}C} = \dfrac{1}{24}$; $\therefore D^{I}CD^{II} = 2°\,23'$, and $\gamma CD' = 87°\,37'$:

$\tan A^{I}CA^{II} = \dfrac{A^{I}A^{II}}{CA^{II}} = \dfrac{3}{8}$; $\therefore$ $A^{I}CA^{II} = 20°\;33'{\cdot}3$, and $\gamma CA' = 69°26'{\cdot}7$.

Furthermore,

$$CA' : CD' : CH = \sqrt{73} : \sqrt{577} : \sqrt{630} = 8{\cdot}5 : 24 : 25{\cdot}1, \text{ nearly.}$$

It is easy to obtain, in the same manner, the projection of the axes for any other values of the rotation and tilt. Thus, for drawing cubic crystals, Professor von Lang (*Lehrbuch der Krystallographie*, 1866), uses axes in which $D^{II}C = 4CA^{II}$, and $BB^{II} = 9B^{I}B^{II}$. Hence, if we assume Fig. 51 to represent a cube projected with these numerical relations, we have:

$$\sin \sigma = 1 \div 9, \text{ and } \sigma = 6°\;23'{\cdot}6.$$

$\gamma CD' = 90° - 1°\;33'{\cdot}5$, $\gamma CA' = 90° - 23°\;58'$.

$CA' : CD' : CH = \sqrt{97} : \sqrt{1296} : \sqrt{1360} = 1 : 3{\cdot}65 : 3{\cdot}74$, nearly.

Having obtained a projection of cubic axes, all simple cubic forms can be drawn as will be shown in Chapter xv.; and a little practice will then enable the student to draw combinations of several forms.

13. To adapt the projected cubic axes to the requirements of a tetragonal crystal, we only change the length of the vertical axis, as shown in Fig. 52. The parameter c on the principal axis is given by the equation, $c = a \tan E$, where E is the angle $001 \wedge 101$. Thus, for cassiterite (SnO_2), $E = 33° 55'$; $\therefore c = C\gamma \tan 33° 55'$, or, more correctly, $= CH \tan 33° 55'$; for CH is the unit length measured on the vertical. If CH be taken to be 25·1 units of length, then c is 16·88 units of the same scale. The points A, A', C in Fig. 52 are at the parametral distances on the projected axes of X, Y, and Z respectively.

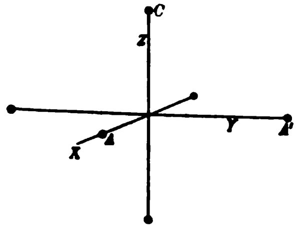

FIG. 52.

14. In a prismatic crystal the length CA' of Fig. 51, measured along OX, has also to be changed and by a similar rule. In Art. 8 it was shown that, for barytes: $a = b \tan 39° 11'$, and $c = b \tan 52° 43'$. But, in Fig. 51, CA' along OX is the same length as CD' measured along OY. Hence, we have to mark off on CA' a length

$$= CA' \tan 39° 11' = CA' \times {\cdot}8151;$$

and similarly, along $C\gamma$ a length $= CH \tan 52° 43' = CH \times 1{\cdot}3135$.

Similarly, for the axes of topaz, take a length $CA' \tan 27° 50'$ along OX, CD' along OY, and a length $CH \tan 43° 31'$ along OZ.

15. To obtain the axes of an oblique crystal, we first replace CA' of Fig. 51 by a new line $OX_{,}$, situated in the plane $\gamma CA'$ and inclined to $C\gamma$ at an angle β. This is done as follows. In Fig. 53, let $C\gamma$ and $CA_{,}$ be two lines, each equal to CA of Fig. 49, and let $\gamma CA_{,} = 90°$. With centre C describe the quadrant of a circle $\gamma QA_{,}$, and let the arc

$$\gamma Q = \wedge \gamma CQ = \beta.$$

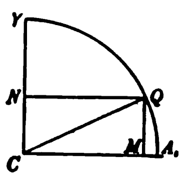

FIG. 53.

Then CQ, when projected in the plane $\gamma CA'$ of Fig. 51, is the direction of the axis $OX_{,}$ required: and $CQ = CA_{,} = CA$, is the unit of length. Draw QM and QN parallel to $C\gamma$ and $CA_{,}$, respectively. Then $CM = CQ \sin MQC = CA_{,} \sin \beta$; and $CN = CQ \cos NCQ = C\gamma \cos \beta$.

Now, by the character of the projection, equal parallel lines are projected as equal parallel lines, and lengths in the same line are

projected as lengths having to one another the same ratio as the original lengths. Hence, $OA_{,}$, OY, OZ, in Fig. 54, being the directions of CA', CD' and $C\gamma$ of Fig. 51, construct in the plane $ZOA_{,}$ a parallelogram $Omqn$ equal and similar to that of Fig. 53; where

$$Om = OA_{,} \sin\beta$$
$$= CA' \sin\beta, \text{ of Fig. 51,}$$
$$On = C\gamma \cos\beta, \quad ,, \quad ,, \quad ,,.$$

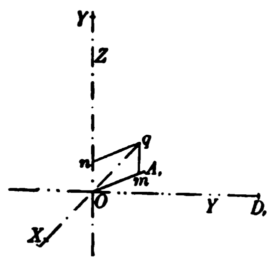

FIG. 54.

Join Oq, and produce it on both sides of the origin. Then Oq is the direction $OX_{,}$, and is the unit of length measured in this direction.

The lengths Oq and $O\gamma$ have now to be changed in the ratios $a \div b$ and $c \div b$; or, as Oq is the shortest length, it may be more convenient to change CD' and $C\gamma$ of Fig. 51 in the ratio of $b \div a$, and $c \div a$, respectively. When such computations are made, the correct value of $C\gamma$, viz. CH of Fig. 50, is used, and not the approximate value $A^{I}H'$.

Example. To obtain the axes of orthoclase in which $\beta = 63° 57'$, and

$$a : b : c = {\cdot}658 : 1 : {\cdot}555,$$

we have $Om = CA' \sin 63° 57' = 8{\cdot}5 \sin 63° 57' = 7{\cdot}637$ units.

$$On = C\gamma \cos 63° 57' = 25{\cdot}1 \cos 63° 57' = 11{\cdot}02 \text{ units.}$$

The parallelogram $Omqn$ is now completed, and Oq drawn. Retaining Oq, the unit length on OX, as the unit parameter, a length $OB_{,}$ is taken on $OY = 24 \div {\cdot}658 = 36{\cdot}47$ units of the scale. Similarly, OC on OZ is made equal to $25{\cdot}1 \times {\cdot}555 \div {\cdot}658 = 21{\cdot}17$ units of the scale.

16. A convenient set of oblique axes can be expeditiously obtained as follows. We start from the position in Art. 9, where the cube has been turned through the angle ρ but not tilted[1]. The positive cubic axes then form a cross, Fig. 60, with unequal horizontal arms, OD and OE; where $OD : OE = 1 : 3$, or, generally, $1 : n$. The arm OE to the right is retained as the axis OY, whilst the cubic axis projected in OD has to be replaced by a new line in the vertical plane ZOX. The vertical arm OA'' is retained as OZ, and its length is $OD\sqrt{10}$, or $OE\sqrt{10} \div 3$, or generally, $OE\sqrt{1+n^2} \div 3$. Hence, in Fig. 55, OC and OB are drawn intersecting one another

[1] This is the same position as that mentioned in Art. 23 before the eye is moved. The drawings are orthographic, for the rays are at 90° to the paper. The projection is only advantageous when β differs appreciably from 90°, and when no face has to be shown which is nearly horizontal.

at right angles, and any arbitrary length OB is cut off on the horizontal arm to the right. This represents the unit of length on OY, just as OE does in Fig. 60. The continuation on the left, which may be denoted by OD, represents the cubic axis perpendicular to OB, and its length is $OB \div 3$. With sides lying in OD and the vertical construct the parallelogram $OMXN$, where

$$OM = OD \sin NOX = OD \sin \beta = OB \sin \beta \div 3,$$

$$ON = OA'' \cos NOX = OB \sqrt{10} \cos \beta \div 3.$$

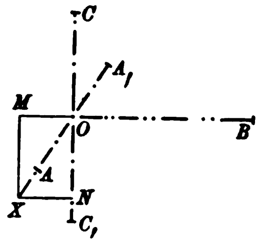

FIG. 55.

The diagonal OX of the parallelogram gives the unit length on the inclined axis OX of an oblique crystal, the lines OB and OC giving the positive directions of the axes of Y and Z. Fig. 55 gives the axes of orthoclase from which the figure, given in Chap. XII., has been drawn. Having obtained the directions of the axes, OA is now taken on OX equal to $OX \times \cdot 658$, OB is the unit of length on OY, and

$$OC = OA'' \times \cdot 555 = OB \sqrt{10} \times \cdot 555 \div 3 = OB \times \cdot 585.$$

The parameters are known in terms of the arbitrary length OB.

17. In the anorthic system the angles between the axes, $YOZ = \alpha$, $ZOX = \beta$, $XOY = \gamma$, may have any values, and vary with the substance. We first find, in the plane $A'CD'$ of Fig. 51, two lines inclined to one another at an angle $\psi = 100 \wedge 010$. This angle between the vertical faces is not to be confused with γ, and is usually obtained by direct measurement. The obtuse angle, $180° - \psi$, is usually placed to the right front.

Suppose CD' of Fig. 51 to be unmoved; we can, by the process employed in Art. 15, find a line $O\xi$ in the plane $A'CD'$ inclined to CD' at the angle ψ. The sides of the parallelogram along CD' and along the prolongation of CA' to the front are: $Cd = CD' \cos \psi$, $Cm = CA' \sin \psi$. The diagonal $C\xi$ lies in the plane $D'CA'$, and its length is the unit length. Further, the plane $\gamma CD'$ is parallel to (100) and $\gamma C\xi$ to (010), and they contain the axes of Y and X, respectively. In the plane $\gamma CD'$ we form a parallelogram, the sides ON' along $C\gamma$, and OM' along CD', being given by $ON' = C\gamma \cos \alpha$, $OM' = CD' \sin \alpha$. The diagonal gives the direction of OY and the unit length on it. A similar parallelogram $OMXN$ is formed in the plane $\gamma C\xi$ in which β replaces α, and $C\xi$ replaces CD'. The diagonal is the axis OX and the unit length on it. When α and β are $>90°$, the sides of the parallelograms are taken so that γCY and γCX are $>90°$. The unit lengths on OX and OZ are now multiplied by the numbers $a \div b$, and $c \div b$, respectively.

18. Suitable projections of the axes of anorthic crystals can be also obtained by a method similar to that given for oblique crystals in Art. 16, in which the cube has not been tilted. Starting with horizontal arms, OE to the right and $OD = OE \div 3$ to the left, we first change the length on the left so that it should represent unit length $O\xi$ on a line making $180° - \psi$, instead of $90°$, with the axis on the right. The direction of the line is still the horizontal arm to the left. Then $O\xi = OE \cos(\psi - \rho) \div 3 \sin\rho$, where $\rho = 18° 26'$; and, as before, OA'' on the vertical axis is equal to $OE\sqrt{10} \div 3$.

In the plane ZOE a parallelogram $OM'YN'$ is made, where

$OM' = OE \sin\alpha$, and $ON' = OA'' \cos\alpha$.

The diagonal OY is then the direction of the axis of Y, and the unit length on it.

Similarly, in the plane $ZO\xi$ the parallelogram $OMXN$ is made, where

$OM = O\xi \sin\beta = OE \cos(\psi - \rho) \sin\beta \div 3 \sin\rho$;

and $ON = OA'' \cos\beta$. The diagonal OX is the axis of X and the unit length on it.

The unit lengths on two of the axes are then easily changed to correspond with the parameters of the crystal to be drawn.

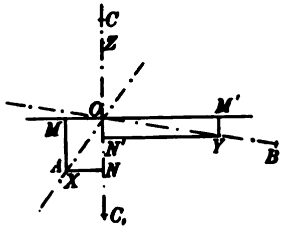

FIG. 56.

Hence the lengths in the paper are all known in terms of the arbitrary length OE.

Example. As an illustration let us take cyanite (Al_2SiO_5) in which we know: $\psi = 73° 56'$, the angles: $\alpha = 90° 5'{\cdot}5$, $\beta = 101° 2'{\cdot}25$; and the parameters

$$a : b : c = {\cdot}899 : 1 : {\cdot}709.$$

Then $O\xi = OE \cos 55° 30' \div 3 \sin 18° 26' = OE \times {\cdot}6$ (very nearly):

$$OA'' = OE \times 1{\cdot}1 \text{ (very nearly).}$$

The rectangular cross has the arms $O\xi$ of 6 units, OE of 10 and OA'' of 11 units of length.

Again, since $\alpha = 90° 5'{\cdot}5$, the axis OY is left in OE, the error being imperceptible. OX is found from the parallelogram $OMXN$ with sides:

$$OM = O\xi \sin 78° 58' = 5{\cdot}88, \text{ and } ON = OA'' \cos 78° 58' = 2{\cdot}125.$$

The final lengths on OX, OY, and OZ, are: $OA = 6{\cdot}27 \times {\cdot}899 = 5{\cdot}63$, $OE = 10$, and $OC = OA'' \times {\cdot}709 = 7{\cdot}8$ units. The angle XOM can also be calculated for the above values; for $\tan XOM = ON \div OM = 2{\cdot}125 \div 5{\cdot}88$. Therefore $XOM = 24° 25'$.

19. To draw rhombohedral and hexagonal crystals, a regular hexagon, $\delta\delta_{\prime\prime}\delta'\ldots$, Fig. 57, is projected in the plane $A'CD'$ of Fig. 51 which contains two lines of unit length at right angles to one another.

Let $O\delta_{\prime} = O\delta'$ be the unit of length, and on the line at right angles to $O\delta_{\prime}$ take $OM_{\prime} = OM' = O\delta_{\prime} \tan 60° = OA\sqrt{3}$, where OA on $OM_{\prime}$ is the unit of length. Complete the rhombus $M_{\prime}\delta' M' \delta_{\prime}$, and

bisect the sides at δ, $\delta_{\prime\prime}$, $\bar{\delta}$, δ''. Join $\delta\delta_{\prime\prime}$ and $\bar{\delta}\delta''$, and produce them and the sides of the rhombus to meet at M, $M_{\prime\prime}$, &c. Then clearly the small triangles $\delta M_{\prime}\delta_{\prime\prime}$, $\delta_{\prime\prime}M\delta'$, &c., are all equal and equilateral; and $\delta\delta_{\prime\prime}\delta'\bar{\delta}\delta''\delta_{\prime}$ is a regular hexagon, each of the sides of which is the unit of length, for $\delta\delta_{\prime\prime}$ is parallel to, and one-half the length of, $\delta_{\prime}\delta'$. Also $OM = OM_{\prime\prime} = \&c. = OM_{\prime} = OA\sqrt{3}$.

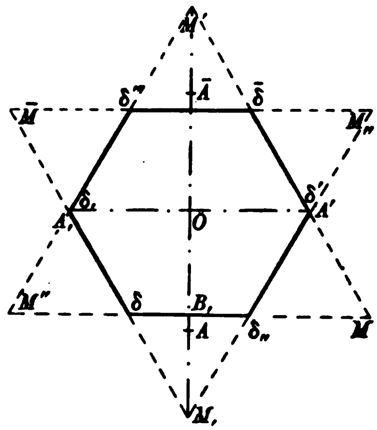

FIG. 57.

But in parallel projections the ratios of lengths on parallel lines remain unaltered in the projection. Hence, let CD' and CA' of Fig. 51 be produced both ways, and let the origin be taken at C. On CD' lengths $= CD'$ are measured off on both sides of C and give the points $\delta_{\prime}\delta'$ of Fig. 57. On CA' points $M_{\prime}$ and M' are then taken at distance $CA'\sqrt{3}$, and these are joined to the points on CD' to complete the rhombus. The sides of this rhombus are then bisected at points δ, $\delta_{\prime\prime}$, $\bar{\delta}$, δ''. The lines $\delta\delta_{\prime\prime}$, $\bar{\delta}\delta''$ are then drawn, and produced to meet the sides of the rhombus produced at M, $M_{\prime\prime}$, &c. The hexagon and the useful points M, $M_{\prime\prime}$, &c., are thus all projected. We also know the relations between lengths along the lines joining the points to the origin, for the sides of the projected hexagon give the unit of length along any line parallel to them, and the projected lengths OM, $OM_{\prime}$, &c. are equal to the unit length multiplied by $\sqrt{3}$.

The unit length on the vertical axis is known to be $C\gamma$ of Fig. 51. To adapt the projection to any particular rhombohedral or hexagonal crystal all that is needed is to multiply $C\gamma$ by c, the linear parameter of the crystal.

In the above, one of the diagonals of the hexagon coincided with CD', and this is a position often selected. If, however, it be desired to make a diagonal coincide with CA', we need only take lengths on $CD' = CD'\sqrt{3}$, and complete the projection in the manner described in the first case. The sides of the two hexagons are inclined to one another at angles of 30°.

20. In Art. 15 a method is given of drawing a line of unit length in an axial plane and inclined at any angle to one of the cubic axes. Hence, rectangular unit axes can be drawn in the plane inclined at any desired angle to the original projected axes. If this be done in the horizontal plane, $A'CD'$ of Fig. 51, and one of the unit lengths be then increased in the ratio $\sqrt{3}:1$, we can project the rhombus $M'A'M_,A_,$ of Fig. 57 in any desired azimuth. The hexagon of Fig. 57 can then be quickly projected. Such a projection will enable us to draw the pyramid $\tau\{hkl\}$, and the rhombohedron $\pi\{hkl\}$, of classes (i) and (ii) of the rhombohedral system.

We shall use the method of Art. 15 to find the rectangular axes of twins; as *e.g.* in one of the twins of staurolite described in Chap. XVIII.

21. It may sometimes be necessary to find a line making a definite angle with a known line in one of the conspicuous planes of the crystal, such as *e.g.* with one of the axes in an axial plane. If the projection of two lines of equal length at 90° to one another be known, the line can be found as follows. Let us suppose the plane to be $\gamma CD'$ of Fig. 51, and let the line required make an angle $\theta<90°$ with CD'. Now the triangles $\gamma CD'$ and $CD'\delta$ are both right-angled isosceles triangles, so that the angles $\gamma D'C$ and $D'C\delta$ are both the projections of angles of 45°. If the line required makes an angle θ upwards and to the right, the triangle $\gamma CD'$ is used; if upwards and to the left, $D'C\delta$. Let us take the first case.

On CD' as side construct a right-angled isosceles triangle $CD'V$ in the plane of the paper, so that $CV=CD'$, and $VD'C=45°$; and through D' draw $D'T$ making the angle θ with CD', and let it meet CV in T. Through T draw Tt parallel to $V\gamma$ to meet $C\gamma$ in t. Then $D't$ is the projection of a line inclined to CD' in the required direction at an angle θ.

From the construction $CT=CD'\tan\theta$, and $CV=CD'$.

$$\therefore CT=CV\tan\theta.$$

Again, from the similar triangles $VC\gamma$, TCt, we have

$$Ct : C\gamma = CT : CV = \tan\theta : 1. \quad \therefore Ct = C\gamma\tan\theta.$$

Hence $D't$ is the required line, since $D'C\gamma$ is a projected right angle.

The process is applicable to any plane if the angles of a definite projected triangle lying in it are known, but it is not often needed. The particular case taken above is, however, of constant use in drawing twins, for it is the readiest way of projecting the twin-axis, and thereby of projecting the axes of the rotated individual.

(iii) *Clinographic Drawings.*

22. Naumann (*Lehrbuch der Krystallographie*, II., p. 400 et seq., 1830), projected a set of rectangular axes of equal length in the following way. One face of a cube is first placed in the paper with one of its edges vertical. The cube is then turned about one of the vertical edges, lying in the paper, through an angle ρ, so that the width of the face to the left is n times that to the right. So far the construction is the same as that of Mohs given in Art. 9. If the paper be turned through 90° until it coincides with the horizontal plane, OA and OB are two of the axes, and their projections on the horizontal line of the drawing are given by OD and OE of Fig. 58. If $n = 3$, then

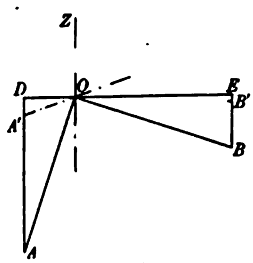

FIG. 58.

$$\tan \rho = OD \div DA = OD \div OE = 1 \div 3;$$

and $\rho = 18° 26'$. This value is generally convenient.

Naumann now supposed the eye to be raised above the horizontal plane so as to look down upon the top of the cube. The vertical axis is unaltered in length and position, for it is in the paper. The extremities of the horizontal axes, originally projected in D and E, Fig. 58, are now depressed in the vertical planes. The amount of depression of each extremity is proportional to its distance from the paper, and varies with the height to which the eye is raised. Let the depression DA' of the extremity A on the axis OX be $OD \div s$, and let σ be the angle the rays make with the perpendicular to the paper. The depression of A is seen from Fig. 59 (a), which gives the lines in the vertical plane through the eye and DA' of Fig. 58, to be

$DA' = DA \tan \sigma$. From Fig. 58,

$$DA = OE = OD \cot \rho;$$

$$\therefore\ DA' = OE \tan \sigma.$$

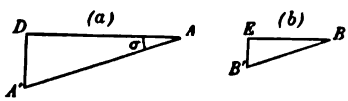

FIG. 59.

By hypothesis,

$$DA' = OD \div s = OE \div ns.$$

$$\therefore\ OE \div ns = OE \tan \sigma;\ \text{and}\ \tan \sigma = 1 \div ns.$$

Again, from Fig. 59 (b), which gives the lines in the vertical plane through the eye and $B'E$, we have $EB' = EB \tan \sigma$; and from Fig. 58, $EB = OE \tan \rho$. $\therefore\ EB' = OE \tan \rho \tan \sigma = OE \div n^2 s$.

The depressions DA' and EB' are, therefore, given in terms of the arbitrary length OE and the numbers n and s.

We can now readily find the angles DOA' and EOB', and the lengths OA', OB'.

For
$$\tan DOA' = DA' \div OD = 1 \div s,$$
$$\tan EOB' = EB' \div OE = \tan \rho \tan \sigma = 1 \div n^2 s.$$

Also,
$$A'O^2 = OD^2 + A'D^2 = OD^2\left(1 + \frac{1}{s^2}\right) = OE^2\frac{(1+s^2)}{n^2s^2}$$
$$= AO^2 \sin^2\rho \left(1 + \frac{1}{s^2}\right) = AO^2 \frac{s^2+1}{s^2(n^2+1)},$$
$$B'O^2 = B'E^2 + OE^2 = OE^2(1 + \tan^2\rho \tan^2\sigma)$$
$$= OE^2\left(1 + \frac{1}{n^4s^2}\right) = AO^2\frac{1+n^4s^2}{n^2s^2(1+n^2)}.$$

The above expressions are perfectly general, and give the values for any rotation and elevation desired in terms of OE, or of OA the cubic edge in the paper.

Naumann employed the values $n = s = 3$, in drawings of all systems, and these values are adopted in this book. Hence, $\rho = BOE = 18°\ 26'$.

Also, $\tan EOB' = \tan\rho \tan\sigma = 1 \div n^2s = 1 \div 27$; $\therefore EOB' = 2°\ 7'$.
$\tan DOA' = 1 \div s = 1 \div 3$; $\therefore DOA' = 18°\ 26'$.

The angle σ, given by $\tan\sigma = 1 \div 9$, is $6°\ 20'$.

Furthermore, $A'O^2 = AO^2 \div 9$; $\therefore A'O = AO \div 3$.

$$B'O^2 = \frac{9}{10}\left(1 + \frac{1}{9^3}\right)AO^2 = \frac{73}{81}AO^2;\ \therefore B'O = AO \times \cdot 9493.$$

23. Naumann gave the following practical rule for drawing the axes. Two lines OZ and NOE, Fig. 60, are drawn at 90° to one another, OZ being vertical and $ON = OE$; and the line NOE is trisected at D and D'. Lines are then drawn through N, D, D', E parallel to OZ. On the vertical through N a point R is taken, so that $NR = ON \div s$, and the line RO is drawn, meeting the verticals through D and D' at the points A', $X_{,}$. The line RO is the axis of X, and the lengths OA', $OX_{,}$ are the unit lengths on it. It is clear that

$$DA' : OD = NR : ON = 1 : s.$$

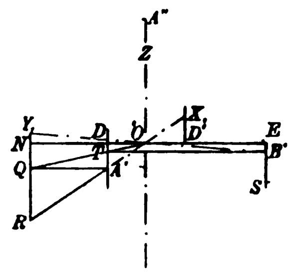

FIG. 60.

Through A' a line $A'Q$ is drawn parallel to ON, and OQ is then drawn meeting DA' in T. Hence

$$DT : DA' \text{ (or } NQ) = OD : ON = 1 : 3;$$

or generally $= 1 : n$. $\therefore\ DT = DA' \div 3 = DA' \div n$, generally.

The line TB' is drawn through T, parallel to NOE, to meet the vertical through E in B'. The line $B'OY_{\prime}$ intercepted between the two extreme verticals, gives the unit lengths OB' and $OY_{\prime}$ on the axis of Y.

For $$\tan EOB' = \frac{EB'}{OE} = \frac{DT}{OE} = \frac{DA'}{nOE} = \frac{1}{n^2 s} = \tan\rho \tan\sigma.$$

To find the length OA'' on OZ, we take S on EB' where $ES = DO$. A length on $OZ = OS$, is the unit length OA'' in the vertical, as is evident from either of the equal triangles BOE, AOD of Fig. 58.

From a set of cubic axes thus projected those of tetragonal and other crystals can be obtained in the manner described in Arts. 13—20.

In Fig. 60, s has, for the sake of greater clearness, been taken $= 3 \div 2$, though this value is rarely, if ever, used. If $1 \div s = 0$, $DA' = 0$, and the eye is not moved. The cubic axes are then projected in OD, OE, OA'' of Fig. 60, and form the irregular rectangular cross from which, in Arts. 16 and 18, the axes of oblique and anorthic crystals are obtained. The drawings are then particular cases of orthographic drawings, for the rays are perpendicular to the paper.

24. The drawing of simple forms, or of simple combinations in systems in which the forms to be shown are not closed, offers little difficulty. But in drawing combinations, in which several forms occur, a good deal of judgment and practice in drawing is needed to know which forms should be drawn first. When a decision on this point is come to, the simple form should be completely drawn, a very hard pencil, or still better a stout needle mounted in a convenient handle, being used. When introducing the faces of the remaining forms, care must be taken to cut off proportional lengths on all homologous edges. This is readily done by means of proportional compasses. No general rule can be given as to whether the predominant form, *i.e.* that which has its faces most largely developed, or one of the subordinate forms, should be first

drawn. Thus, if we desire to show the regular tetrahedron $\kappa\{111\}$ having its edges truncated by narrow faces $\{100\}$ of the cube, it is best to draw the cube first; the faces of the tetrahedron are then easily inserted by drawing lines through points on the cubic edges parallel to the diagonals of the cubic faces and modifying alternate coigns of the cube. To cut off equal lengths from the coigns on the cubic edges, the proportional compasses are fixed at the desired ratio—say to cut off one-tenth—the long legs being made to span each edge in turn, the short legs give the length from the coign on this edge. On the other hand, in a combination of $\kappa\{111\}$ with the rhombic dodecahedron $\{110\}$ small, the tetrahedron should be first drawn.

25. Until some skill has been attained it is best to project the axes and to draw the crystal on a very large scale. The drawing can be reduced either by photography, or by the following method

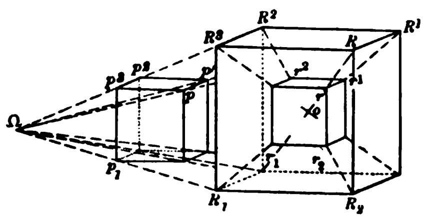

Fig. 61.

illustrated by Fig. 61. All the coigns are joined to any point in the paper, which may be either at the origin, or at any point without the drawing. The proportional compasses are then set so as to give the required reduction in the length of any line. Thus, if the drawing were to be reduced by a-half, the compasses would be set to bisect each line. By spanning each line from a coign to the fixed point with the long legs and reversing the compasses we get a series of points, as shown in the diagram, which, when joined by lines parallel to those of the original drawing, give a diagram half the dimensions of the original one. In the diagram one of the smaller cubes is one-half, and the other two-fifths, of the large cube. Thus,

$$\Omega p : \Omega R = \Omega p' : \Omega R^1 = pp' : RR^1 = 1 : 2.$$

Similarly, $\quad Or : OR = Or^1 : OR^1 = rr^1 : RR^1 = 2 : 5.$

Clearly the process can be reversed and a small figure magnified in any required ratio.

To avoid confusion the interrupted lines from Ω are not carried through the large cube, and the back edges of the interior cube have been omitted.

26. It is often desirable to show the faces at the back of a crystal. This is usually done by means of lines of dots or very short strokes. When the crystal is centro-symmetrical the edges at the back are very easily drawn as follows. All the front faces having been introduced, the coigns are pricked through on to tracing paper and the figure copied. The tracing paper is then *inverted*, that is to say, is turned half-way round in its plane, so that the edges bounding the figure again exactly fit. The coigns are now pricked through the tracing paper on to the original drawing and give the coigns at the back of the figure. It is clear that lines through the centre of the figure would join opposite coigns and be bisected at the centre, and that the process therefore gives the positions of parallel faces.

CHAPTER VII.

LINEAR AND STEREOGRAPHIC PROJECTIONS.

(i) *Linear Projection.*

1. THE diagrams given in plans, orthographic and clinographic drawings, aid the student to realize crystal-forms, and to recognize the symmetry and habit of crystals of different substances; but they do not enable him to grasp all the zonal relations of a crystal, or to determine the relations between the dihedral angles and the indices of the faces. For these purposes diagrams involving a more highly abstract representation of a crystal are generally used.

2. The first we shall describe is that known by German Crystallographers as *linear projection.* The plane of projection is parallel to some important face of the crystal, such as a plane of symmetry or an axial plane, or that perpendicular to the triad axis of a rhombohedral crystal. All faces and edges are shifted to parallel positions so as to pass through a fixed point, not lying in the plane of projection, which is called the *centre of projection.* The straight line in which the transposed face meets the plane of projection is called the *trace* of the face. Parallel faces will, when shifted, coincide in a plane parallel to each of them, and are therefore represented by a single trace. Two or more traces meet in the point in which the line of intersection of the corresponding planes meets the plane of projection. The line joining this point to the centre of projection is therefore a zone-axis, and the point is called a *zonal point*[1]. The points, in which different traces

[1] It may be denoted by enclosing in crotchets the letters which indicate the traces meeting in it.

intersect the trace of any particular face P, are zonal points which fix the directions of zone-axes parallel to the face P; and all the zones to which this face is common must give zonal points situated in its trace. Again, the line joining any two zonal points is the trace of a possible face; for the plane which contains the two zone-axes, joining the centre of projection to the zonal points, is a possible face.

The projection is useful in several ways. It may be used:—(i) for determining the directions of zone-axes, or edges of the crystal; (ii) for testing whether any face is in a particular zone, for the trace must then pass through the zonal point; (iii) for determining the symbol of a face parallel to two zone-axes. It can also be used for numerical calculations, but its suitability for such computations is so inferior to that of the stereographic projection that we shall not use it for the purpose.

3. *Example.* We shall illustrate the projection by giving the steps by which the linear projection, Fig. 63, is made. It represents the crystal of barytes of which Fig. 62 is a plan. A plane parallel to c (001) is taken for that of projection, and the centre is taken at a point C on the zone-axis, OZ, perpendicular to this face and at a distance c (the parameter) from the paper. The point C is not shown in Fig. 63, for all points on OZ are projected in O, where OZ meets the paper.

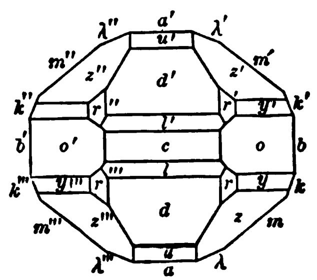

FIG. 62.

The faces a and a', when shifted parallel to themselves so as to pass through C, coincide in a plane through OZ which meets the paper in $YOY_,$. This trace is labelled a (100). Similarly, the trace in which the plane through C, parallel to b and b', meets the paper is given by the axis $XX_,$ at right angles to $YY_,$.

It is, furthermore, clear that every face parallel to the zone-axis OZ must, when it is transposed so as to pass through C, meet the paper in a trace which passes through O. Hence, from a knowledge of the angles given in Chap. III. Art. 9, we can draw the traces of the faces m, k, λ, &c., by means of a protractor. The traces k and k' make angles of 22°·15′ with OX; whilst the pairs m, m' and λ, λ' are inclined to OY at angles of 39° 11′ and 22° 10′, respectively.

Again, the traces of all tautozonal faces meet in the zonal point. If,

moreover, the zone-axis is parallel to the plane of projection, the zonal point is at an infinite distance, and all the traces are parallel. Hence, the traces

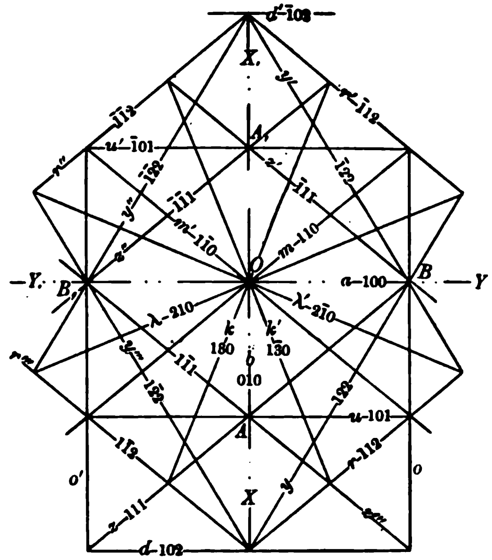

FIG. 63.

of z and z'', which are in a zone with the faces m and c, are parallel to the trace of m, and make angles of 39° 11′ with OY. The trace z may be drawn through any point B on OY. The length OB is taken arbitrarily and gives the scale of the projection, and the distance c of the centre of projection is determined by the length OB taken (Chap. VI. Art. 8). Again, from the triangle AOB, which the trace z forms with the axes, we have

$$OA \div OB = \tan(OBA = 39^\circ\ 11').$$

Hence, OA is known in terms of OB, and is a fixed length. The face z'' passes through $B_,$ and $A_,$ where $OB_, = -OB$, and $OA_, = -OA$. The homologous traces z' and z''' are now drawn through B and $B_,$ parallel to m', and clearly pass through $A_,$ and A, respectively. The symbols of the faces are attached to the traces.

The faces a, z, y, o, y', z', a' are all in a zone having CB for zone-axis, since B is the point in which the traces of a and z intersect. That a and a' are in the zone is obvious, for they are parallel to the plane of symmetry which bisects the angles yy', zz', and is perpendicular to o. The face o is parallel to OX, and its trace must be the line through B parallel to OX. For a similar reason the trace of o' is the parallel line through $B_,$. Their symbols are (011) and (0$\bar{1}$1), respectively.

Again, the faces z, u, z''' are in a zone with the faces b and b', which are parallel to the plane of symmetry bisecting the angle zz'''. The face u is also parallel to OY. Hence, its trace is the line u drawn through A parallel to OY, and its symbol is (101). The homologous face gives the parallel trace u' ($\bar{1}01$) passing through $A_,$.

The traces, o, m', u all meet in a point, and the faces are therefore tautozonal; and similarly for the other sets which meet at the corners of the rectangle formed by the traces of o and u. Close inspection of Fig. 62 shows that the edge [ro] is parallel to the edges into which m' and m''' are foreshortened; and it is clear that all lines in these latter faces will be projected in the same lines. When we meet with such edges as [ro] in a plan, we know that the faces r and o must be in a zone with the faces m' and m'''. Hence, the trace of r passes through the zonal point in which m' and o intersect. But r is also in the zone [$mzrc$]. Hence, its trace is parallel to that of m. It must meet OX at a distance $2OA$, for the triangle formed by the traces u, r, and OX is similar and equal to BOA. Similarly, the trace r meets OY at $2OB$. The face r therefore intercepts on the axes lengths $2OA$, $2OB$, OC. A parallel face is given by OA, OB, $OC \div 2$. Hence, its symbol is (112). The traces of the homologous faces are now easily drawn.

The edges of y, r, d, r''', y''' are all parallel to one another and to the lines b. Hence, the faces are all tautozonal. They therefore all pass through the points in which the trace r meets OX. But d is parallel to OY. Its trace is, therefore, inserted; and the face has the symbol (102). But the trace d meets that of z at the point in which this latter meets that of o'. Hence, the faces z, d, o', are tautozonal.

The trace and symbol of y are now easily found, for y is common to the zones [zyo] and [dry]. Hence, its trace passes through B and through the point of intersection of r and d; and the intercepts of the face are $2OA$, OB, OC. The symbol of y is therefore (122). The homologous traces can be now drawn.

By drawing lines through A and B parallel, respectively, to λ and k, the student can easily prove that the symbols of these faces are (210) and (130). The trace of λ is seen to pass through the intersection of those of y and r', and that of k through those of r' and z'. The symbols can therefore be obtained by Weiss's zone-law.

The trace of l (104) has not been inserted, as it would unduly extend the diagram, for it meets the axis of X at a distance $4OA$. The face c is parallel to the paper and cannot be shown as a trace.

A linear projection is sometimes inconveniently large; for a face, inclined at a small angle to the paper, meets it in a trace at a correspondingly great distance from the origin. In practice, this is a serious drawback.

It should be noticed that a knowledge of the parameter c, or of the angle which the face z makes with the paper, has not been needed. The projection would be the same, except in scale, whatever be the length OC.

4. Linear projections are of great assistance in making plans and other drawings of crystals. Thus the plan, Fig. 62, is readily made from the linear projection. In the preceding Article reference was made to the plan as standing for the crystal; but in actual practice a linear projection would be drawn directly from the crystal after such measurements and determinations of zones as were needed had been made, and the plan would then be drawn from the linear projection. For the edges in the plan are the lines joining O to the zonal points in which the corresponding traces meet. The edges are parallel to the traces when these are parallel to one another, for the zonal point is at an infinite distance. It is also clear that the edge of intersection of c (001) with any face is parallel to the trace of the latter face.

Linear projections may be advantageously used in order to get the directions of the edges of a crystal, when an orthographic or clinographic drawing has to be made; and the method is often less laborious than that given in Chap. v. Art. 4. All the faces are transposed parallel to themselves so as to pass through a point on one of the axes of reference at a distance from the origin equal to the parameter on this axis; as, for instance, the point C on the axis OZ, where $OC = c$. The traces are then drawn in the projected plane XOY. If the axes of X and Y and the parameters on them have been correctly projected, the traces are as easily drawn as in the example worked out in Art. 3. The directions of the edges are given by the lines joining C to the intersections of the traces. The scale of the linear projection is, in this and similar cases, determined by the lengths of the projected axes. Or we may suppose all the faces to pass through A on OX, where $OA = a$, and the traces to be drawn in the plane YOZ. It may, occasionally, be convenient to use partial linear projections in more than one of the axial planes in drawing the same crystal. Such partial projections would be needed to give a few edges only. Care must, of course, be taken not to make mistakes as to the plane in which the traces lie, or as to the axis on which the centre of projection is taken.

5. *Example.* We shall illustrate the application of linear projection to the drawing of crystals, by describing the steps involved in drawing the simple crystal of anorthite, given in Fig. 65.

The forms to be represented are: M {010}, P {001}, y $\{20\bar{1}\}$, l {110}, T $\{1\bar{1}0\}$, p $\{\bar{1}11\}$, o $\{\bar{1}\bar{1}1\}$. The parameters and the following angles are known: $Mh = 010 \wedge 100 = 87° \ 6'$, $\alpha = 93° \ 13'{\cdot}\dot{3}$, $\beta = 115° \ 55'{\cdot}5$.

Projecting the axes by the method of Chap. VI. Art. 18, we draw, in Fig. 64, MM' and $CC_{,}$ at right angles, and construct the parallelograms $OMXN$ and $OM'YN'$ with the sides given below. OE, or OM', being any arbitrary length, we take:

$$O\xi = OE\frac{\cos(87^\circ 6' - 18^\circ 26')}{3\sin 18^\circ 26'} = OE\frac{\cos 68^\circ 40'}{3\sin 18^\circ 26'}.$$

$$OM = O\xi\cos 25^\circ 55'\cdot 5 = OE \times \cdot 36;$$

$$ON = OA''\sin 25^\circ 55'\cdot 5 = OE\sqrt{10}\sin 25^\circ 55'\cdot 5 \div 3 = OE \times \cdot 46;$$

$$OM' = OE\cos 3^\circ 13'\cdot\dot{3} = OE, \text{ very nearly};$$

$$ON' = OA''\sin 3^\circ 13'\cdot\dot{3} = OE\sqrt{10}\sin 3^\circ 13'\cdot\dot{3} \div 3 = OE \times \cdot 18;$$

$$OA'' = OE\sqrt{10} \div 3 = OE \times 1\cdot 06.$$

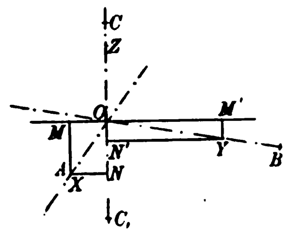

FIG. 64.

Then OX and OY are unit lengths in the axes of X and Y. These lengths and OA'' have now to be changed in the ratios of the parameters, $\cdot 64 : 1 : \cdot 55$, or in the ratios $1 : 1 \div \cdot 64 : \cdot 55 \div \cdot 64$. The length $OX = OA$ being unchanged, the length OY is divided by $\cdot 64$ to give OB, and the length $OC = OA'' \times \cdot 55 \div \cdot 64 = OE \times \cdot 91$.

A linear projection is now made in the plane XOY, Fig. 65, all the faces being shifted to pass through C on OZ at distance c. The trace y is parallel to OY and bisects $OX_{,}$; the trace of o $(\bar{1}\bar{1}1)$ is oo meeting the axes of Y and X at $B_{,}$ and $X_{,}$, respectively; and the trace of l (110) is Ol parallel to oo. Similarly, the traces of p and T are pp through B and $X_{,}$, and OT parallel to pp. The edges of the crystal are parallel to the axes, and to the lines joining C to the zonal points.

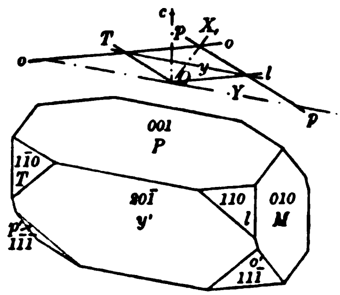

FIG. 65.

The parallelepiped PMy' is first drawn with edges parallel to the axes of X and Y and to the line joining C to the point of intersection of $OX_{,}$ with the trace y. It may have any dimensions; but, if it is to represent a crystal of particular habit, the edges should approximate in length to those in the crystal, when the other faces are neglected.

Faces p. Equal lengths (one-fifth, say) are now cut off by proportional compasses at opposite ends of the upper and lower edges $[Py]$. Through these points the edges $[Pp]$ and $[P_{,}p']$ are drawn parallel to the trace pp, for the face P is parallel to XOY. Through the same points, edges are drawn parallel to the line joining C to the zonal point $[ypl]$; of which $[p'y']$ is alone shown. The edges $[Pp]$ and $[py]$ meet the face M in points on the edges $[MP]$ and $[My]$, which, when joined, give the edge $[Mp]$. The parallel edge $[M_{,}p']$ is determined in the same manner. The edges so obtained will be found to be parallel to the line joining C to the zonal point $[pX_{,}o]$.

Faces o. Equal lengths (one-fourth, say) are now cut off at the other ends of the same edges $[Py]$. Lines parallel to the trace oo give the edges $[Po]$,

$[P,o']$. The remaining edges of the face o' are obtained in a manner similar to that given for p'.

Prism-faces. The faces l and T are easily drawn in a similar manner. They meet P in edges parallel to the traces Ol, and OT; and y' in edges parallel to the lines joining C to the zonal points $[ylp]$ and $[yTo]$, respectively. The last vertical edge $[MT,]$ has to be equal and parallel to $[M,T]$. It is most easily drawn by determining the distance from the coign of the parallelepiped PMy, intercepted by $[M,T]$ on the edge $[PM,]$, and measuring off this length from the opposite coign of PMy, along the parallel edge $[P,M]$. The vertical line through the point so found is the edge required.

If it is desired to show by dotted lines the parallel edges behind the paper, the drawing is covered with tracing-paper and the coigns marked on it. The tracing-paper is then inverted, and the coigns in the edges bounding the figure are again brought into strict coincidence. The positions of the coigns in the tracing give the coigns behind the paper. They can be now pricked through with a needle-point, and the lines joining them to one another and to the coigns in the bounding edges give the edges at the back of the figure.

(ii) *The Stereographic Projection.*

6. This projection was introduced by F. E. Neumann as offering a simple way of representing the data of crystal measurement, and was adopted by Miller and used by him in the *Treatise on Crystallography*, 1839; and in Brooke and Miller's *Mineralogy*, 1852. All the normals to the faces are supposed to pass through the origin, and a sphere of convenient radius is described with the origin as centre. The points in which the normals meet the sphere are the *poles* of the faces. In diplohedral crystals each normal is perpendicular to a pair of parallel faces situated on opposite sides of the origin. We shall take the point on the sphere to be the pole of the face, when the two lie on the same side of the origin. We have seen that, by equations (1) of Chap. IV. Art. 15, the direction of the normal is fully determined if the axes, the parameters, and the indices of a face are known; and that, keeping in mind the relations given in (1), the indices may be said to be those of the normal. Clearly they may, in a like manner, be applied to the pole, for the angles XOP, &c., are the arcs of great circles on the sphere between the pole—which may also be denoted by P—and the *axial points* X, Y, Z. The points, where the axes OX, OY, OZ meet the sphere, are not necessarily poles of possible faces, and the distinction between the axial points and the poles of the faces parallel to pairs

of the axes must, as was indicated in Chap. IV. Art. 19, be carefully borne in mind. The poles of the axial planes YOZ, ZOX, XOY are usually denoted by the letters A, B and C, respectively, and are often inscribed in diagrams in small Italics which take up less room; but, even in this latter case, capital letters will often be used in the text, in order to avoid confusing the poles with the parameters.

Again, since all normals are drawn through the centre of the sphere, and those to all tautozonal faces lie in a plane perpendicular to the zone-axis, the poles of tautozonal faces must lie in a great circle. But we know that a zone-axis $[uvw]$ and a face (hkl) parallel to it are connected by the equation, $hu + kv + lw = 0$. Hence, if the face-indices be used to denote the poles of the faces, we can employ the zone-symbol to denote the great circle in which the poles lie, provided that the relation implied is that given by the above equation. There will then be no ambiguity in stating that the pole (hkl) lies in the *zone-circle* $[uvw]$. It is merely another way of stating that the face is parallel to that zone-axis.

All the propositions of Chap. V. can, now, be stated as propositions between poles and zone-circles. Thus, the point of intersection of two zone-circles is a possible pole; for the radius through this point is perpendicular to both the zone-axes, and is therefore perpendicular to the possible face parallel to the axes.

7. For the purpose of representing the sphere on the paper the stereographic projection is used, owing to the simplicity of the construction; for every circle on the sphere is projected on the paper in a straight line or circle. The reader will find proofs of this statement in text-books of Spherical Trigonometry, and in E. Reusch's *Die Stereographische Projection*; Leipzig, 1881.

The plane of the paper is supposed to pass through the centre of the sphere, and the eye to be placed *on the surface* of the sphere at one extremity of the diameter perpendicular to the paper: the other extremity of this diameter will be called the opposite point. The great circle in which the sphere meets the paper is called the *primitive circle*, or shortly the *primitive*. The projections of poles are then the points in which the lines joining the poles to the eye meet the paper. It is clear that the projections of poles situated on the hemisphere opposite to the eye will fall within the primitive circle; that the projections of poles which lie on the primitive circle

will coincide with the poles themselves; and that lines to poles, situated on the same side of the paper as the eye, will meet the paper outside the primitive. To illustrate the projection, let a terrestrial globe be taken to represent the sphere, let the equator be the primitive circle, and let the North, or South, Pole be the position of the eye. Suppose Fig. 66 to represent a meridian-circle, and the eye to be placed at E, the South Pole. The line Eq to a point q on the upper semicircle meets the diameter of the equator within the circle at Q; the line to the point L on the equator meets the diameter at L itself, and that to a point h on the lower semicircle has to be prolonged to meet the diameter outside the circle at H. It is, furthermore, clear that all points on this meridian must be projected in points lying in the diameter OL; and the same must be true for points on all meridian-circles. Similarly, the projections of the points on *any* circle which lies in a plane passing through the eye must be a straight line. For the lines joining the points to the eye all lie in the plane of the circle, and two planes intersect one another in a straight line. If the circle meets the primitive, the straight line must pass through the two points of intersection.

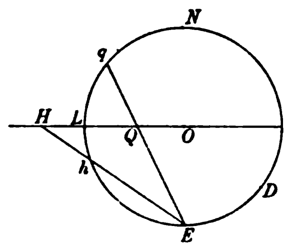

FIG. 66.

8. The poles usually shown in stereographic projections are those on the upper hemisphere *above* the paper. The eye is therefore supposed to be below the paper. The positions of the projected poles are usually indicated by dots. Any other points which may be needed, such as those of the extremities of zone-axes, may be indicated by a small cross or other mark. When it is desired to indicate the positions of poles below the paper, and this is often desirable when the crystal is not centro-symmetrical, the convention is usually made that the position of the eye is changed to the end of the diameter above the paper, and the positions of poles on the lower hemisphere as thus seen are indicated by circlets.

9. The next point to be considered is the orientation of the crystal with respect to the sphere. The crystal is invariably placed, unless the contrary is specially stated, so that some important zone-axis—often an axis of symmetry—is the diameter through the eye.

The normals to faces constituting the zone are all at right angles to the axis and, therefore, lie in the plane of the paper; consequently, the poles are easily placed in their correct position by means of a protractor, when the angles between them have been measured.

In many cases the zone-axis through the eye is an axis of symmetry, or perpendicular to a plane of symmetry. In these cases the zone-axis meets the sphere in possible poles, and the centre is the projection of a possible, or actual, pole. Zone-circles can, then, be drawn through the axis and each pole in the primitive, as in Fig. 74. Such zone-circles must be projected in the diameters of the primitive, and correspond to the meridian-circles in the illustration in Art. 7. *Diametral zones*, as they may be called, through known poles in the primitive are, when present, a great aid in making *stereograms*, if we may use this word to designate the stereographic projection of the poles of a crystal.

10. PROBLEM 1. To determine the projection of a point lying on a great circle through the eye, when its arc-distance from the opposite point is known.

Suppose a section through the eye, the centre, and the given point q, to be shown in Fig. 66. This section can be compared with that made by the plane of a meridian-circle on a globe. The line joining the point q on the meridian to the eye (at the South Pole) meets the equatorial diameter in a point Q. All great circles on the sphere are equal, for they have the same radius as the sphere. If the lines in the meridian-plane were fixed, as they would be in a model in which the meridian-circle was of brass and the lines EN, OQ, EQq were threads, it is clear that the figure and model could be turned about the diameter OQ without changing the positions of any points in this line or their relation to the points on the meridian-circle. In such a rotation the eye moves with the meridian and retains its position on this circle. Suppose the meridian, and the lines in its plane, to be turned about OQ until it coincides with the equator, the primitive in the projection. Fig. 66 may now be supposed to be drawn in the primitive, and the eye to be in the paper at E, the extremity of the diameter perpendicular to OQ. The opposite point N is the other extremity of this diameter. Hence we have the following rule:

In the plane of the primitive draw the diameter NOE perpendicular to the trace OQ of the meridian-circle containing the point q to be projected. Let the points where the diameter meets the

primitive be E and N. Mark off by a protractor $Nq = a$, the given arc, and join Eq. The point of intersection, Q, of Eq with the trace OQ is the projection of the point q on the sphere.

11. PROBLEM 2. Given the projection Q of a point q on the sphere, to find that of the other extremity of the diameter through q.

By Fig. 67, and considerations similar to those employed in Prob. 1, the point required is easily found. Draw the diameter OQ, and produce it indefinitely on the side away from Q. Then draw a diameter OE perpendicular to OQ meeting the primitive in E. At E make the right angle QEs, and produce Es to meet OQ in S. The point S is that required. The angle at E, being a right angle, is the angle in a semicircle. Therefore qOs is a diameter. This diameter may be supposed to be drawn in a meridian-circle which has been turned, in the manner described in Prob. 1, about the equatorial diameter OQ until it coincides with the equator.

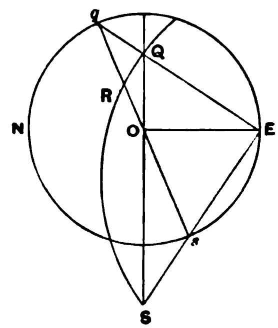

FIG. 67.

12. The anorthic system is the only one in which the construction, given in Prob. 1, does not enable us to fix a number of poles. In other systems a number of faces will, as a rule, lie in zones the axes of which can be conveniently placed in the primitive; and the angles the poles in such zones make with the opposite point can be determined either by direct measurement, or by extremely simple calculation. In cases where a considerable amount of calculation is needed to get the distance of a pole from the opposite point, it will be best to apply the method of Art. 20 to determine the position of the projected pole.

When the projections within the primitive of two or three poles have been found, those of other poles can be obtained by drawing the zone-circles through pairs of known poles. It is clear that any two poles fix the position of a zone-circle; for only one plane can be drawn through these poles and the centre of the sphere.

13. PROBLEM 3. Given the projections R and Q of two poles, to draw the projection of the great circle through the poles.

(*a*) Determine by the method given in Prob. 2 (using Fig. 67) the projection S of the point s at the other extremity of the diameter through one of the given poles. It is most convenient in practice to take that projected pole Q which is furthest from the centre. The circle required must pass through the point S; for all great circles through a given pole q have the same diameter and contain the point s at the extremity opposite to the pole. Hence we have three points in the paper R, Q, and S, and the circle through them is readily drawn by Euclid IV. 5.

The centre of the circle is most readily found as follows. Open out a pair of compasses to any convenient extent and with each of the three points as centre describe circles. The lines, joining the points in which each pair of circles intersects, pass through the centre required.

(*b*) When the points Q and R are at very unequal distances from the centre a more rapid method is illustrated by Fig. 68. Describe with any convenient radii two circles[1] through the given points, cutting the primitive in points ll' and $rr_{,}$, respectively. Draw the lines ll' and $rr_{,}$ to meet in x, and through x draw the diameter $GOSx$. The circle through R and Q meets the primitive in the points G and S. We thus have at least three points on the circle, and its centre and radius are obtained in the manner given under (*a*).

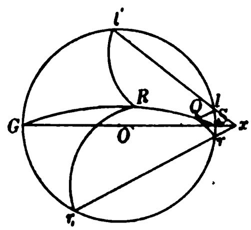

FIG. 68.

14. A zone-axis is the diameter of the sphere perpendicular to the plane of a zone-circle. The point in which the zone-axis meets the sphere above the paper is a pole of the zone-circle, and is at 90° to every point in it. To determine the arcs on a projected zone-circle, the projection of the pole is required. The pole of a zone-circle is *not*, as a rule, the pole of a possible crystal-face.

PROBLEM 4. To find P, the projection of the pole of a great circle which is projected in the known circle GMH.

Let the known circle meet the primitive in G and H, Fig. 69. Bisect the arc GH at M, and draw the straight line MO. This line is the trace of the plane of a great circle through the eye

[1] To avoid confusing the figure, the arcs of the circles ll' and $rr_{,}$ between Q and R are not drawn. All that is needed is to determine the points l, l', r and $r_{,}$, with accuracy.

perpendicular to the great circle which is projected in GMH. It is also clear that G and H are the poles of the great circle MOP, for the two diameters GH and MO are at right angles to one another. Hence, P the projection of the pole required is in MO at 90° from M. Draw the line GM, and produce it to meet the primitive at m. From m measure off a quadrant mp, and join Gp, meeting MO in P. If the figure is rotated about the diameter MO until G comes to the position of the eye, the points m and p come into the meridian projected in MOP; and, the angle mp being 90°, the point p is now in the extremity of the zone-axis. The point P is therefore known.

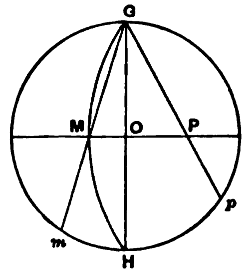

FIG. 69.

15. PROBLEM 5. A zone-circle and a known point in it being projected in the known circle GRH and the point R, respectively; to find the projection, Q, of a point in the zone-circle at a given angular distance from the known point.

The projection P of the pole of the zone-circle is first found by the preceding construction. A straight line is drawn through PR, Fig. 70, meeting the primitive in r. From r an arc $rq = a$, the given angle, is measured on the primitive by a protractor. The straight line qP meets the circle GRH in the point Q required. For a proof of this the reader is referred to works on Spherical Trigonometry, or to Reusch's *Stereogr. Projection.*

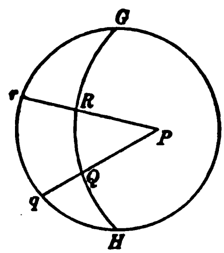

FIG. 70.

COR. 1. Conversely, the angle a between two projected poles can be determined by reversing the preceding construction; that is, the pole P of the great circle through the poles having been found, straight lines PR and PQ are drawn and continued to meet the primitive in r and q. The arc rq is then measured by a protractor.

16. PROBLEM 6. To draw the projection of a zone-circle when the projection P of the extremity of the zone-axis is known.

It is clearly only necessary to reverse the construction given in Problem 4 for finding the point P. Draw, in Fig. 69, the diameter

MOP and then the diameter *GOH* at right angles to it. The circle required passes through *G* and *H*, for both points are at 90° to *P*. Now draw *GP* and produce it to meet the primitive in *p*. Mark off on the primitive a quadrantal arc *mp*, and join *mG*. The point *M*, in which *mG* and *OP* intersect, is on the circle required; which can now be drawn since we know three points on it.

17. PROBLEM 7. A known zone-circle and point in it being projected in the circle [*GRH*] and *R*, respectively; to draw through *R* the projection of a great circle making an angle β with the known zone-circle.

The angle between two planes, and therefore between two zone-circles, is the same as that between the two lines perpendicular to them. Hence, the angle β is that between the two poles of the given and the required circles. Let *P*, Fig. 71, be the point of projection of the pole of [*GRH*]; if we can find *Q*, that of the required circle, the construction can be completed by the preceding problem. Construct, by Prob. 6, the circle [*PQ*] of which *R* is the pole. (It is not shown in the figure, as it would needlessly complicate it.) The circle must contain both *P* and *Q*: for *R* is at 90° to every point on this circle. Mark off *Q* on it at distance β from *P* (Prob. 5). Then construct the circle *LRM* of which *Q* is the pole (Prob. 6). The problem is now solved. It is necessary to know on which side of the original circle the required one is to lie, and therefore to know on which side of *P* the arc β between *P* and *Q* is to be measured. When this is known the circle is drawn accordingly.

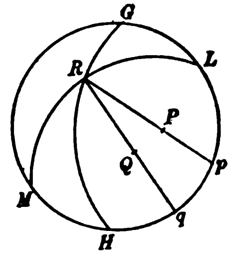

FIG. 71.

18. PROBLEM 8. To find the projection of a small circle, which has its centre at a known point on the sphere and a given arc a for arc-radius.

Draw, in Fig. 72, a diameter *OP* through the projection *P* of the known point. The small circle cuts the meridian circle, of which the diameter *OP* is the projection, in two points at angular distances a from the centre (projected in *P*) of the small circle. Through *E*, an extremity of the diameter perpendicular to *OP*, draw *EP* to meet the primitive at *p*. Cut off on the primitive two arcs *pr* and $pr_{,}$, each equal to a; and join *Er*, $Er_{,}$. The points *R* and $R_{,}$, in which these lines meet the

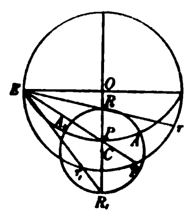

FIG. 72.

diameter OP, are the projections of points on the small circle. Bisect $RR_{,}$ at C, then C is the centre, and CR the radius, of the projection of the small circle on the sphere.

19. The preceding problem gives an easy method for finding the projection of any pole, when the angular distances from two known poles are given. For the pole must lie on small circles drawn with the given arc-radii about each of the known poles. The projections of these small circles are obtained by the preceding construction. The *two* points of intersection satisfy the data. To decide which point is that required is generally easy from knowledge of the crystal, and the position of the pole with respect to other poles.

This construction is usually applied to make a stereogram of an anorthic crystal. The poles in the primitive are placed by means of a protractor, and then a few poles are projected within the primitive from a knowledge of their angular distance from any two poles in the primitive which are not the extremities of a diameter. The points P and p coincide when the pole is in the primitive. Consequently r and $r_{,}$ are marked off on the primitive at the required angles from P, and R, $R_{,}$, and C are then found as before.

20. If, however, the angles from the unknown pole to those in the primitive exceed 45°, the above construction is inconvenient, for the line $Er_{,}$ then meets the diameter at a very distant point. In such a case it is best to draw the tangents at r and $r_{,}$ to the primitive. The point of intersection is the centre C of the small circle. For, since p the centre on the sphere of the small circle lies on the primitive, the two circles cut one another at right angles, and the radius of one circle at the point of intersection is a tangent to the other.

Example. Thus, in Fig. 73, the poles M, $M_{,}$, T of anorthite are placed in the primitive, and the point n is required, where $M_{,}n = 47° \; 24'$, and $Tn = 53° \; 14' \cdot 5$. Arcs $M_{,}a$, $M_{,}a_{,}$ on the primitive are measured off equal to $M_{,}n$, and the tangents at a, $a_{,}$ are drawn to meet at C. A circle with centre C, and with radius Ca (the length of the tangent), is a small circle, points on which are at 47° 24′ from $M_{,}$. A similar construction is made on either side of T, the arcs $T\beta_{,}$, $T\beta_{,,}$ being 53° 14′·5. A third point β on ZT can be found by joining $E\beta_{,}$, where $TE = 90°$. The circle with radius $D\beta$ and centre D cuts the circle $aa_{,}$ at the required point n within the primitive.

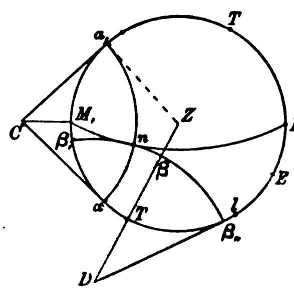

Fig. 73.

21. *Example.* We proceed to describe the construction of the stereogram, Fig. 74, of the crystal of barytes, already discussed in Art. 8 of this Chapter and in Chap. VI. Art. 8. The axis OZ, parallel to the faces a, m, b, is taken as the

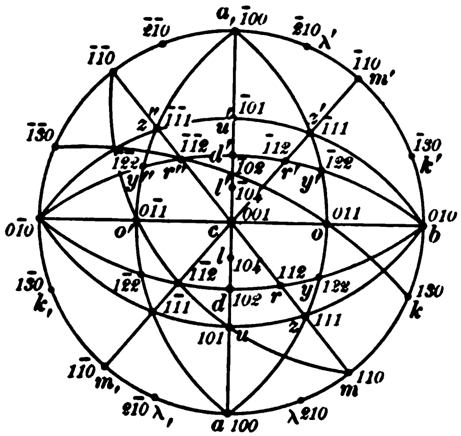

FIG. 74.

diameter through the eye; consequently the poles, a, λ, m, &c., lie on the primitive. The pole a is placed at the lowest point, b at 90° to the right. The other poles λ, m, k, &c., are put in by a protractor from the table of angles given in Chap. III. Art. 9. The zones [$audlc$], [boc], [mrc], and [$cr'z'$] are then drawn as straight lines through the centre.

The position of a pole, o, z, or d, on one of these lines is then determined by Prob. 1. Let us take o. Then angles $=co$ (52° 43′) are measured on the primitive from the point $a_{,}$, and the points so obtained are joined by straight lines to the pole a; for the points a, $a_{,}$ are at 90° from the great circle [$bcb_{,}$]. The points of intersection with [$bcb_{,}$] are the poles o and o'.

The circles in which the zones [$azyo$], [$ao'a_{,}$], are projected can then be drawn. The centre of the circle through $aoa_{,}$ is easily found by drawing circles (larger than the primitive, so that lines of construction will not confuse the stereogram) about $a_{,}$ and o, as centres. The line joining their points of intersection meets the line bc in the centre, x, required. The compasses are opened out to the extent xa and the circle described. The centre, x, is not shown; it is merely an auxiliary point in the construction. One of the needle points of the compasses is next placed at a and the centre on the line bc of the circle $ao'a_{,}$ is found, and the circle itself then drawn. The intersections of these circles with the zones [mc], [$m_{,}c$] fix the positions of z, z', z'', z'''.

The zone-circle [bzu] is next determined in a similar manner. Its centre is a trifle further from the point c than u', and the construction can be easily tested. The homologous circle [$bz'u'$] is drawn with the same radius. The poles u and u' are therefore fixed.

By drawing the zone-circle [muo'] and its homologues, the points r, r', &c., are fixed. It is then possible to draw the zone-circles [$byrd$] and [$by'r'd'$], which

by their intersections with the zones [azo], [auc] fix the positions of y, y', &c., and of d, d'.

The student will find that the following zones exist, and that circles through the poles can be drawn; viz. [kor''], [$\lambda zu'$], [zly''], [$o'ly$], [$\lambda dr''$], one of which has alone been drawn. The possibility of drawing a circle through three poles establishes the fact that the corresponding faces are tautozonal.

22. The indices of the faces on the crystal can now be all determined by the zonal relations of the circles and poles in the stereogram, and will be found to be those given in Chap. v. Art. 15 and in Chap. vi. Art. 8.

The poles a (100), b (010), c (001) of the axial planes coincide with the axial points X, Y, Z since the planes are at right angles. The face of which z is the pole is the parametral plane (111). The homologous poles above the paper are z' ($\bar{1}11$), z'' ($\bar{1}\bar{1}1$), z''' ($1\bar{1}1$).

The poles m, &c., in the primitive are those of faces parallel to OZ, and the last index is zero. The pole m is the intersection of the zone [001, 111] with [ab]=[001]. Hence m is (110). The homologous poles are m' ($\bar{1}10$), m'' ($\bar{1}\bar{1}0$), $m_{,}$ ($1\bar{1}0$).

Similarly, the pole o is the intersection of [bc]=[100] with [az]=[$0\bar{1}1$]. Hence, o has the symbol (011), and o' ($0\bar{1}1$). For similar reasons the poles u and u' are (101) and ($\bar{1}01$).

Again, λ, z, u' lie on the same circle and are therefore tautozonal. The symbol of the zone-circle is [111, $\bar{1}01$]=[$1\bar{2}1$]. Hence, since the face λ is parallel to OZ, and the last index zero, its symbol is (210).

The pole r is the intersection of the zone-circles [mc]=[$1\bar{1}0$] and [$m_{,}uo$]=[$11\bar{1}$]. Combining the two symbols by the rule, given in table (23) of Chap. v., we find for r the symbol (112) or ($\bar{1}\bar{1}\bar{2}$). The pole r (112) lying in the first octant must be taken. The opposite pole ($\bar{1}\bar{1}\bar{2}$) below the paper is necessarily in the same zone-circles.

The indices of d and y can now be found, for both are in [br]=[$20\bar{1}$]. The face d is also parallel to OY, or in the zone [ac]=[010]. Hence d is (102), and d' ($\bar{1}02$). Furthermore, y is also in the known zone-circle [az]=[$0\bar{1}1$]. Hence y is (122).

Again, when constructing the stereogram, k was placed by the protractor. It will be found to lie in the circle through the poles r and o', i.e. in [$3\bar{1}\bar{1}$]. The pole k has the symbol (130). Again, from the circle [zy'']=[$4\bar{3}\bar{1}$], l can be shown to be (104).

Generally, the numerical values of $a : b : c$ would be now determined. In this case, they have already been found in Chap. vi. Art. 8 from the elementary geometry of the crystal.

The student's attention may be called to the fact that the indices of a pole, situated in a zone between two known poles, are the sum of simple multiples of corresponding indices of the known poles (Chap. v. Art. 8, Equations (13)). Thus, the indices of l are the sum of those of y'' added to double those of z.

CHAPTER VIII.

THE ANHARMONIC RATIO OF FOUR TAUTOZONAL FACES.

1. LET there be four tautozonal faces $P_1(h_1k_1l_1)$, $P_2(h_2k_2l_2)$, $P_3(h_3k_3l_3)$, $P_4(h_4k_4l_4)$, no two of which are parallel. Let H_1K_1BA and H_2K_2DC, Fig. 75, be the two planes P_1 and P_2; and let OX, OY, be two of the axes of reference, so that $OH_1 = a \div h_1$, $OK_1 = b \div k_1$, $OH_2 = a \div h_2$, $OK_2 = b \div k_2$. We proceed to establish the relation, given in equation (9), between the angles made by the four faces with one another and the indices of the faces.

Let the plane xOy of the figure be perpendicular to the zone-axis, OT, and meet the line of intersection of the faces P_1 and P_2 in e. Let the planes through OT and each of the axes meet the plane xOy in the lines Ox and Oy, respectively. In the planes TOX and TOY, draw the lines H_1A, H_2C, K_1B, K_2D, parallel to the zone-axis

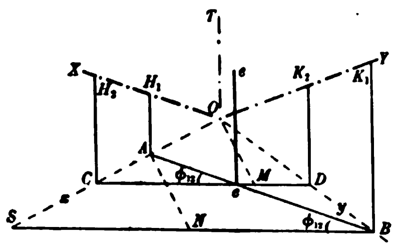

FIG. 75.

OT; and let them meet the lines Ox, Oy in the points A, C, B and D. Join AB and CD; and draw BS parallel to CD. From O and A draw OM and AN both perpendicular to CD, and let them meet CD and BS in M and N, respectively. Let the angle between the normals to P_1 and P_2 be denoted by ϕ_{12}, and let those between other pairs of normals be indicated in a similar manner by attaching subscripts to indicate the faces involved.

Now $\wedge AeC = \wedge ABN$, is equal to the angle between the normals to P_1 and P_2, and is therefore ϕ_{12}. Hence, from the triangle ABN, we have

$$AN = AB \sin \phi_{12} \ldots\ldots (1).$$

From the similar triangles OMC, ANS, we have

$$\frac{AN}{OM} = \frac{AS}{OC} = \frac{OS - OA}{OC} = \frac{OS}{OC} - \frac{OA}{OC} \ldots\ldots (2).$$

From the similar triangles OBS, ODC, $\dfrac{OS}{OC} = \dfrac{OB}{OD}$; and from the similar triangles OBK_1, ODK_2, $\dfrac{OB}{OD} = \dfrac{OK_1}{OK_2} = \dfrac{b}{k_1} \div \dfrac{b}{k_2} = \dfrac{k_2}{k_1}$.

$$\therefore \quad \frac{OS}{OC} = \frac{k_2}{k_1} \ldots\ldots (3).$$

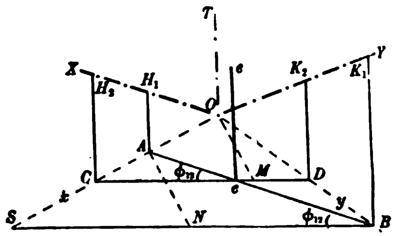

Fig. 75.

Again, from the similar triangles OAH_1, OCH_2, we have

$$\frac{OA}{OC} = \frac{OH_1}{OH_2} = \frac{a}{h_1} \div \frac{a}{h_2} = \frac{h_2}{h_1} \ldots\ldots (4),$$

$\therefore$ from equations (1), (2), (3) and (4), we have

$$\frac{AB}{OM} \sin \phi_{12} = \frac{AN}{OM} = \frac{k_2}{k_1} - \frac{h_2}{h_1} = \frac{h_1 k_2 - k_1 h_2}{h_1 k_1} = \frac{w_{12}}{h_1 k_1} \ldots\ldots (5);$$

denoting by u_{12}, v_{12}, w_{12}, the zone-indices of $[P_1 P_2]$ derived by taking $(h_1 k_1 l_1)$ and $(h_2 k_2 l_2)$ in table (10) of Chap. v. Art. 4.

In equation (5) the angle ϕ_{12} and the indices h_1, k_1, h_2, k_2 are involved with OM $(=p_2)$, the normal on P_2, and the length AB intercepted in P_1 between the lines Ox and Oy. These two lengths are arbitrary and indeterminate, for the faces may be at any distance from the origin. They must, therefore, be eliminated.

2. A similar figure, and an exactly similar set of relations to those given in equations (1)—(5), are obtained, if the face P_3 is combined with P_1. By replacing P_2 by P_3 we do not change the axes, the lines Ox and Oy, or AB, for these do not depend on the position of P_2. But ϕ_{12} has to be replaced by ϕ_{13}, the angle between P_1 and P_3, and $p_2 = OM$ by $p_3 = OM_3$, the normal on face P_3. It is clear also that the indices h_3, k_3 of P_3 replace h_2, k_2, those of P_2.

By comparing the present procedure with that followed in Chap. v. Art. 3, the student can easily write the five equations in which the face P_3 replaces P_2. We need only give the final equation correspondiug to (5). It is

$$\frac{AB}{p_3}\sin\phi_{13} = \frac{k_3}{k_1} - \frac{h_3}{h_1} = \frac{h_1k_3 - k_1h_3}{h_1k_1} = \frac{w_{13}}{h_1k_1} \text{...........(6)}.$$

The zone-index, entering into this equation, is w_{13}, that obtained by table (10) of Chap. v. Art. 4 from faces P_1 and P_3.

Dividing equation (5) by equation (6), we eliminate AB, the indeterminate length depending on the exact distance of P_1 from the origin, and have

$$\frac{p_3}{p_2}\frac{\sin\phi_{12}}{\sin\phi_{13}} = \frac{h_1k_2 - k_1h_2}{h_1k_3 - k_1h_3} = \frac{w_{12}}{w_{13}} \text{..................(7)}.$$

3. The length AB has been eliminated only by introducing a new indeterminate length p_3. But the two indeterminate lengths in (7) depend on the distances of P_2 and P_3 from the origin and not on P_1. This suggests a ready method of eliminating both p_2 and p_3. It is only necessary to take some fourth face $P_4\,(h_4k_4l_4)$ in the zone, which is not parallel to any one of the three already employed. If this be done, and faces P_2, P_3 be combined with P_4 as they were with P_1, it is clear that an equation exactly similar to (7) will be obtained, in which the only difference arises from the replacement of P_1 by P_4. Representing the angles which P_4 makes with P_2 and P_3 by ϕ_{42} and ϕ_{43}, and replacing h_1, k_1 by h_4, k_4, respectively, we have

$$\frac{p_3}{p_2}\frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{h_4k_2 - k_4h_2}{h_4k_3 - k_4h_3} = \frac{w_{42}}{w_{43}} \text{..............(8)}.$$

Dividing equation (7) by (8), we have

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{h_1k_2 - k_1h_2}{h_1k_3 - k_1h_3} \div \frac{h_4k_2 - k_4h_2}{h_4k_3 - k_4h_3} = \frac{w_{12}}{w_{13}} \div \frac{w_{42}}{w_{43}} \text{...(9)}.$$

This important relation[1], connecting the angles between four faces in a zone with their indices, is the foundation of much of our future discussion. It will be extensively used in the solution of the problems which arise in the determination (1) of the symbol of a crystal-face, and (2) of the angles between faces of which the symbols are known; and likewise in establishing various general relations.

4. The trigonometrical compound ratio forming the left side of the equation is known as the *anharmonic ratio* of four planes intersecting in, or parallel to, a common straight line, and clearly depends only on the angles between the planes. The numbers on the right side, being zone-indices, are whole numbers. Hence, we have established that *the anharmonic ratio of any four tautozonal faces, no two of which are parallel, is a commensurable number.*

5. If any one of the angles ϕ_{12}, &c., becomes 180°, the corresponding sine is zero, and the anharmonic ratio becomes zero, or infinite, according as the angle appears in the numerator or denominator. It is easy to see that the corresponding value of w on the right becomes zero at the same time, for the indices of parallel faces only differ in sign. Thus, if the first and third face are parallel, $(h_3k_3l_3)$ is $(\bar{h}_1\bar{k}_1\bar{l}_1)$; and $w_{13} = h_1k_3 - k_1h_3 = -h_1k_1 + k_1h_1 = 0$. The equation then becomes indeterminate and meaningless.

We can, also, show that the equation becomes indeterminate if the faces P_1 and P_4, or the faces P_2 and P_3, are parallel. Thus let us suppose the faces P_4 and P_1 to be parallel. Then $\phi_{42} = 180° - \phi_{12}$, and $\phi_{43} = 180° - \phi_{13}$. Also $h_4 = -h_1$, $k_4 = -k_1$, $l_4 = -l_1$.

Consequently, we have on the left side

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin(180° - \phi_{12})}{\sin(180° - \phi_{13})} = \frac{\sin\phi_{12}\sin\phi_{13}}{\sin\phi_{13}\sin\phi_{12}} = 1:$$

and on the right side

$$\frac{w_{12}}{w_{13}} \div \frac{w_{42}}{w_{43}} = \frac{h_1k_2 - k_1h_2}{h_1k_3 - k_1h_3} \div \frac{h_4k_2 - k_4h_2}{h_4k_3 - k_4h_3} = \frac{h_1k_2 - k_1h_2}{h_1k_3 - k_1h_3} \div \frac{-h_1k_2 + k_1h_2}{-h_1k_3 + k_1h_3} = 1.$$

[1] The proposition was established and first published by Miller in the *Treatise on Cryst.*, 1839. It appears from manuscript notes, bearing the date 1831 but first published after his death, that Gauss also had discovered that the trigonometrical compound ratio was rational (*Werke*, II. p. 308, 1863). He gives the proposition as a relation of four zone-circles, and his expression is equivalent to that given in (27) of Art. 19 of this Chapter. The elegant geometrical proof, given in Arts. 1—3, is due to G. Cesàro (*Rivista di Min. e. Crist. Ital.* v. 1889).

We have thus retained no angles on the left and no indices on the right, and the equation has reduced to the meaningless identity, $1 = 1$. The same result can be easily shown to hold when faces P_2 and P_3 are parallel.

6. In distinguishing the four faces as P_1, P_2, P_3 and P_4, nothing has been said to limit the order in which they actually occur on the crystal. The faces may be taken *in any order whatever*, and the student can easily prove for himself that there are *six* different possible arrangements of the four faces. The only point which must be carefully attended to is this:—that the numbers W must be calculated from the indices arranged in table (10) of Chap. v. Art. 4 in exactly the same order in which the faces occur in the angles on the left. The student will, however, find it convenient to select the extreme faces for P_1 and P_4, and the pair occupying intermediate positions for P_2 and P_3. This is not the order most commonly adopted in mathematical text-books. The advantage of taking the faces in the order recommended arises from the fact that the anharmonic ratio is then positive; and difficulties as to the directions in which the angles are to be taken and as to the signs of the trigonometrical ratios are avoided. Errors may arise in other arrangements, when an angle has to be calculated by the aid of equation (9) from a knowledge of the symbols of the faces and certain angles.

7. If we take a stereographic projection of the crystal, we have for the zone a great circle with four poles in it, no two of which are at the extremities of a diameter. The angles ϕ_{12}, ϕ_{13}, &c., are the arcs intercepted between the poles; and as we can transfer, by equations (1) of Chap. IV. Art. 15, the indices of the face to the normal and therefore to the pole, we may state the law of Art. 4 in a slightly different form as follows:—the anharmonic ratio of any four poles in a zone-circle, no pair of which are at 180° to one another, is rational and is given by equation (9) of Art. 3.

Hence, if the poles of tautozonal faces are projected on a stereographic projection, we have, as will be shown in Art. 10, a ready way of determining the symbols of the poles, and therefore of the faces, when those of at least three faces in the zone are known and the angles between all the faces have been measured; for we can, in turn, take each of the unknown poles with the three known ones to form an anharmonic ratio.

8. Again, from Chap. v. Art. 4, we see that $h_1k_2 - k_1h_2 = w_{12}$ is the zone-index referring to the axis OZ as deduced from the faces P_1 and P_2; and similarly, for the other numbers w_{13}, &c. The two axes involved in the proof, given in Arts. 1—3, were OX and OY. Also the auxiliary lines Ox and Oy were the orthogonal projections of this pair of axes on the plane containing the normals to the faces. As far as the proof is concerned, any other pair of the axes and their projections on the plane of the normals might equally well have been taken. Thus, we might take OY and OZ, and the projections would accordingly be represented by Oy and Oz. We should have similar triangles, like those in the figure, connecting $\sin\phi_{12}$ with the intercepts on OY and OZ. The only change would clearly be to introduce the indices l_1, l_2, &c., instead of h_1, h_2, &c. And similarly, if the axes OX and OZ were taken, we should replace k by l throughout.

Taking the first of the two pairs just given, we have

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{l_1k_2 - k_1l_2}{l_1k_3 - k_1l_3} \div \frac{l_4k_2 - k_4l_2}{l_4k_3 - k_4l_3}$$

$$= \text{(changing signs throughout)}\ \frac{k_1l_2 - l_1k_2}{k_1l_3 - l_1k_3} : \frac{k_4l_2 - l_4k_2}{k_4l_3 - l_4k_3}$$

$$= \frac{u_{12}}{u_{13}} : \frac{u_{42}}{u_{43}} \text{ (see Chap. v. Art. 4)} \ldots\ldots (9^*)$$

Similarly, when the axes OX and OZ are taken together,

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} : \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{h_1l_2 - l_1h_2}{h_1l_3 - l_1h_3} : \frac{h_4l_2 - l_4h_2}{h_4l_3 - l_4h_3}$$

$$= \text{(changing signs throughout)}\ \frac{v_{12}}{v_{13}} \div \frac{v_{42}}{v_{43}} \ldots\ldots (9^{**})$$

But, since in each case the angles on the left side are always the same, the expressions must have the same values, and it is immaterial which pair of axes and which zone-indices are used.

9. The identity of the three expressions for the same anharmonic ratio can be proved independently from the relations (16) and (17) given in Chap. v. Art. 10. It was there shown that for faces, P_1, P_2, P_3, in the same zone

$$\frac{u_{12}}{u_{13}} = \frac{v_{12}}{v_{13}} = \frac{w_{12}}{w_{13}} = \frac{hu_{12} + kv_{12} + lw_{12}}{hu_{13} + kv_{13} + lw_{13}};$$

where h, k, l are any numbers, which (if rational) may be the indices of a pole *not* lying in the zone.

Similarly, for the same zone we must have

$$\frac{u_{42}}{u_{43}} = \frac{v_{42}}{v_{43}} = \frac{w_{42}}{w_{43}} = \frac{hu_{42} + kv_{42} + lw_{42}}{hu_{43} + kv_{43} + lw_{43}}.$$

We may also represent the above ratios by the expressions

$$\left[\frac{P_1P_2}{P_1P_3}\right] = \frac{\begin{vmatrix} h_1k_1l_1 \\ h_2k_2l_2 \end{vmatrix}}{\begin{vmatrix} h_1k_1l_1 \\ h_3k_3l_3 \end{vmatrix}}, \text{ and by } \left[\frac{P_4P_2}{P_4P_3}\right] = \frac{\begin{vmatrix} h_4k_4l_4 \\ h_2k_2l_2 \end{vmatrix}}{\begin{vmatrix} h_4k_4l_4 \\ h_3k_3l_3 \end{vmatrix}}.$$

These latter are convenient ways of expressing the ratios when known numbers are introduced. They mean only that we calculate the zone-indices u_{12}, v_{12}, &c., from each pair of face-indices by table (10) of Chap. v. Art. 4. Now some of the zone-indices may be zero but not all three: in such cases, a glance at one pair of face-indices in the expressions suffices to show which zone-index has a finite value, and the corresponding zone-index is then used throughout. We can, therefore, express the anharmonic ratio by the equations (9), (9*), (9**), in which the zone-indices, or their equivalents given above, are introduced on the right side. We shall frequently write the anharmonic ratio in the form last suggested; viz.

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{\begin{vmatrix} h_1k_1l_1 \\ h_2k_2l_2 \end{vmatrix}}{\begin{vmatrix} h_1k_1l_1 \\ h_3k_3l_3 \end{vmatrix}} \div \frac{\begin{vmatrix} h_4k_4l_4 \\ h_2k_2l_2 \end{vmatrix}}{\begin{vmatrix} h_4k_4l_4 \\ h_3k_3l_3 \end{vmatrix}} \text{............} (10).$$

We shall also, for the sake of brevity, use the symbol

$$\text{A.R. } \{P_1P_2P_3P_4\}$$

to represent the anharmonic ratio of four tautozonal faces or of four poles in a zone-circle, and we shall generally suppose P_1 and P_4 to be the extreme faces or poles.

10. Since the indices of each face occur in the first degree both in the numerator and denominator of the right side of equation (9), or of the equivalent equation (10), it follows that an equation of the first degree in the indices of any one of the faces is given by it. If all the angles and the indices of *three* of the faces are known, we can, using the expression involving w_{12}, &c., obtain an equation of the first degree connecting h_3, k_3, for the face P_3 (say) the indices of which are required. For, since the angles are known, we can

compute the left side of (9). Let the computed number be m. Then, from (9),

$$m = \frac{h_1k_2 - k_1h_2}{h_1k_3 - k_1h_3} \times \frac{h_4k_3 - k_4h_3}{h_4k_2 - k_4h_2},$$

$$\therefore \frac{h_4k_3 - k_4h_3}{h_1k_3 - k_1h_3} = m\frac{h_4k_2 - k_4h_2}{h_1k_2 - k_1h_2} = n\,(\text{say});$$

where n is easily found, since the numbers on the right are all known.

Hence, $$h_3(nk_1 - k_4) = k_3(nh_1 - h_4)\ldots\ldots\ldots\ldots(11).$$

But $P_3(h_3k_3l_3)$ lies in the known zone $[u_{12}v_{12}w_{12}]$,

$$\therefore h_3u_{12} + k_3v_{12} + l_3w_{12} = 0\ldots\ldots\ldots\ldots(12).$$

The simplest whole numbers satisfying equations (11) and (12) are the indices of P_3.

From equation (11) we have

$$\frac{h_3}{nh_1 - h_4} = \frac{k_3}{nk_1 - k_4} = (\text{by symmetry})\ \frac{l_3}{nl_1 - l_4}\ldots\ldots(13).$$

These equations are of the same form as (13) of Chap. v. Art. 8; and, when n has been computed, they give the symbol of P_3.

But, as already stated in Art. 9, the faces may be parallel to a zone-axis lying in XOY, when the zone-indices w_{12}, w_{13}, &c., are all zero. In this case equations (11) and (12) become identical, and one of the other equations, (9*) or (9**), must be taken. An equation of the same form as (11), but involving a different pair of the indices of P_3, is then obtained which, combined with (12), gives the required indices.

Example. Thus, in the crystal of anorthite, Fig. 76, M is (010), P (001), T' ($\bar{1}10$), and x ($\bar{1}01$); and the face p lies in the zones $[Mx]=[101]$, and $[TPT']=[110]$. Therefore, by Weiss's zone-law, the symbol of p is ($\bar{1}11$).

By measurement of the zone $[PpgT']$, we have $Pp = 54° 17'$, $Pg = 80° 18'$, $PT' = 110° 40'$. Taking the faces in order to be those indicated in the formula by suffixes 1, 2, 3, and 4, and therefore taking g to be $(h_3k_3l_3)$ or (hkl), since the subscript, 3, is no longer needed to indicate the particular face, we have

$$\frac{\sin Pp}{\sin Pg} \div \frac{\sin T'p}{\sin T'g} = \left|\begin{array}{c}001\\ \bar{1}11\\ \hline 001\\ hkl\end{array}\right| \div \left|\begin{array}{c}\bar{1}10\\ \bar{1}11\\ \hline \bar{1}10\\ hkl\end{array}\right|.$$

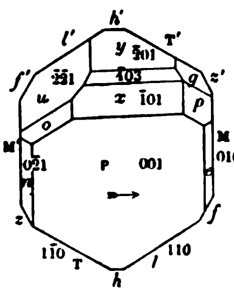

Fig. 76.

On determining the zone-indices from [Pp] it will be seen that: $u_{12} = -1$, $v_{12} = -1$, and $w_{12} = 0$. We must, therefore, avoid the zone-indices w. Taking the u's, we find: $u_{12} = -1$, $u_{13} = -k$, $u_{42} = 1$ and $u_{43} = l$.

Introducing the values of the angles and of the zone-indices into the equation, we have

$$\frac{\sin 54^\circ 17'}{\sin 80^\circ 18'} \times \frac{\sin 30^\circ 22'}{\sin 56^\circ 23'} = \frac{-1}{-k} \times \frac{l}{1} = \frac{l}{k}.$$

$$\begin{array}{rr} L \sin 54^\circ 17' = 9{\cdot}90951 & L \sin 80^\circ 18' = 9{\cdot}99375 \\ L \sin 30^\circ 22' = \underline{9{\cdot}70375} & L \sin 56^\circ 23' = \underline{9{\cdot}92052} \\ \bar{1}{\cdot}61326 & \bar{1}{\cdot}91427 \\ & \underline{\bar{1}{\cdot}61326} \\ & \log 2 = \quad {\cdot}30101 \end{array}$$

$\therefore k = 2l$, which is satisfied by making $k = 2$ and $l = 1$.

But, since the face g is in the zone [PT']=[110], $h + k = 0$.

Hence, the symbol of g is ($\bar{2}21$).

The student should notice that the equation, $h + k = 0$, is the same as is obtained by making $w_{43} = 0$; and, since all the zone-indices w are zero, this equation might have been thus obtained and not from the direct application of Weiss's law.

11. When, however, two poles P_1 and P_3 (say) are alone known in the zone, equations (9) and (10) do not give us relations limited to poles. All they tell us is that, if either P_2 or P_4 is a pole, then they are both poles. But if P_2 is not a pole, P_4 is not one either. We can easily make any number of rational anharmonic ratios of which any two poles P_1 and P_3 are members but in which the other points are not poles. Thus, whatever be the angle P_1P_3, we get a rational anharmonic ratio if P_2 bisect the angle P_1P_3 and P_4 be at 90° to P_2. For, in this case, the left side of equation (9) becomes—

$$\frac{\sin P_1P_2}{\sin P_1P_3} \div \frac{\sin P_4P_2}{\sin P_4P_3} = \frac{\sin P_1P_2 \sin P_4P_3}{\sin 2P_1P_2} = m \text{ (say)};$$

since $\qquad \sin P_4P_2 = \sin 90^\circ = 1, \text{ and } P_1P_3 = 2P_1P_2.$

But $\qquad P_4P_3 = 90^\circ - P_2P_3 = 90^\circ - P_1P_2;$

and $\qquad \sin 2P_1P_2 = 2 \sin P_1P_2 \cos P_1P_2.$

Hence, $\qquad m = \dfrac{\sin P_1P_2 \cos P_1P_2}{2 \sin P_1P_2 \cos P_1P_2} = \frac{1}{2}.$

The value of the anharmonic ratio is therefore $1 \div 2$, a commensurable number. But it is absurd to suppose that we can have a pole bisecting the angle between *any two poles whatever.* For were this the case, we could go on continually bisecting the angle until at last the angles between adjacent faces made infinitely

small angles with one another, and this is only true of curved surfaces such as spheres. The assumption contravenes the fundamental notion of crystalline structure and form.

12. *Example.* From the angles of the crystal of barytes given in Chap. III. Art. 9, the symbols of all the faces can now be easily determined.

To the faces a, b, c and z, the symbols (100), (010), (001) and (111) have been assigned. The intersections of the first three are therefore the axes, and z gives the ratios of the parameters $a : b : c$. The symbols of m, o and u, were then found by the intersections of known zone-circles (Chap. VII. Art. 22) to be (110), (011) and (101), respectively. We, therefore, know the symbols of three faces in each of the measured zones.

(i) In the zone $[amb]$ we take a to be P_1, $m = P_2$, $b = P_4$, and P_3 to be one of the unknown faces with symbol $(hk0)$.

Hence,
$$\frac{\sin am}{\sin aP_3} \div \frac{\sin bm}{\sin bP_3} = \begin{vmatrix} 100 \\ 110 \\ \bar{1}00 \\ hk0 \end{vmatrix} \div \begin{vmatrix} 010 \\ 110 \\ 0\bar{1}0 \\ hk0 \end{vmatrix}.$$

The student will, by trial with the symbols of a and m, readily find that $u_{12} = v_{12} = 0$, and that $w_{12} = 1$. The zone-indices w, referring to the axis OZ, must, therefore, be taken throughout. They are: $w_{13} = k$, $w_{42} = -1$, $w_{43} = -h$. Furthermore, since $ab = 90°$, $\sin bm = \cos am$, and $\sin bP_3 = \cos aP_3$.

$$\therefore \tan am \div \tan aP_3 = \frac{1}{k} \div \frac{-1}{-h} = \frac{h}{k}.$$

If now P_3 is made to coincide with λ, $aP_3 = a\lambda = 22° \, 10'{\cdot}5$. Introducing this angle and $am = 39° \, 11'{\cdot}25$ into the above equation, we determine $h \div k$, and the symbol of λ.

Thus,
$$\begin{aligned} L \tan (am = 39° \, 11'{\cdot}25) &= 9{\cdot}91127 \\ L \tan (a\lambda = 22° \, 10'{\cdot}5 \;) &= \underline{9{\cdot}61022} \\ \log 2 &= \;\; {\cdot}30105. \end{aligned}$$

$\therefore h \div k = 2$, and λ has the symbol (210).

If P_3 is made coincident with the face k, $aP_3 = ak = 67° \, 45'{\cdot}5$. Therefore, introducing this value into the equation, we have

$$\begin{aligned} L \tan am &= \;\; 9{\cdot}91127 \\ L \tan (ak = 67° \, 45'{\cdot}5) &= \underline{10{\cdot}38819} \\ \log ({\cdot}3335) &= \bar{1}{\cdot}52308. \end{aligned}$$

Hence, $h \div k = 1 \div 3$, which is satisfied by making $h = 1$, and $k = 3$. These, being the simplest integers in the required ratio, are the required indices. The symbol of the face k is therefore (130).

(ii) A similar formula can be easily obtained by the student for the zone $[audl]$, by following strictly the steps adopted in (i) for $[amb]$. The formula is $\tan au \div \tan aP = h \div l$. He can then prove d to be (102) and l to be (104).

(iii) The zone [$azyo$] gives a similar formula, for $ao=90°$. Thus, taking a to be P_1, y to be $P_2(hkl)$, z to be P_3 and o to be P_4, we have

$$\frac{\sin ay}{\sin az} \div \frac{\sin oy}{\sin oz} = \begin{vmatrix} 100 \\ hkl \\ \bar{1}00 \\ 111 \end{vmatrix} \div \begin{vmatrix} 011 \\ hkl \\ 0\bar{1}1 \\ 111 \end{vmatrix}.$$

But, $\sin oy = \sin(90° - ay) = \cos ay$, and $\sin oz = \cos az$; also $u_{12}=0$, $v_{12}=-l$, $w_{12}=k$. Hence, computing the v's and w's, we have: $v_{13}=-1$, $w_{13}=1$, $u_{42}=l-k$, $v_{42}=h$, $w_{42}=-h$, $v_{43}=1$, $w_{43}=-1$. And, since $u_{12}=0$; $u_{43}=l-k=0$.

The anharmonic ratio becomes,

$$\frac{\tan ay}{\tan az} = \frac{-l}{-1} \div \frac{h}{1} = \frac{l}{h}, \text{ using the zone-indices } v;$$

$$\text{and } = \frac{k}{h}, \text{ using the zone-indices } w.$$

Introducing the values of the angles ay, az, and computing, we have

$$L \tan(ay = 63° \, 59') = 10{\cdot}31150$$
$$L \tan(az = 45° \, 41'{\cdot}5) = 10{\cdot}01042$$
$$\log 2 = {\cdot}30108.$$

$\therefore l \div h = k \div h = 2.$

These equations are all satisfied by making $k=l=2$, and $h=1$. Being the simplest integers, they are taken as the indices, and the symbol of y is (122).

The symbol of r is then found by Weiss's zone-law to be (112), and the angle it makes with c (001) can be computed from the A.R. {$crzm$}, when cz is known.

13. The attention of the student is directed to the simplification, in the examples just given, of the general expression involving the trigonometrical ratios, by its conversion into the ratio of two tangents. This is always possible when two of the poles are at 90° to one another, for as was seen two of the angles are the complements of the other pair. When, therefore, two poles at 90° to one another can be found in a zone, the student will find the labour of computation much reduced if these two poles are selected as a pair of those entering into the anharmonic ratio. Zones, having such poles at 90° to one another, are of frequent occurrence in all crystals except those belonging to the anorthic system.

14. The anharmonic ratio can, likewise, be applied to the determination of the angle between one of the poles and any of the others, when the angles between the three latter poles are known as well as the indices of all the poles. To avoid trouble with signs, the anharmonic ratio should be arranged, as already recommended in Art. 6, so that P_1 and P_4 occupy extreme positions whilst P_2 and P_3 are placed between them. It is not necessary, however, that they should follow one another in the order of the numbers;

although, for the sake of convenience, we shall assume that they do so, and that ϕ_{12} is less than ϕ_{13}. Let P_2 be the pole of which the arc-distance from the others is required; and let $P_1P_3 = \phi_{13}$, $P_4P_3 = \phi_{43}$, be known angles. Then, since the symbols of the faces are known, the right side of equation (10) can be computed; call it m.

Then $$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = m.$$

$$\therefore \frac{\sin\phi_{12}}{\sin\phi_{42}} = m\frac{\sin\phi_{13}}{\sin\phi_{43}} = \tan\theta \text{ (say)} \text{ .... } (14).$$

The terms on the right are known, and the value can be easily computed by means of logarithm-tables. It must be some number greater than zero, for by the arrangement of the anharmonic ratio we are at liberty to regard all the terms as positive. Now, any number greater than zero can be represented as the tangent of an angle, θ, less than 90°. Furthermore, on computation, the expression on the right will be found to be greater or less than unity, according as the angle ϕ_{12} is greater or less than ϕ_{42}. If it is less than unity, the equation is in the most convenient form; if it is greater than unity, an expression less than unity can be obtained by inverting the equation. We may, therefore, suppose $\phi_{12} < \phi_{42}$, and that the equation is in the required form.

Now, $$1 - \frac{\sin\phi_{12}}{\sin\phi_{42}} = 1 - \tan\theta,$$

and $$1 + \frac{\sin\phi_{12}}{\sin\phi_{42}} = 1 + \tan\theta.$$

Dividing the former by the latter, we have

$$\frac{\sin\phi_{42} - \sin\phi_{12}}{\sin\phi_{42} + \sin\phi_{12}} = \frac{1-\tan\theta}{1+\tan\theta} = \tan(45° - \theta).$$

But, by well-known trigonometrical formulæ (Todhunter's *Trig.* p. 55, 1859),

$$\sin\phi_{42} - \sin\phi_{12} = 2\sin\frac{\phi_{42}-\phi_{12}}{2}\cos\frac{\phi_{42}+\phi_{12}}{2};$$

$$\sin\phi_{42} + \sin\phi_{12} = 2\cos\frac{\phi_{42}-\phi_{12}}{2}\sin\frac{\phi_{42}+\phi_{12}}{2}.$$

Hence, $$\frac{\sin\dfrac{\phi_{42}-\phi_{12}}{2}\cos\dfrac{\phi_{42}+\phi_{12}}{2}}{\cos\dfrac{\phi_{42}-\phi_{12}}{2}\sin\dfrac{\phi_{42}+\phi_{12}}{2}} = \tan(45° - \theta),$$

$$\therefore \tan\frac{\phi_{42}-\phi_{12}}{2} = \tan\frac{\phi_{42}+\phi_{12}}{2}\tan(45° - \theta), \text{ ...... } (15).$$

But $\phi_{42} + \phi_{12} = \phi_{14} = 2\alpha$ (say), is a known angle. The terms on the right are therefore known, and the value can be computed by the aid of logarithm-tables. The angle $\phi_{42} - \phi_{12} = 2\gamma$ (say), is, therefore, determined.

Thus we have
$$\phi_{42} + \phi_{12} = 2\alpha,$$
$$\phi_{42} - \phi_{12} = 2\gamma;$$
$$\therefore\ \phi_{42} = \alpha + \gamma, \quad \phi_{12} = \alpha - \gamma.$$

The same transformation of equation (10) also applies when P_1 or P_4 is the pole, the angular distance of which from the others is unknown. Thus, if ϕ_{23}, ϕ_{34} are both known and ϕ_{12} is needed, equation (10) is changed to
$$\frac{\sin\phi_{12}}{\sin\phi_{13}} = m\,\frac{\sin\phi_{42}}{\sin\phi_{43}} = \tan\theta_1 \text{ (say)}.$$
By the same transformations as were given in detail in the first case, we have
$$\tan\frac{\phi_{13} - \phi_{12}}{2} = \tan\frac{\phi_{13} + \phi_{12}}{2}\tan(45° - \theta_1).$$
Here the known angle is $\phi_{13} - \phi_{12} = \phi_{23}$.
$$\therefore\ \tan\frac{\phi_{13} + \phi_{12}}{2} = \tan\frac{\phi_{13} - \phi_{12}}{2} \div \tan(45° - \theta_1);\ldots\ldots(16).$$
Hence, $\phi_{13} + \phi_{12}$ is determined.

15. ***Example.*** In the crystal of orthoclase, Fig. 77, the zone $[cyx]$ is measured, and the indices c (001), $c_{,}$ (00$\bar{1}$), x (10$\bar{1}$), y (20$\bar{1}$), are known. It is required from the given angles, $c_{,}x = 50°\ 16'{\cdot}5$, $c_{,}y = 80°\ 18'$, to determine the angle the face a (100) makes with c (001). The face a and its parallel $a_{,}$ ($\bar{1}$00) are not present on the crystal, but the angle ac is an important element of the crystal, viz. the angle between the axes OZ and $OX_{,}$.

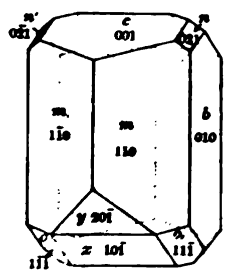

Fig. 77.

Let us take a (100) to be P_1, y to be P_2, $x = P_3$ and $c_{,} = P_4$ of the general equation (10).

Hence,
$$\frac{\sin ay}{\sin ax} \div \frac{\sin c_{,}y}{\sin c_{,}x} = \frac{\begin{vmatrix}100\\20\bar{1}\end{vmatrix}}{\begin{vmatrix}100\\10\bar{1}\end{vmatrix}} \div \frac{\begin{vmatrix}00\bar{1}\\20\bar{1}\end{vmatrix}}{\begin{vmatrix}00\bar{1}\\10\bar{1}\end{vmatrix}} = \frac{1}{1} \div \frac{-2}{-1} = \frac{1}{2}.$$
$$\therefore\ \frac{\sin ay}{\sin ax} = \frac{\sin(c_{,}y = 80°\ 18')}{2\sin(c_{,}x = 50°\ 16'{\cdot}5)} = \tan\theta.$$

$\log 2 =$	$\cdot 30103$	$L\sin 80°\ 18' =$	$9{\cdot}99375$
$L\sin 50°\ 16'{\cdot}5 =$	$9{\cdot}88600$		
	$10{\cdot}18703$		$1\underline{0}{\cdot}18703$
		$L\tan 32°\ 39' =$	$9{\cdot}80672.$

$$\therefore\ \theta = 32°\ 39', \text{ and } 45° - \theta = 12°\ 21'.$$

Also $ax - ay = xy = 30° \ 1'{\cdot}5$ is known.

$\therefore$ employing formula (16), $\tan \frac{1}{2}(ax + ay) = \tan 15° \ 0'{\cdot}75 \div \tan 12° \ 21'$.

$$L \tan 15° \ 0'{\cdot}75 = 9{\cdot}42843$$
$$L \tan 12° \ 21' \ = \underline{9{\cdot}34034}$$
$$\therefore L \tan \tfrac{1}{2}(ax + ay) = 10{\cdot}08809 = L \tan 50° \ 46'{\cdot}25.$$
$$\therefore ax + ay = 101° \ 32'{\cdot}5,$$
$$ax - ay = \ 30° \ \ 1'{\cdot}5.$$

$\therefore ax = 65° \ 47'$. Hence, $ac_{,} = ax + xc_{,} = 116° \ 3'{\cdot}5$, and $ac = 63° \ 56'{\cdot}5$.

It may happen that an arrangement of the four poles is taken which leads to disadvantageous angles. Thus, for instance, if the above poles are taken in the order $\{cayx\}$, so that the unknown pole a occupies a middle place, the angle θ of expression (15) is $44° \ 34'$. The angle $45° - \theta$ is then $26'$, and an error of a few seconds causes a considerable difference in the tangent. It was for this reason that a was, in the example, taken as one of the extreme poles. Very small angles, or angles of nearly 90°, should be as far as possible avoided; for, in such cases, a slight error in the observation may become multiplied in the course of the computation. When disadvantageous angles cannot be avoided in a direct computation, the liability to error may be minimised by taking some intermediate possible pole with the known ones. The position of this pole being determined by the method of Art. 14, the pole can then be combined with two of those first given and the one required. An angle is thus obtained which gives, by addition or subtraction, the angle required. No general rule for such artifices can be given.

16. In computing the angle between a face with known symbol, and three other known faces in the zone, five different cases may be met with. The transformation of equation (10), necessary to compute the angle which the face makes with any one of the others, depends on the arrangement of the known angles. For the sake of brevity let the poles be indicated by the points 1, 2, 3, 4 in the zone-circles shown in Fig. 78. Suppose ϕ_{14} to be less than 180°, and let the known arcs be those between the poles joined by interrupted arcs; then the different cases can be given as follows:

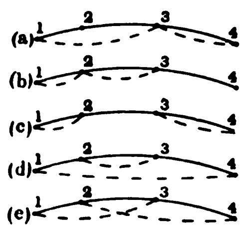

Fig. 78.

(*a*) ϕ_{13}, ϕ_{24} being known, to find ϕ_{12};
(*b*) ϕ_{13}, ϕ_{23} ,, ,, ,, ,, ϕ_{34};
(*c*) ϕ_{12}, ϕ_{34} ,, ,, ,, ,, ϕ_{23};
(*d*) ϕ_{14}, ϕ_{23} ,, ,, ,, ,, ϕ_{12};
(*e*) ϕ_{13}, ϕ_{24} ,, ,, ,, ,, ϕ_{12}.

Cases (*a*) and (*b*) are discussed in Art. 14, and case (*b*) occurred in the example of Art. 15.

17. The expression for the anharmonic ratio can, in cases (a) and (b), be also transformed into an expression involving cotangents as follows:

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{1}{m}\,(\text{say}),$$

$$\therefore \frac{1}{m} = \frac{\sin\phi_{43}\sin\phi_{12}}{\sin\phi_{13}\sin\phi_{42}} = \frac{\sin\phi_{43}\sin\overline{\phi_{13}-\phi_{23}}}{\sin\phi_{13}\sin\overline{\phi_{43}+\phi_{23}}}$$

$$= \frac{\sin\phi_{43}(\sin\phi_{13}\cos\phi_{23}-\sin\phi_{23}\cos\phi_{13})}{\sin\phi_{13}(\sin\phi_{43}\cos\phi_{23}+\sin\phi_{23}\cos\phi_{43})}$$

$$= \frac{\sin\phi_{43}\sin\phi_{13}\sin\phi_{23}(\cot\phi_{23}-\cot\phi_{13})}{\sin\phi_{13}\sin\phi_{43}\sin\phi_{23}(\cot\phi_{23}+\cot\phi_{43})}$$

$$= \frac{\cot\phi_{23}-\cot\phi_{13}}{\cot\phi_{23}+\cot\phi_{43}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

If then P_2 is the unknown pole,

$$(m-1)\cot\phi_{23} = \cot\phi_{43} + m\cot\phi_{13} \quad\ldots\ldots\ldots\ldots\ldots(18).$$

The angles and number on the right side are known; hence ϕ_{23} can be found by the help of a table of natural cotangents. Similarly, for case (b), $\cot\phi_{43}$ is unknown and all the other numbers are known; so that ϕ_{43} is found from the tables.

18. In cases (c), (d) and (e), the unknown angle cannot be obtained by either of the transformations given in Arts. 14 and 17. They can, however, be all dealt with by one and the same process. For, by taking in (c) the pole $\overline{P}_1$ opposite to P_1, we have the known arc $P_2\overline{P}_1 = 180° - \phi_{12}$, completely enclosing the arc ϕ_{34}. The case becomes therefore identical with (d). Similarly, (d) can be transformed to the arrangement (e) by taking the pole opposite to P_2 with the others.

Now the A.R. is

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \times \frac{\sin\phi_{43}}{\sin\phi_{42}} = \frac{1}{m}\,(\text{say}).$$

In (c) both the angles in the denominator, in (e) both the angles in the numerator, and in (d) all four angles, are unknown. Now

$$2\sin A\sin B = \cos(A-B) - \cos(A+B) \quad (\text{Todh. Trig. p. 60}).$$

Applying this to the equation given above, we have

$$\frac{1}{m} = \frac{\cos(\phi_{12}-\phi_{43}) - \cos(\phi_{12}+\phi_{43})}{\cos(\phi_{13}-\phi_{42}) - \cos(\phi_{13}+\phi_{42})} \quad\ldots\ldots(19)$$

But, $$\phi_{13}-\phi_{42}=\phi_{12}+\phi_{23}-(\phi_{23}+\phi_{43})=\phi_{12}-\phi_{43};$$
$$\phi_{13}+\phi_{42}=\phi_{12}+\phi_{23}+(\phi_{23}+\phi_{43})=2\phi_{23}+\phi_{12}+\phi_{43};$$
$$\phi_{12}+\phi_{43}=\phi_{13}+\phi_{42}-2\phi_{23}.$$

Hence, in case (c), we have

$$\cos(2\phi_{23}+\phi_{12}+\phi_{43})=m\cos(\phi_{12}+\phi_{43}) + (1-m)\cos(\phi_{12}-\phi_{43})\ldots(20).$$

In case (e), we have

$$m\cos(\phi_{13}+\phi_{42}-2\phi_{23})=(m-1)\cos(\phi_{13}-\phi_{42})+\cos(\phi_{13}+\phi_{42})\ldots(21).$$

The angles and numbers on the right side of each of these equations are known. The cosines can be extracted from tables of natural cosines. The number m is usually a simple one so that the computation is not laborious. Hence, the value of $\cos(2\phi_{23}+\phi_{12}+\phi_{43})$, or of $\cos(\phi_{13}+\phi_{42}-2\phi_{23})$, can be found from the tables, and the unknown angle ϕ_{23} is determined.

In case (d), the equation (19) is resolved into a form slightly different from that given in (20) or (21). For,

$$\phi_{12}-\phi_{43}=\phi_{12}-(\phi_{41}-\phi_{12}-\phi_{23})=2\phi_{12}-\phi_{41}+\phi_{23};$$
$$\phi_{12}+\phi_{43}=\phi_{41}-\phi_{23};$$
$$\phi_{13}-\phi_{42}=\phi_{12}-\phi_{43}=2\phi_{12}-\phi_{41}+\phi_{23};$$
$$\phi_{13}+\phi_{42}=\phi_{41}+\phi_{23}.$$

Hence, equation (19) becomes

$$(m-1)\cos(2\phi_{12}-\phi_{41}+\phi_{23})=m\cos(\phi_{41}-\phi_{23})-\cos(\phi_{41}+\phi_{23})\ldots(22).$$

The expression on the right, involving only the known value of the anharmonic ratio and known angles, can therefore be computed. Hence, the cosine on the left is found, and the unknown angle ϕ_{12} can be obtained.

Such cases as (c), (d) and (e), do not often occur in practice. For, in calculating angles from known indices, we generally have several angles in the zone either known or to be computed; and the calculations can be done in a series so that the poles used can in each case be brought under case (a) or (b), when the more convenient logarithmic transformation of the equation can be adopted.

Miller, in page 13 of his *Tract on Crystallography*, 1863, points out that we can frequently employ cases (c) or (d) when one of the known angles is a right angle. Thus, if $\phi_{12}=90°$ in case (c), equation (20) becomes

$$\sin(2\phi_{23}+\phi_{43})=(2m-1)\sin\phi_{43}\ldots\ldots\ldots\ldots(23).$$

If in case (d) ϕ_x becomes $90°$, then equation (22) becomes

$$(m-1)\sin(2\phi_{12}-\phi_{13}) = (m-1)\sin\phi_{13} \quad\text{..............}(24).$$

19. The anharmonic ratio expressed in terms of the sines of the arcs between the poles may be replaced by one involving the angles between the zone-circles drawn through the four tautozonal poles and any pole S, not lying in the zone.

For, from the triangles P_1SP_2, P_1SP_3, of Fig. 79 we have

$$\frac{\sin P_1P_2}{\sin P_2S}=\frac{\sin P_1SP_2}{\sin P_1P_2S}, \quad \frac{\sin P_1P_3}{\sin P_3S}=\frac{\sin P_1SP_3}{\sin P_1P_3S};$$

$$\therefore \frac{\sin P_1P_2}{\sin P_1P_3}=\frac{\sin P_1SP_2}{\sin P_1SP_3}\cdot\frac{\sin P_1P_3S}{\sin P_1P_2S} \quad\text{..............}(25).$$

Similarly, from the triangles P_4SP_2, P_4SP_3, we have

$$\frac{\sin P_4P_2}{\sin P_4P_3}=\frac{\sin P_4SP_2}{\sin P_4SP_3}\cdot\frac{\sin P_4P_3S}{\sin P_4P_2S} \quad\text{..............}(26).$$

But $\sin P_1P_2S=\sin P_4P_2S$ and $\sin P_1P_3S=\sin P_4P_3S$. Hence, dividing (25) by (26),

$$\frac{\sin P_1P_2}{\sin P_1P_3}\div\frac{\sin P_4P_2}{\sin P_4P_3}=\frac{\sin P_1SP_2}{\sin P_1SP_3}\div\frac{\sin P_4SP_2}{\sin P_4SP_3}=\frac{u_{12}}{u_{13}}\div\frac{u_{42}}{u_{43}}=\&c.$$

$$=\frac{hu_{12}+kv_{12}+lw_{12}}{hu_{13}+kv_{13}+lw_{13}}\div\frac{hu_{42}+kv_{42}+lw_{42}}{hu_{43}+kv_{43}+lw_{43}} \quad\text{.....}(27),$$

where (hkl) is the symbol of S (see Art. 9).

We shall use, as an abbreviation, the symbol A.R. $\{S.1234\}$ for the anharmonic ratio given in (27) when expressed in terms of the angles between great circles passing through a pole S on the sphere.

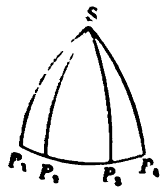

FIG. 79.

Each of the terms on the right side can be re-arranged so as to involve the indices of two zones passing through S and one or other of the pairs of poles P_1, P_4; or P_2, P_3. Thus, the poles having the same symbols as before, let us take $[u_2v_2w_2]$ to be the symbol of the zone $[SP_2]$ and $[u_3v_3w_3]$ that of $[SP_3]$. Then

$$hu_{12}+kv_{12}+lw_{12}=h(k_1l_2-l_1k_2)+k(l_1h_2-h_1l_2)+l(h_1k_2-k_1h_2)$$
$$=h_1(lk_2-kl_2)+k_1(hl_2-lh_2)+l_1(kh_2-hk_2)=h_1u_2+k_1v_2+l_1w_2.$$

Similarly, $\qquad hu_{13}+kv_{13}+lw_{13}=h_1u_3+k_1v_3+l_1w_3;$

the remaining pair of expressions can be transformed in a similar manner.

Hence, A.R. $\{1234\}=$ A.R. $\{S.1234\}$

$$=\frac{h_1u_2+k_1v_2+l_1w_2}{h_1u_3+k_1v_3+l_1w_3}\div\frac{h_4u_2+k_4v_2+l_4w_2}{h_4u_3+k_4v_3+l_4w_3} \quad\text{................}(28).$$

In Fig. 79 the poles have been arranged in numerical order, but this is not necessary; and they may be taken in any order.

Transformation of axes.

20. It is necessary occasionally to transfer the representation of the forms on a crystal from one set of axes and parameters to another. The equations required to give the new symbols of the faces are very simple; and, as was shown by Miller (*Tract on Cryst.* p. 14), they are easily obtained by means of the anharmonic ratio.

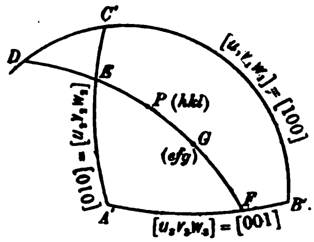

FIG. 80.

In Fig. 80, let the zone-circles $[B'C'] = [u_1v_1w_1]$, $[C'A'] = [u_2v_2w_2]$, $[A'B'] = [u_3v_3w_3]$, in the original representation become [100], [010], and [001], respectively. Let the pole G have the known symbols (efg) and $(e'f'g')$ when referred to the old and new axes. It is required to find the new symbol $(h'k'l')$ of a pole P which originally had the known symbol (hkl).

Draw the zone-circle $[GP]$, and let it meet the new axial zones in D, E, and F. From (28) of the preceding article, we have

$$\text{A.R. } \{PDEG\} = \text{A.R. } \{C'.\,PDEG\} = \frac{\sin PD}{\sin PE} \div \frac{\sin GD}{\sin GE}$$

$$= \frac{hu_1 + kv_1 + lw_1}{hu_2 + kv_2 + lw_2} \div \frac{eu_1 + fv_1 + gw_1}{eu_2 + fv_2 + gw_2} \ldots (29).$$

But the anharmonic ratio, given by the trigonometrical ratios on the left, depends only on the angles between the four poles. Its value cannot, therefore, vary with the arbitrary choice of axes and parameters. Hence, the right side of the equation has a constant definite value independent of the axial constants. But when $[B'C']$, $[C'A']$, become [100], [010]; and G and P become $(e'f'g')$ and $(h'k'l')$, respectively, we have

$$\text{A.R. } \{PDEG\} = \frac{h'}{k'} \div \frac{e'}{f'} = \frac{h'}{e'} \div \frac{k'}{f'} \ldots\ldots\ldots\ldots\ldots (30).$$

Hence,
$$\frac{h'}{e'} \div \frac{k'}{f'} = \frac{hu_1 + kv_1 + lw_1}{eu_1 + fv_1 + gw_1} \div \frac{hu_2 + kv_2 + lw_2}{eu_2 + fv_2 + gw_2}.$$

This equation is satisfied by making

$$\left.\begin{aligned}\frac{h'}{e'}&=\frac{hu_1+kv_1+lw_1}{eu_1+fv_1+gw_1},\\ \frac{k'}{f'}&=\frac{hu_2+kv_2+lw_2}{eu_2+fv_2+gw_2},\\ \text{and, by symmetry,}\qquad \frac{l'}{g'}&=\frac{hu_3+kv_3+lw_3}{eu_3+fv_3+gw_3}\end{aligned}\right\}\qquad\text{(31).}$$

The latter equation can be also deduced from A.R. $\{PEFG\}=$ A.R. $\{A'.PEFG\}$ in the same way as the first two have been obtained.

The above equations (31) give $h'k'l'$ in terms of known numbers; and since, by hypothesis, u_1, v_1, w_1, &c., are zone-indices, and therefore commensurable numbers, the values of $h'k'l'$ are also commensurable.

The equations can, however, be expressed in another form. It will be remembered that u_1, v_1, w_1, &c., are usually expressed in their simplest ratios; and, furthermore, it is clear that any common factor is cancelled in the anharmonic ratio, for the same set of zone-indices, u_1, v_1, w_1, or one of the others, occurs in both the numerator and denominator. We are, therefore, at liberty to assign to the zone-indices any rational values we choose provided their ratios remain unchanged. We can, therefore, give them values which satisfy the following equations:

$$\left.\begin{aligned}e'&=eu_1+fv_1+gw_1,\\ f'&=eu_2+fv_2+gw_2,\\ g'&=eu_3+fv_3+gw_3\end{aligned}\right\}\qquad\text{(32).}$$

Then h', k', l', are given by the equations:

$$\left.\begin{aligned}h'&=hu_1+kv_1+lw_1\\ k'&=hu_2+kv_2+lw_2\\ l'&=hu_3+kv_3+lw_3\end{aligned}\right\}\qquad\text{(33).}$$

In equations (33) the values of u_1, v_1, w_1, &c., which satisfy (32) must be introduced. The two sets, (32) and (33), are the form in which the solution is given by Miller. In many ways the more general equations (31) are preferable, for the numbers actually given in the problem can be at once introduced without fear of error.

Authorities occasionally differ in the axes and parametral plane adopted for crystals of all systems except the cubic. In a tetragonal crystal the principal axis is always OZ, but OX and OY may be different lines. In a prismatic crystal the same lines are always taken as axes, but their arrangement as the axes of X, Y and Z,

may be different. In the oblique system the axis OY is always at right angles to the other pair, but OX and OZ may be varied. In the anorthic system all the axes may be different. In the rhombohedral and hexagonal systems the parametral plane is always perpendicular to the triad, or hexad, axis, whilst the Millerian axes may be varied. An instance of the application of the transformation, in the case of the hexagonal crystals of apatite, is given in Chap. XVII.

Example. To illustrate the method we take a crystal of axinite, belonging to the anorthic system. Miller takes for axial planes p (010), v (100), m (001), and for parametral plane x (111). He then gives the following: l (120), i (131), y (101), r (011). In Dana's *Mineralogy*, l is (100), p $(1\bar{1}0)$, m (001), y (021) and i $(3\bar{1}1)$; using Miller's letters to indicate the faces. It is required to find formulæ of transformation from the one notation to the other.

Dana's	[100]=[my]	is	Miller's	[010]
,,	[010]=[lm]	,,	,,	$[2\bar{1}0]$
,,	[001]=[lp]	,,	,,	[001].

Hence, taking Miller's axes as the original set and Dana's symbols to be those affected with dashes in the equations given above, and remembering that i (131) of Miller becomes Dana's $(3\bar{1}1)$, we have from (31)

$$\left.\begin{aligned} h' &= e'\frac{k}{3} = k \\ k' &= f'\frac{2h-k}{2-3} = 2h-k \\ l' &= g'\frac{l}{1} = l \end{aligned}\right\} \quad \text{.............................} (34).$$

If the two sets of equations (31) and (32) are used, we have

$$\left.\begin{aligned} e' &= fv_1 \\ f' &= eu_2 + fv_2 \\ g' &= gw_3 \end{aligned}\right\} \qquad \left.\begin{aligned} h' &= kv_1 \\ k' &= hu_2 + kv_2 \\ l' &= lw_3 \end{aligned}\right\}.$$

But $(e'f'g')$ is $(3\bar{1}1)$ and (efg) is (131); also $u_2 : v_2 = 2 : -1$, $\therefore u_2 = -2v_2$.

Hence,
$$3 = 3v_1; \quad \therefore v_1 = 1,$$
$$-1 = u_2 + 3v_2 = v_2(3-2) = v_2,$$
$$1 = w_3.$$

Introducing these values into the second set of equations, we have finally the equations already given

$$\left.\begin{aligned} h' &= k \\ k' &= 2h-k \\ l' &= l \end{aligned}\right\}.$$

Hence, Dana's symbols for all the faces given can be immediately deduced.

The formulæ (31), or the equivalent pair (32) and (33), are sufficiently general for all practical problems. It is easy, however,

to see that still more general equations can be obtained in an exactly similar way from the anharmonic ratio by assuming that the zone $[u_1 v_1 w_1]$ becomes $[u'\ v'\ w']$, and similarly for the two other zones. There is, however, nothing to be gained by the more cumbrous formulæ which result from such an assumption. As illustrated by the example, the amount of variation in the selection of axial planes and parameters by different authors is always very slight and the formulæ required are extremely simple.

21. On page 15 of the *Tract*, Miller gives the correlative formulæ for finding the zone-indices corresponding to a change of axes and parameters. The process of deduction from the anharmonic ratio is exactly similar to that given in the preceding article. The equations are

$$\left.\begin{aligned} u' &= eu+fv+gw \\ v' &= hu+kv+lw \\ w' &= mu+nv+pw \end{aligned}\right\} \text{..............................(35)},$$

where (efg) becomes (100), (hkl) becomes (010) and (mnp) (001). Also the *exact* numerical values of the face-indices have to be obtained from

$$\left.\begin{aligned} r' &= er+fs+gt \\ s' &= hr+ks+lt \\ t' &= mr+ns+pt \end{aligned}\right\} \text{..............................(36)},$$

$[r's't']$ and $[rst]$ being known zone-indices in the two representations of the same zone-axis. From the equations (36) the exact values of the face-indices to be used in (35) are obtained.

In this proposition, also, the earlier stage in the solution gives formulæ in which the numerical data can be inserted without the computation involved in (36).

Thus,

$$\left.\begin{aligned} u' &= r'\,\frac{eu+fv+gw}{er+fs+gt} \\ v' &= s'\,\frac{hu+kv+lw}{hr+ks+lt} \\ w' &= t'\,\frac{mu+nv+pw}{mr+ns+pt} \end{aligned}\right\} \text{..............................(37)}.$$

CHAPTER IX.

CONDITIONS FOR PLANES AND AXES OF SYMMETRY, AND RELATIONS BETWEEN THE ELEMENTS OF SYMMETRY.

1. A PARTICULAR and important case of anharmonic ratios is that known as *harmonic ratio*, of which the values are one of the commensurable numbers 2, -1, and $1 \div 2$, according to the order in which the planes, or poles, are taken. Hence, if four planes, having a common line of intersection, form a harmonic ratio and three of them are possible faces, the fourth plane is also a possible face. Harmonic ratios occur very frequently in zones of crystals of all systems. Thus the four faces (100), (101), (001), ($\bar{1}01$) form a harmonic ratio in crystals of every system, quite irrespective of the zone-axes selected as axes of reference and of the values assigned to any two of the angles involved in the ratio. Similarly, (100), (121), (021), ($\bar{1}21$) form a harmonic ratio. The student can easily prove that the right side of equation (10) of Chap. VIII. is $1 \div 2$[1] in each of the above cases. On the left side of the equation the values of two of the angles between three of the faces may be introduced when, by one of the processes given in Chap. VIII. Arts. 14—18, the inclination of the fourth face to any of the others can be calculated.

2. A very special case of harmonic ratios occurs when two of the planes are at right angles to one another, and one of these planes lies exactly midway between the remaining pair. This

[1] In mathematical text-books the value of a harmonic ratio is generally taken to be -1. If the poles are situated as in Fig. 78, this value corresponds to the arrangement—

$$\frac{\sin\phi_{12}}{\sin\phi_{14}} \div \frac{\sin\phi_{32}}{\sin\phi_{34}} = \left[\frac{P_1P_2}{P_1P_4}\right] \div \frac{[P_3P_2]}{[P_3P_4]}.$$

If the reader computes the right side of this equation for the two sets of poles given for illustration, he will find the value to be -1. We shall retain the arrangement recommended in Chap. VIII., which corresponds to the value $1 \div 2$.

relation can also be expressed by stating that one pair of planes bisects the angle between the other pair internally and externally, the bisectors being, consequently, at right angles to one another. Thus we may suppose planes P_1 and P_3 to be at right angles, and P_3 to bisect the angle ϕ_{24}, so that $\phi_{23}=\phi_{34}$. The left side of equation (10) of Chap. VIII. has in this case the value $1 \div 2$, whatever may be the indices of the planes and the value of the angle ϕ_{23}. For, making

$$\phi_{12}=90^\circ-\phi_{23}, \quad \phi_{23}=\phi_{34}, \quad \phi_{42}=2\phi_{23},$$

we have

$$\frac{\sin\phi_{12}}{\sin\phi_{13}} \div \frac{\sin\phi_{42}}{\sin\phi_{43}} = \frac{\cos\phi_{23}\sin\phi_{43}}{\sin\phi_{42}} = \frac{\cos\phi_{23}\sin\phi_{23}}{2\sin\phi_{23}\cos\phi_{23}} = \tfrac{1}{2}.$$

The relation is, therefore, true of any four planes parallel to a given line, when one pair of the planes are the internal and external bisectors of the angles between the other pair. Before it can be applied to a crystal we must know that *at least three* of the planes are possible faces[1]. When three of the planes are faces, possible or actual, then the fourth is also a possible face of the crystal. Thus, if, in Fig. 81, a face R of a crystal bisects the angle between two other faces Q and S—the zone-axis being perpendicular to the paper—the plane P in the same zone and at right angles to R is also a possible face. It bisects the supplement of the angle QS. Also, if P, Q, R are possible faces, PR being 90°, then S making with R an angle equal to RQ is a possible face.

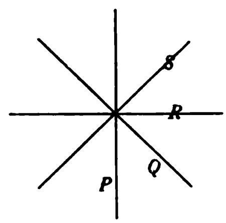

FIG. 81.

On the other hand it was shown, in Chap. VIII. Art. 11, that the planes bisecting the internal and external angles between two faces are not as a rule possible faces. Hence, Q and S being faces, if R is not a face neither is P a possible face.

In Chap. VIII. Art. 7, the anharmonic ratio was stated as a relation between four tautozonal poles, and the same must be true of harmonic ratios. If, then, four points in a zone-circle, no two of which are opposite, form a harmonic ratio and three of them are poles, the fourth is also a possible pole of the crystal. Further, if the arc between a pair of poles is bisected at a possible pole, the point at 90° to this latter pole is also a possible pole of the crystal.

[1] We shall often speak of a plane through any point or line as a face when it is parallel to a possible face of a crystal.

3. Prop. 1. To prove that a plane of symmetry in a crystal is a possible face perpendicular to a possible zone-axis.

(a_1) A face P on a crystal, which is not parallel or perpendicular to a plane of symmetry Σ, is repeated over it in a like face P', so that the line of intersection of P and P' is in, or parallel to, Σ. This line is, therefore, an edge of the crystal. The same is true of every other pair of faces which are reciprocal reflexions in the plane of symmetry; their edges are also in, or parallel to, the plane of symmetry. Hence, this plane is parallel to several edges or zone-axes, and is therefore parallel to a possible face (Chap. v. Art. 11).

(a_2) By the aid of the poles on the sphere we can put the above proof in a different form, which is, however, essentially the same. Let, in Fig. 82, the centre of the sphere be taken at a point in the plane of symmetry, which, therefore, cuts the sphere in a great circle (ANB). Let P be a pole on one side of (ANB). It must be repeated in a like pole P' on the other side, so that the zone-circle [PP'] is necessarily at right angles to (ANB). Also, if M is the point of intersection of the great circles, the arc $PM = MP'$.

Similarly, any other pole Q is repeated over (ANB) in a like pole Q', so that the zone-circle [QQ'] is at right angles to (ANB); and arc $QN = NQ'$, N being the point of intersection of (ANB) and [QQ'].

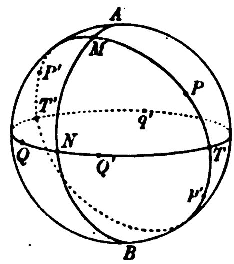

Fig. 82.

The two zone-circles [PP'] and [QQ'], being both perpendicular to (ANB), must meet in a point T which is at 90° to (ANB) and must be a possible pole (Chap. vii. Art. 6).

We have therefore established the first part of the proposition; viz. that a plane of symmetry is parallel to a possible face of the crystal.

(b_1) Again, in Fig. 82, T, P, P' are three tautozonal poles of the crystal, and M is a point in the zone such that $MT = 90°$, and $PM = MP'$. Hence, the four points form a harmonic ratio, and M is, therefore, a possible pole. In the same way the four points T, Q', N, Q form a harmonic ratio and three, T, Q', Q, are possible poles, no two of which are opposite. Therefore, N is a possible pole. Hence (ANB), containing two poles, is a possible zone-circle, and the normal to its plane through T is a possible zone-axis.

(b_2) The same result can be obtained from a consideration of

the crystal-edges independently of the first condition. The edges are repeated over a plane of symmetry in pairs, which are reciprocal reflexions, such that lines drawn through a point in the plane Σ parallel to the edges lie in a plane at right angles to Σ. Hence, the plane, parallel to two homologous edges which are symmetrical with respect to a plane Σ, is parallel to a possible face and is perpendicular to Σ. The same is necessarily true of every other pair of homologous edges. Hence, we must have a number of possible faces all perpendicular to the plane of symmetry; and their lines of mutual intersection must be all perpendicular to it. The plane of symmetry is therefore perpendicular to a possible zone-axis.

4. Prop. 2. To prove that an axis of symmetry is a possible zone-axis.

Let the line $O\Delta$ in Fig. 83 be a dyad axis. Through any point O in it draw a line OB parallel to any edge of the crystal which is neither parallel nor perpendicular to $O\Delta$, and suppose the plane containing $O\Delta$ and OB to be that of the paper. Then, by the definition, a semi-revolution about $O\Delta$ must bring OB to the position $OB_{,}$, which is parallel to an edge of the crystal similar to the first one. But $OB_{,}$ is necessarily in the same plane with $O\Delta$ and OB, for a second rotation of 180° in the same direction will bring $OB_{,}$ back to OB; and this is, clearly, not possible, if $OB_{,}$ stand out of the paper to the front or the back. Hence, OB and $OB_{,}$ being zone-axes, their plane is parallel to a possible face.

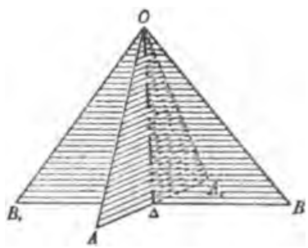

Fig. 83.

Similarly, any other zone-axis OA, which is neither parallel nor perpendicular to $O\Delta$ and which is not in the plane $OB\Delta$, is repeated in another zone-axis $OA_{,}$; and the plane $AO\Delta A_{,}$ also contains $O\Delta$. As the plane $AO\Delta A_{,}$ contains two zone-axes, it is parallel to a possible face.

Hence, $O\Delta$ is parallel to two possible faces, and is, therefore, a possible zone-axis.

Cor. 1. In the case of a tetrad axis, we can turn the crystal about it through 90° four times in succession until the crystal returns to the original position. Hence, the edge $LM_{,}$ of Fig. 84 takes up three new positions, which are obtained by turning

the crystal about the tetrad axis through 90°, through 2 × 90° = 180°, and through 3 × 90° = 270°. One of these positions, when the crystal has been turned through 180°, is clearly LM'. But the face parallel to the edges LM, LM', is also parallel to the tetrad axis OL. Similarly, the plane OLM is parallel to a possible face. Hence, a tetrad axis is a possible zone-axis.

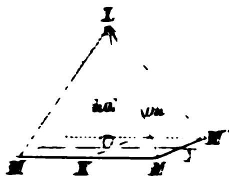

Fig. 84.

COR. 2. The same is also true in the case of a hexad axis: for if the crystal is turned thrice in the same direction, each time through 60°, a position is attained which is the same as that given by a single rotation of 180° about the same axis. Thus the edges $VA_{,}$, VA', Fig. 85, are coplanar with the hexad axis OV. Similarly, the planes OAV, $OA_{,,}V$ contain two edges and are parallel to possible faces. Hence, a hexad axis is, also, a possible zone-axis.

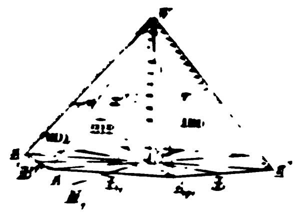

Fig. 85.

The same argument would apply to any axis of symmetry of even degree, but we shall see in Prop. 6 that no axis can have a degree higher than six.

COR. 3. Suppose an axis of symmetry of odd degree, n, to exist in a crystal, n being > 3. Let the continuous lines VA, VA_1, &c. of Fig. 86 represent the polar edges of a pyramid formed by the homologous and interchangeable faces of such a crystal when it is projected on a plane perpendicular to the axis of symmetry OV.

Produce the lines AA^1, A^1A^2, &c., in which the faces meet the plane of projection; and let alternate pairs

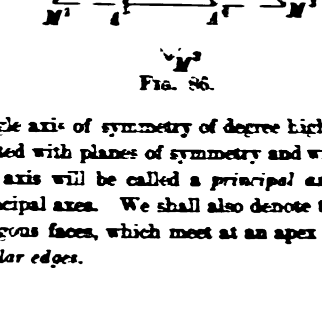

Fig. 86.

[1] Several classes of crystals have a single axis of symmetry of degree higher than 2, which may, or may not, be associated with planes of symmetry and with dyad axes perpendicular to it. Such an axis will be called a *principal axis*. Thus, in Figs. 84, 85, OL and OV are principal axes. We shall also denote the edges of a pyramid, formed by the homologous faces, which meet at an apex on the principal axis, such as V of Fig. 85, *polar edges*.

of these lines meet in the points M, M^1, &c. The lines VM, VM^1, &c., clearly bisect the angles A^3VA^4, A^4VA^5, &c. But VM, VM^1, &c., are the orthogonal projections on the paper of the possible edges in which alternate faces of the pyramid would meet, if produced. Hence, the plane containing the opposite possible edges VA and VM also contains OV, for VA and VM are co-linear. Similarly, the planes containing other opposite pairs VA^1 and VM^1, &c., also contain OV. If then axes of odd degree, higher than three, were possible, they would be possible zone-axes.

The above proofs are not applicable to the case of a triad axis.

5. PROP. 3. To prove that a plane perpendicular to an axis of symmetry is a possible face.

Let, in Fig. 87, $O\Delta$ be a dyad axis, and let a face EFG of the crystal, not parallel or perpendicular to the axis, meet it at Δ. There must be a second face EFH, which is interchangeable with EFG when the crystal is turned about $O\Delta$ through 180°. Let EF be the line of intersection of the two planes. When the crystal has been turned through 180° about $O\Delta$, the edge EF retains its direction, for the two faces have changed places: the only difference is that points E and F equally distant from Δ have changed places. Hence the edge EF is perpendicular to $O\Delta$.

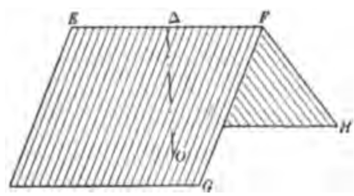

FIG. 87.

Similarly, any other face whatever, not parallel or perpendicular to $O\Delta$, and not in a zone with EFG and EFH, must be associated with a second like face, so that the two faces meet in an edge perpendicular to $O\Delta$.

Hence $O\Delta$ is perpendicular to two edges which are not parallel. It is, therefore, perpendicular to the possible face which is parallel to these edges.

COR. 1. In a manner similar to that employed in Cors. 1 and 2 of Art. 4, it can be shown that tetrad and hexad axes are perpendicular to possible faces. For two successive rotations in the same direction, each of 90°, about a tetrad axis interchange a pair of faces intersecting in a line perpendicular to the axis; and similarly, three successive rotations, each of 60°, about a hexad axis interchange a similar pair of faces which intersect in an edge

perpendicular to the axis. Hence a number of non-parallel edges can be perpendicular to tetrad and hexad axes. These axes must therefore be perpendicular to possible faces.

COR. 2. On the supposition of an axis of symmetry of odd degree, $n > 3$, it has been seen in Prop. 2, Cor. 3, that VM, VM^1, &c., would be possible edges. Then VMM^1, &c., would be possible faces belonging to a second pyramid. But, from Fig. 86, it is clear that the pair of opposite possible faces VAA^1, VMM^1 intersect one another in a straight line parallel to AA^1 and MM^1 and perpendicular to OV. The same is true of all the opposite pairs of possible faces. Hence, OV is perpendicular to a number of possible edges, and is therefore perpendicular to a possible face.

6. It is not possible to establish, as a consequence of the law of rational indices, that a triad axis is, when alone, a possible zone-axis, or that it is perpendicular to a possible face.

Let, in Fig. 88, VO be a triad axis, and VM a zone-axis of the crystal which is not parallel or perpendicular to OV. Let MO be drawn perpendicular to the triad axis and meet it at O. Rotation of the crystal about VO through 120° must bring VM to $VM_{,}$ and OM to $OM_{,}$, so that the triangle $VOM_{,}$ is not coplanar with VOM. A further rotation of 60° is required to bring $OM_{,}$ into the continuation $O\mu$ of OM. But this is not a possible angle of rotation about a triad axis, nor is there an edge $V\mu$ associated with VM. A second rotation of 120° in the same direction as the first brings the triangle into the position $VOM_{,,}$ so that $OM_{,,}$ lies as far beyond $O\mu$ as $OM_{,}$ was short of it. A third rotation of 120° clearly brings the line back to VM.

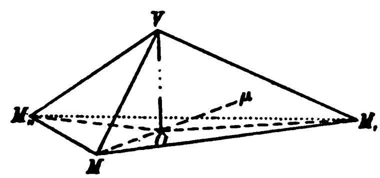

FIG. 88.

The proof of Prop. 2 depends, in the case of an axis of symmetry of even degree, on the fact that two homologous zone-axes are coplanar with the axis; and in the case of an axis of odd degree, $n > 3$, the proof depends on the fact that two non-homologous edges, such as VA and VM, are coplanar with the axis. Here the homologous zone-axes form the edges of a trigonal pyramid, and no pair of them lies in a plane containing the triad axis. Nor can an auxiliary pyramid be formed by the edges of alternate faces.

Again, the proof that a possible face is perpendicular to an axis

of symmetry depends on the fact that pairs of possible faces intersect in lines perpendicular to the axis. The homologous faces, such as $MVM_{,}$, $M_{,}VM_{,,}$ in Fig. 88, intersect in $VM_{,}$ which is not at 90° to the triad axis, for $OM_{,}$ is at 90° to VO. Hence, we cannot prove that the triad axis is perpendicular to a possible face.

But, although it is not possible to prove that a triad axis, when it is alone or associated only with a centre of symmetry, is a zone-axis perpendicular to a possible face, we can prove that it is so when it is associated with a plane, or with axes, of symmetry. It is also found to be an actual zone-axis, and perpendicular to faces, in all the known crystals in which triad axes exist. Hence, we shall *assume* that, where a triad axis exists, it is a possible zone-axis and perpendicular to a possible face.

Hence, generally, an axis of symmetry is (1) parallel to a possible zone-axis, and (2) perpendicular to a possible face.

7. Prop. 4. If a crystal has any two of the three elements of symmetry—a plane, a centre, and an axis of even degree perpendicular to the plane—then it must also have the third element.

(*a*) Let, in Fig. 89, $ABCD$ be a plane of symmetry, Σ; and let O, a point in Σ, be the centre of symmetry. Let OP be the normal on any face ABE, and OP_1 the normal on the corresponding homologous face ABF on the opposite side of the plane Σ; and let $O\Delta$ be the normal on the plane $ABCD$.

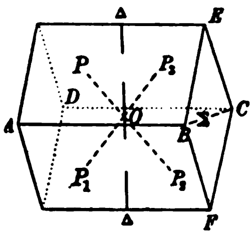

Fig. 89.

The plane containing OP and OP_1 is perpendicular to Σ, and must contain $O\Delta$. Now, since the crystal has a centre of symmetry there must be faces at both ends of a normal continued through the centre. Hence, OP_2 and OP_3, the continuations of OP and OP_1, are perpendicular to faces FCD and ECD which are parallel to ABE and ABF respectively (Euclid XI. 14). The four faces form a four-sided prism, like the roof of a long building and its reflection in a horizontal plane passing through the eaves.

It is clear that rotation through 180° about the normal $O\Delta$ interchanges OP and OP_3, OP_1 and OP_2, and the corresponding faces. For $O\Delta$ is the external bisector of the angle POP_1, and, consequently, bisects the angles POP_3 and P_1OP_2.

The same relations must hold for each set of four faces

symmetrical with respect to $ABCD$ and the centre O. They must be interchangeable in pairs about the normal $O\Delta$ on $ABCD$. Hence rotation through 180° about the line $O\Delta$ interchanges homologous faces. But $O\Delta$, being the normal on a plane of symmetry, is a zone-axis perpendicular to a possible face (Prop. 1). Hence, $O\Delta$ is an axis of symmetry of at least twofold rotation. The possibility of rotations through 60°, or 90°, is not excluded. For three rotations in the same direction, each of 60°, about $O\Delta$ bring OP to the position OP_3; and similarly, two successive rotations, each of 90°, are equivalent to a single rotation of 180° about the same axis. The axis may, therefore, be one of dyad, tetrad, or hexad symmetry; *i.e.* it is an axis of symmetry of even degree.

(*b*) When a crystal has a centre of symmetry O, and an axis of symmetry of even degree $O\Delta$, it can, by the aid of the same diagram, be shown that the plane $ABCD$, perpendicular to $O\Delta$, is a plane of symmetry. For rotations through 180° about $O\Delta$ are always possible and interchange pairs of normals OP and OP_3, where all three lines are co-planar and $O\Delta$ bisects the angle POP_3. Owing to the centre of symmetry each normal is diplohedral, and the faces at opposite ends of a normal are necessarily parallel. But parallel planes intersect a third in parallel lines (Euclid XI. 16). Hence the edges AB and CD are each parallel to the edge $E\Delta$, in which the faces ABE and DCE meet. The edges are therefore all perpendicular to the plane containing $O\Delta$, OP, OP_3, and therefore to $O\Delta$. The plane, containing the edge AB and the line through O in the plane POP_3 which is perpendicular to $O\Delta$, bisects the angle POP_1, and therefore the angle between the faces of which OP and OP_1 are the normals.

A similar relation can be found for every other set of two pairs of parallel faces, which are also interchangeable, in pairs, by rotation through 180° about $O\Delta$. Hence, the plane $ABCD$ is a plane of symmetry; for it bisects the angles between homologous pairs of faces, and is parallel to a possible face and perpendicular to a zone-axis.

(*c*) If a plane of symmetry, $ABCD$, and an axis of symmetry of even degree, $O\Delta$, perpendicular to $ABCD$ coexist in a crystal, then parallel faces must be invariably present, *i.e.* the crystal is centro-symmetrical. The proof is obtained from the diagram, already used, by taking the relations of the lines and faces in a different order. OP and OP_1 are symmetrical with respect to $ABCD$ which

bisects the angle between them and is perpendicular to their plane. Hence, the plane P_1OP contains $O\Delta$. Again, $O\Delta$ bisects the angle POP_2, and OP_2 lies in the plane containing OP and $O\Delta$. Hence, OP_1, OP, OP_2 all lie in one plane, and $POP_1 + POP_2 = 180°$. Therefore OP_1 and OP_2 are opposite directions on the same straight line. The faces ABF and DCE at the opposite ends P_1 and P_2 must therefore be parallel; and similarly, ABE and DCF are parallel faces. The same is true of every other set of faces which are homologous with respect to Σ and to $O\Delta$. The crystal is therefore centro-symmetrical.

Hence, the presence of *any two* of the three elements of symmetry—a centre, a plane and an axis of *even* degree perpendicular to the plane of symmetry—always involves the presence of the third element.

8. Cor. The proof given above also establishes a very important relation of the triad axis; viz. that *in a centro-symmetrical crystal having an axis of triad symmetry, the plane perpendicular to the axis cannot be a plane of symmetry.* For the axis perpendicular to a plane of symmetry in a centro-symmetrical crystal must be an axis of symmetry of *even* degree. There is nothing to exclude the possibility of a triad axis and a plane of symmetry perpendicular to it occurring together in a crystal; but in such a case the crystal cannot be centro-symmetrical, and parallel faces, if they occur, are to be regarded as belonging to different forms and will often be distinguished by a difference of character; *i.e.* such parallel faces are not homologous.

9. We shall now consider the limitations to the number of planes of symmetry possible in a zone. We shall, thereby, see that the arrangements possible are such that the least angles between planes of symmetry are 90°, 60°, 45° and 30°.

Prop. 5. To prove that the only angles, less than 60°, possible between planes of symmetry are 45° and 30°.

Let, in Fig. 90, P and Q be the poles of planes of symmetry such that no pair of planes of symmetry in their zone make with one another a less angle than $PQ = \phi$ (say). Now, P must be repeated on the other side of Q in a similar pole P_1 where $P_1Q = QP$; and similarly, Q must be repeated in a similar pole Q_2. Suppose the planes of symmetry

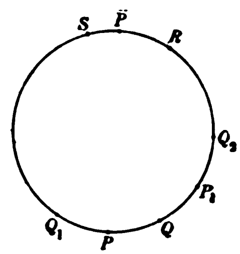

Fig. 90.

to be perpendicular to the paper, so that the poles P, Q, P_1, &c., all lie in the primitive, where $PQ = QP_1 = P_1Q_2 = \&c. = \phi$. But, since P and Q are the poles of planes of symmetry, so must also the poles P_1, Q_2, &c., be poles of planes of symmetry. Let the successive poles be inserted in the zone until R, the last pole before $\bar{P}$, is reached, where $\bar{P}$ is the opposite pole to P. Now the next step gives either $\bar{P}$, or a pole S beyond it. But $\bar{P}$ is also the pole of a plane of symmetry—that first taken—and $R\bar{P}$, $S\bar{P}$, are both less than $RS = \phi$. This contravenes the limitation on the poles P and Q first taken; viz. that no planes of symmetry in the zone make with one another a less angle than ϕ. Hence, in proceeding beyond R the next pole must be $\bar{P}$, and $R\bar{P} = \phi$. The angle ϕ is therefore an exact submultiple of 180°. Call the number n, then $n\phi = 180°$.

If, now, n is greater than 3, there must be at least three other poles between P and $\bar{P}$, and we can apply the anharmonic ratio of four tautozonal poles to investigate the possible values of n and ϕ. But, if ϕ is 60° or 90°, the anharmonic ratio is inapplicable, since pairs of poles are at 180° to one another. Supposing $\phi < 60°$, we have

$\dfrac{\sin PP_1}{\sin PQ} \div \dfrac{\sin Q_2P_1}{\sin Q_2Q} = m$, where m is some positive rational number.

$$\text{Then,}\qquad m = \frac{\sin PP_1 \sin Q_2Q}{\sin PQ \sin Q_2P_1} = \frac{\sin^2 2\phi}{\sin^2 \phi} = \frac{4\sin^2\phi\cos^2\phi}{\sin^2\phi}$$
$$= 4\cos^2\phi = 2(1+\cos 2\phi) \ldots\ldots (1).$$

Let $2\phi = \theta$, $\therefore n\theta = 360°$; and $\cos\theta = \cos 2\phi = (m \div 2) - 1 \ldots (2)$.

The problem involves, therefore, the determination of the angles less than 120°, which have rational cosines and are exact submultiples of 360°. These angles are 90° and 60°, as is proved in Art. 10. In these cases, $\cos\theta = \cos 90° = 0$, and $\cos\theta = \cos 60° = 1 \div 2$. Hence, when $\theta = 90°$, $\phi = 45°$ and $n = 4$; and when $\theta = 60°$, $\phi = 30°$ and $n = 6$. The value of the anharmonic ratio is, in the former case, $m = 2$; and in the latter, $m = 3$.

10. *Proof.* Series (3) (Todhunter's *Trig.*, p. 226, 1859), and the properties of equations having integral coefficients, enable us to prove that the only angles $< 120°$, which are exact submultiples of 360° and have rational cosines, are 90° and 60°.

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \&c. \ldots (3).$$

The rth term is

$$(-1)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!}(2\cos\theta)^{n-2r},$$

the coefficient of $(2\cos\theta)^{n-2r}$ being integral.

But, when $n\theta = 360°$, $\cos n\theta = 1$; and writing x for $2\cos\theta$, we have the equation

$$x^n - nx^{n-2} + \frac{n(n-3)}{2!}x^{n-4} - \&c. = 2 \dots\dots\dots\dots (4).$$

Further, the commensurable roots of an equation of the nth degree, which has integral coefficients and unity for that of the highest term, are integral and exactly divide the constant term. It follows that the possible values of x are limited to ± 2, ± 1, and 0.

If $2\cos\theta = \pm 2$, θ is 0° or 180°. The former value is absurd, and the latter gives $\phi = 90°$. This is a possible angle between planes of symmetry, but it is not admissible as a solution of equation (1), for it gives a meaningless anharmonic ratio.

If $2\cos\theta = \pm 1$, θ is 60° or 120°, and ϕ is 30° or 60°. The former establishes that the least angle possible between planes of symmetry is 30°, and that we cannot have more than six such planes in a zone. The latter value (60°) is a possible angle between planes of symmetry, but it is not admissible as a solution of (1); for, if four successive poles are taken at 60° to one another, the extreme pair are at 180°, and the anharmonic ratio is meaningless.

If $2\cos\theta = 0$, $\theta = 90°$, $\phi = 45°$ and $n = 4$. This is the only other solution possible, and is that given when n is even and the last term of series (3) is 2.

It follows that the greatest number of planes of symmetry in a zone is six in one case and four in the other[1]. Crystals may, also, have only two, or three, tautozonal planes of symmetry.

[1] The proof of this important proposition given in the text is due to Professor Story-Maskelyne, who gave it in a course of lectures attended by the author in 1869. Professor von Lang (*Kryst.* p. 75, 1866) gives an expression, equivalent to (1), limiting the angles of *isogonal zones*, i.e. of zones in which equal angles recur. He points out that 30°, 45°, 60° and 90°, satisfy the equation; but he does not prove them to be the only angles which do so. In a classical memoir, published in *Acta Soc. Sci. Fennicæ*, IX. p. 1, 1871, but read on 19 March, 1867, Professor Axel Gadolin finds that the angles of rotation about axes of symmetry are subject to equation (2); and he proves, by the method used in Art. 10, that 60° and 90° are the least possible angles. His method of arriving at equation (2) is, however, not altogether satisfactory; for it is not applicable to the case of a tetrad axis. A different method has, therefore, been adopted in Art. 11.

COR. 1. No crystal can have five tautozonal planes of symmetry; for the least angle between them is 36°, and this angle does not satisfy equation (2).

COR. 2. Planes of symmetry inclined to one another at angles of 45°, and also at 30°, cannot coexist in one and the same zone. It is clear that a zone-circle cannot be divided by poles at distances of 30° to one another and also at distances of 45° without some of the poles making angles of less than 30° with one another.

11. PROP. 6. To prove that tetrad and hexad axes are the *only* axes of symmetry of degree higher than three which are possible in crystals.

Suppose A_n to be an axis of symmetry of degree n (> 3) occurring alone in a crystal, and let it be placed vertically. By Props. 2 and 3, A_n is a possible zone-axis perpendicular to a possible face. If then a face meets the axis at a point V, we have a pyramid of n similar faces all passing through the apex V and meeting the horizontal base in possible edges.

We proceed to show that n possible faces can be found parallel to A_n and inclined to one another in succession at an angle $\phi = 180° \div n$.

When n is even, $n \div 2$ faces through opposite polar edges are possible, each of which contains the axis of symmetry. Thus, in

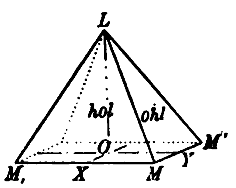

FIG. 91.

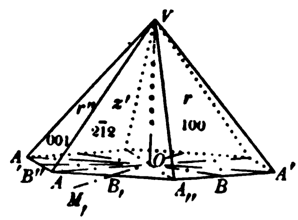

FIG. 92.

Fig. 91, the planes $M_{,}LM'$, MLO, both containing the tetrad axis OL, are possible faces inclined to one another at an angle of 90°. Similarly, in Fig. 92, the planes $A_{,}VA'$, AVO, $A_{,,}VO$, containing the hexad axis of symmetry OV, are possible faces inclined to one another at angles of 60°. Generally, therefore, we have $n \div 2$ possible tautozonal faces inclined to one another in successive pairs at an angle $\theta = 360° \div n$.

If now the alternate faces of the pyramid are extended to meet one another, we obtain in Fig. 91 horizontal edges through L, parallel to OX and OY, which bisect the angles $M_{,}OM$, MOM'. Through these edges and the axis OL possible faces can be drawn; and we have four faces through the axis OL inclined to one another in succession at angles of 45°. In the general case, when n is greater than 4, we obtain new polar edges, such as $VM_{,}$ of Fig. 92, through which and the axis $n \div 2$ possible faces can be drawn bisecting the angles between adjacent faces of the first set. Thus, for instance, the face $OVM_{,}$ bisects the angle between OVA and $OVA_{,,}$. We thus have, altogether, n tautozonal faces through A_n inclined to one another in succession at an angle $\phi = 180° \div n$.

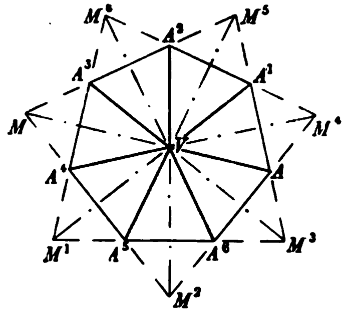

FIG. 93.

The same can be shown to be the case, when n is odd and > 3. Thus, in Fig. 93, the face through the axis and the possible edge VM bisects the angle between the possible faces through VA^3 and VA^4 parallel to the axis.

Fig. 90 may now be taken to represent the poles of the n tautozonal faces through the axis A_n, inclined to one another in succession at angles ϕ, where $n\phi = 180°$. Forming the anharmonic ratio, A.R. $\{PQP_1Q_2\}$, of four consecutive poles, we obtain equations (1) and (2). Hence, as proved in Art. 10, the possible values of $\theta = 2\phi$ are restricted to 60° and 90°; and the axis can only be a hexad or a tetrad axis. Similarly, 72° not being a possible value of θ, a pentad axis is inadmissible.

The only axes of symmetry possible in crystals are therefore those of two-, three-, four-, and six-, fold symmetry.

12. *Isogonal zones.* Zones in which one of the crystallometric angles, 60°, 45° or 30°, recur are common in cubic crystals and in crystals having a principal axis. We can, however, show that, even when the faces are not planes of symmetry, no pair of faces can include an angle of 30°, or 60°, in a zone in which 45° recurs in succession; and, vice versâ, that 45° cannot occur in a zone in which either 30°, or 60°, recurs in succession.

Thus, in Fig. 94, let $PQ = QP_{,} = P_{,}Q_{,} = \&c. = \phi$, where ϕ is either 45° or 30°. Take any possible pole R in the zone, and let $\wedge PR = a$. Also, let S be a point such that $RS = \phi_{,}$.

Then the A.R. $\{PRQP_{,}\}$ and the A.R. $\{PSQP_{,}\}$ give the equations

$$\frac{\sin PR}{\sin PQ} \div \frac{\sin P_{,}R}{\sin P_{,}Q} = \frac{\sin a}{\sin (2\phi - a)} = m \text{ (a rational number)} \ldots\ldots\ldots (5),$$

$$\frac{\sin PS}{\sin PQ} \div \frac{\sin P_{,}S}{\sin P_{,}Q} = \frac{\sin (a + \phi_{,})}{\sin (2\phi - \overline{a + \phi_{,}})} = n \text{ (say)} \ldots\ldots\ldots\ldots (6).$$

(*a*) If ϕ is 45°, then (5) gives

$$\frac{\sin a}{\sin (90° - a)} = \tan a = m \ldots\ldots\ldots\ldots\ldots (7);$$

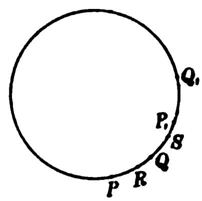

FIG. 94.

and (6) gives

$$\tan (a + \phi_{,}) = n = \frac{\tan a + \tan \phi_{,}}{1 - \tan a \tan \phi_{,}}$$

$$= \text{(by substitution from (7))} \frac{m + \tan \phi_{,}}{1 - m \tan \phi_{,}} \ldots (8).$$

If $\phi_{,}$ is 45°, then $\tan \phi_{,} = 1$, and n is rational. Hence, in a zone having 45° recurring, we can have a pole S at 45° to any other known pole R.

If, however, $\phi_{,} = 30°$, then $\tan \phi_{,} = 1 \div \sqrt{3}$; and from (8)

$$n = \frac{m\sqrt{3} + 1}{\sqrt{3} - m} = \frac{(m\sqrt{3} + 1)(\sqrt{3} + m)}{3 - m^2} = \frac{4m + (m^2 + 1)\sqrt{3}}{3 - m^2} \ldots\ldots\ldots (9).$$

Hence, n contains a term involving the surd $\sqrt{3}$, and is irrational.

Therefore S at 30° to R is not a possible pole; and it is not possible to find two poles inclined at 30° to one another in a zone in which 45° recurs in succession.

(*b*) Equations (5) and (6) were established before any value was assigned to ϕ. Let us now take $\phi = 30°$,

$$\therefore \frac{1}{m} = \frac{\sin (60° - a)}{\sin a} = \frac{\sin 60° \cos a - \sin a \cos 60°}{\sin a} = \frac{\sqrt{3} \cot a - 1}{2}.$$

$$\therefore \cot a = \frac{2}{\sqrt{3}} \left(\frac{1}{m} + \frac{1}{2} \right) = \frac{m_{,}}{\sqrt{3}} \text{ (say)} \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where $m_{,} = 2 \left(\frac{1}{m} + \frac{1}{2} \right)$, a rational number.

Similarly, from (6), we have

$$\cot (a + \phi_{,}) = \frac{2}{\sqrt{3}} \left(\frac{1}{n} + \frac{1}{2} \right) = \frac{n_{,}}{\sqrt{3}} \text{ (say)} \ldots\ldots\ldots\ldots\ldots (11),$$

where $n_{,} = 2 \left(\frac{1}{n} + \frac{1}{2} \right)$.

By expanding the left side of (11) and substituting for $\cot a$ from (10), we have

$$\frac{n_{,}}{\sqrt{3}} = \frac{\cot a \cot \phi_{,} - 1}{\cot a + \cot \phi_{,}} = \frac{m_{,} \cot \phi_{,} - \sqrt{3}}{m_{,} + \sqrt{3} \cot \phi_{,}} \ldots\ldots\ldots\ldots\ldots (12).$$

If $\phi_{,}=30^\circ$, $\therefore \cot\phi_{,}=\sqrt{3}$;

and
$$\frac{n_{,}}{\sqrt{3}}=\frac{m_{,}-1}{m_{,}+3}\sqrt{3}\,;\ \therefore\ n_{,}=3\,\frac{m_{,}-1}{m_{,}+3}\ \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

Hence, $n_{,}$ and n are rational, and a pole S is possible at 30° to any other pole R in a zone in which 30° recurs in succession.

If, however, $\phi_{,}=45^\circ$, $\therefore \cot\phi_{,}=1$; and
$$\frac{n_{,}}{\sqrt{3}}=\frac{m_{,}-\sqrt{3}}{m_{,}+\sqrt{3}}=\frac{(m_{,}-\sqrt{3})^2}{m_{,}^2-3}\ \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Hence, $n_{,}$ is irrational, and therefore n. There cannot, therefore, be a pole S at 45° to a pole R in a zone in which 30° recurs in succession.

By making $\phi_{,}$ negative in the above equations, we can obtain similar relations which hold for the case in which S is on the same side of R as P.

Similarly, it can be proved that 45° and 60° do not occur together in a zone, in which one or other of the angles is repeated in succession.

13. Prop. 7. *Euler's Theorem.* Given two axes of rotation, OA and OB, intersecting at O within a body; it is required to prove that successive rotations about them of 2α and 2β respectively are equivalent to a single rotation through an angle 2γ about a third axis OC, and to find the position of OC and the angle 2γ.

Let a sphere of any radius be described about the point of intersection O as centre: and let A and B, Fig. 95, be the two points at which the axes emerge when the rotation about each of them appears to the observer to be in the same direction, either both with, or both against, the hands of a clock. Let AB be the arc in which a great circle through A and B meets the sphere.

Through B draw the great circles BC, BA' and $BC_{,}$, where $\wedge ABC=\wedge CBA'=\wedge A'BC_{,}=\beta$, one-half the angle of rotation about the axis OB. On $A'B$ cut off an arc $A'B=AB$; and through A' draw the great circles $A'C$ and $A'C_{,}$ on opposite sides of $A'B$, and each making with $A'B$ an angle α, one-half the angle of rotation about A. Then OC is a line, rotation about which through the angle $ACA'\,(=2\gamma)$ is equivalent to successive rotations, counter-clockwise, of 2β about OB and 2α about OA.

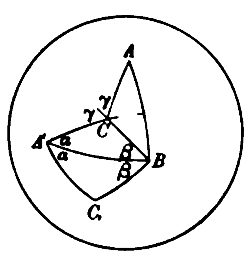

Fig. 95.

The three spherical triangles ABC, CBA', $A'BC_{,}$ have their sides and angles equal, each to each. For, in ABC and $A'BC$, the two sides AB, BC are equal to the two sides $A'B$, BC; and the included angle ABC to the angle $A'BC$. Hence, $A'C=AC$; and

$\wedge A'CB = \wedge ACB$, and $\wedge CAB = \wedge CA'B$. Again, in the triangles $A'BC$, $A'BC_{,}$ two angles of the one are equal to two angles of the other, and the adjacent side $A'B$ is common. Hence, the other angles and sides are equal; viz. $A'C = A'C_{,}$ and $BC = BC_{,}$.

If the body and sphere (supposed to be rigidly connected together) are turned, counter-clockwise, about OB through 2β, A is brought to A' and C to $C_{,}$. If, afterwards, the body and sphere are turned through 2α, again counter-clockwise, about OA' (the new position of the axis OA), the point $C_{,}$, where C had been left by the first rotation, is brought back to C. After both rotations the radius OC retains its original position; and the body is in the same position as it would be in after a single rotation, counter-clockwise, about OC through the angle ACA'; for A' is the final position of A.

If, however, the rotations are taken clockwise in the same order, a new point $C_{,,}$, situated on the other side of AB, is the extremity of the equivalent axis of rotation; where $C_{,,}$ is the vertex of a triangle equal and similar to CAB, having AB for base and the angles at A and B equal to α and β respectively.

If the body is first turned about OA and then about OB, the same two axes OC and $OC_{,,}$ are obtained. The former is the equivalent axis, when the rotations are clockwise; the latter, when they are counter-clockwise.

Hence, A and B being the points at which the axes meet the sphere when the rotations about them appear to be in the same direction, the positions of the equivalent axes of rotation are given by C and $C_{,,}$, the vertices of the two triangles which have AB for base and the angles at A and B equal respectively to α and β, one-half the angles of rotation about the axes. The angle of rotation 2γ is, in each case, double the external angle of the triangle at the vertex.

The proposition has only been proved for pure rotations, but it holds true for axes of screw rotation.

14. It is now clear that, if there are two axes of symmetry in a crystal inclined to one another at an angle other than 180°—the latter angle giving only two ends of one and the same axis—there must be at least one other axis of symmetry, of which the position and the angle of rotation are determined by the above theorem. For suppose the axes of symmetry to be parallel to OA and OB of the preceding proposition; and let rotations of 2α and 2β about OA and OB, respectively, interchange, in each case, homologous faces

and edges. Then rotation about OB through 2β brings OA to OA', and interchanges homologous faces. The crystal has, therefore, the same aspect as at first. Rotation in the same direction about OA' (the new position of OA) through 2α also interchanges homologous faces, and leaves the aspect of the crystal the same as at first. But the final position of the crystal can be obtained by a rotation of 2γ in the same direction about OC, which is, therefore, a third axis of symmetry. It is necessary to prove that the least angle of rotation about OC is, in each case, an exact submultiple of 360°; and that OC satisfies the conditions for axes of symmetry given in Props. 2, 3 and 6.

The same is true of the other equivalent axis $OC_{\prime\prime}$, which gives a fourth axis of symmetry when this line is not (as is often the case in crystals) in the continuation through the centre of the axis OC. The new axis, or axes—when C and $C_{\prime\prime}$ are not extremities of a diameter—may, when combined in a similar manner with each of the original axes, give the positions of new axes of symmetry. In order to obtain all the axes of symmetry, the process has to be repeated until every combination is shown to give an axis already established. It has thus been shown that the greatest number of axes of symmetry, which occur together in any crystal, is thirteen.

15. PROP. 8. If two planes of symmetry are present in a crystal, and are inclined to one another at the least angle, ϕ, between planes of symmetry in their zone, then their line of intersection is an axis of symmetry, the least angle of rotation about which is 2ϕ, and the degree of which is $n = 360° \div 2\phi$.

Let the two planes of symmetry meet the sphere in the great circles Σ and S, Fig. 96, and let their line of intersection be perpendicular to the primitive. Then over S the plane Σ must be repeated in a like plane of symmetry Σ_1. Let the points in which the adjacent planes S, Σ, &c., meet the primitive be A, B, A_1, &c.

Let P, P_1, P_2, P_3, &c., be the projections of homologous poles symmetrical with respect to Σ, S, Σ_1, &c.; and let the great circles through C and each of these poles meet the primitive in D, D_1, D_2, D_3, &c.

Since Σ and S are planes of symmetry, they are parallel to possible faces and perpendicular to possible zone-axes. Hence the plane of the primitive is parallel to a possible face, and the radius through C is a possible zone-axis—the line of intersection of Σ and S. It, therefore, satisfies the conditions (Arts. 4—6) for an axis of symmetry. It is now necessary to show that rotation about it through a definite submultiple of 360° interchanges like faces.

Now the arc joining a pair of homologous poles, such as P_1 and P_2, is perpendicular to, and bisected by, the plane of symmetry S with respect to which they are symmetrical. Hence, we have $P_1L = LP_2$, and CL is common to the two spherical triangles CLP_1, CLP_2. The angle CLP_1 is also equal to CLP_2, each being a right angle. The two triangles are therefore equal in all respects, and $CP_1 = CP_2$, and $\wedge\ P_1CL = \wedge\ P_2CL$. But these angles are the same as the arcs BD_1, BD_2 measured on the primitive. Hence $BD_1 = BD_2$. Again, for a similar reason, $CP = CP_1$, and the angle $PCA = \wedge\ P_1CA$, or the arc

$$DA = \text{arc } AD_1.$$

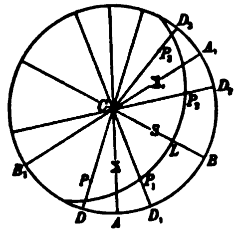

Fig. 96.

But, $AB = BA_1 = \&c. = \phi = 180° \div n$.

Hence,

$$DD_2 = DD_1 + D_1D_2 = 2AD_1 + 2D_1B = 2AB = AA_1 = BB_1 = 2\phi.$$

Also, $$D_1D_3 = D_1D_2 + D_2D_3 = 2(BD_2 + D_2A_1) = 2BA_1 = 2\phi.$$

Hence, if the sphere is turned through 2ϕ about the diameter through C, the pairs of similar points A, A_1; D, D_2; B_1, B; &c., interchange places simultaneously, and the zone-circles CA, CA_1; CP, CP_2; CP_1, CP_3; change places respectively. But $CP = CP_1 = CP_2 = \&c.$ Hence, the similar planes of symmetry and the homologous poles are interchanged at the same time. The line of intersection of the two planes S and Σ is, therefore, an axis of symmetry, the least angle of rotation about which is 2ϕ; and the degree is $n = 360° \div 2\phi$.

Cor. i. If the least angle between pairs of tautozonal planes of symmetry is 90°, the line of intersection is a dyad axis, and the distribution of homologous faces is such as is shown in Fig. 19 and in Fig. 97. The two like planes Σ and Σ_1 of the general proof fall into the same plane, and the two planes S and Σ are not like or interchangeable. In Fig. 97, one plane is XOZ, the other is YOZ.

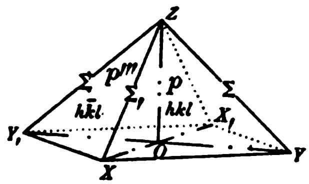

Fig. 97.

Cor. ii. If the least angle between the tautozonal planes of symmetry is 60°, the axis is one of triad symmetry. There are three planes of symmetry in the zone which are like and interchangeable planes. The distribution of homologous faces with respect to the planes and axis of symmetry is shown in Fig. 98.

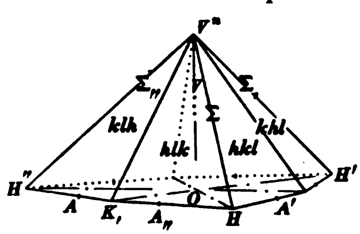

Fig. 98.

COR. iii. If the least angle between S and Σ is 45°, there are four tautozonal planes of symmetry consisting of a pair of like S planes and a pair of like Σ planes. The like planes are interchangeable with one another, but those of one set are not interchangeable with those of the other. The line of intersection is a tetrad axis. The distribution of homologous faces is illustrated by the faces meeting at the apex L in Fig. 99. The planes of symmetry S and S' are LOM and $LOM_{,}$, the planes Σ are LOH and LOK', and the axis OL is the tetrad axis.

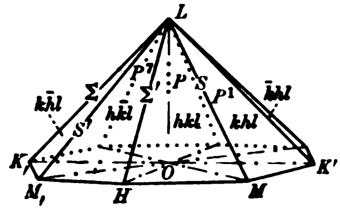

FIG. 99.

COR. iv. When the angle $\phi = 30°$, there are three like planes S at 60° to one another and three like planes Σ also at 60° to one another, so arranged that the planes of one set bisect the angles between pairs of the other set. The S planes are interchangeable with one another, as are likewise the Σ planes, but those of one triad are not interchangeable with those of the other. The line of intersection is an axis of hexad symmetry. The distribution of homologous poles and faces is given in Figs. 100 and 101. As will be explained in Chap. XVII, two sets of symbols, $\{hkl\}$ and $\{pqr\}$, are needed to give all the homologous faces. In Fig. 100, the planes of symmetry Σ are those passing through the centre and the points X, Y and Z, respectively, on opposite

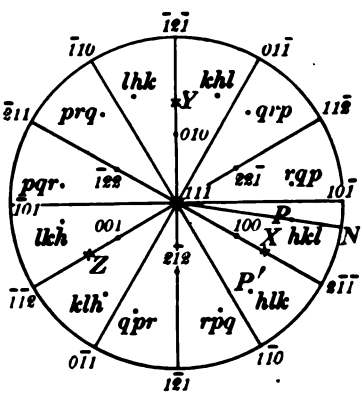

FIG. 100.

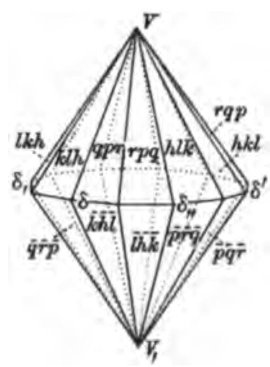

FIG. 101.

sides of which poles of $\{hkl\}$ are symmetrically placed; as also are the poles of $\{pqr\}$. The other triad of planes S bisect the angles between planes of the first triad, and in Fig. 101 pass through the polar edges $V\delta$, $V\delta_{,}$, $V\delta_{,,}$. They have a pole and face of the set $\{hkl\}$ symmetrically placed to one of $\{pqr\}$.

COR. v. Since 30° is the least angle possible between planes of symmetry, it follows that no crystal can have an axis of symmetry, in which planes of symmetry intersect, of higher degree than six. Moreover, since 36° is not a possible angle between planes of symmetry (Art. 10), a pentad axis cannot be the line of intersection of planes of symmetry.

16. PROP. 9. Given an axis of symmetry of degree n, not exceeding 6, and one plane of symmetry parallel to it, to show that there are $(n - 1)$ other planes of symmetry all parallel to the axis.

By rotation about the axis the given plane Σ is brought into $(n-1)$ other positions which must be those of like planes of symmetry and the angle between which is $360° \div n = 2\phi$. But, when n is even, we thus get only $n \div 2$ different planes of symmetry; for, if $m = n \div 2$, the result of m successive rotations of 2ϕ is to turn the plane of symmetry through 180°, when it again coincides with the original plane. But the last proposition has shown that the degree n of an axis of symmetry, which is the intersection of planes of symmetry, is given by the equation, $n = 360° \div 2\phi$, where 2ϕ is the least angle of rotation about the axis and is double that between adjacent planes of symmetry. Hence, since $n\phi = 180°$, there must be n distinct planes of symmetry all intersecting in, or parallel to, the axis of symmetry.

17. PROP. 10. If a dyad axis δ is perpendicular to an axis of symmetry A_n of degree n, there must be $(n - 1)$ other axes of even degree all perpendicular to A_n.

If the axis is a triad axis, rotation about it brings δ into two new and different positions, so that there are three like dyad axes at right angles to A_3.

In the case of an axis A_n of even degree, where $n = 2m$, each rotation about A_n through $360° \div n$ brings δ into the position of a like dyad axis. But the mth rotation gives no new direction, for the whole rotation is $m \times 360° \div 2m = 180°$, and the axis is in the prolongation across the origin of the original axis. Hence, rotation about A_n, gives only m different dyad axes. But, by Euler's theorem, two rotations—one about A_n through $360° \div n$ followed by one of 180° about δ—are equivalent to a single rotation about Δ, where Δ is the vertex of the triangle shown in Fig. 102.

The angle at Δ is clearly a right angle, and the axis Δ is at least a dyad axis perpendicular to A_n. The same is true of a combination of A_n with each of the δ axes. There are, therefore, m like dyad axes Δ, each of which bisects the angle between pairs of δ axes. The axes Δ, Δ, may, when $n=4$, be tetrad axes.

18. Prop. 11. If two axes of symmetry of even degree parallel to a certain face are inclined to one another at the least angle, $\phi = 180° \div n$, between axes parallel to the face; then there are n axes of even degree parallel to the same face inclined to one another in succession at the angle ϕ, and an axis of symmetry of degree n perpendicular to the face to which the n axes are parallel.

Let δ and Δ be two dyad axes inclined to one another at the least angle, $\phi = 180° \div n$, between axes parallel to their plane. Describe a sphere, and take two diameters δδ, ΔΔ, Fig. 102, parallel to the axes, and let them lie in the plane of the primitive and emerge at δ and Δ. Rotation through 180° about the dyad axis Δ brings the axis δ into a position in which it is parallel to the diameter δ,δ,, so that we need only consider rotations about the diameters of the sphere. Similarly, a semi-revolution about δ,δ, gives us the diameter Δ,Δ, parallel to an axis of symmetry. Repeating the process, a diameter R is reached such that at the next repetition the original diameter, but in a reversed direction, or a diameter T beyond it is attained. But if rotation about R gives T, then the angle between R and T is ϕ. But the original diameter lies between them and is also parallel to a dyad axis.

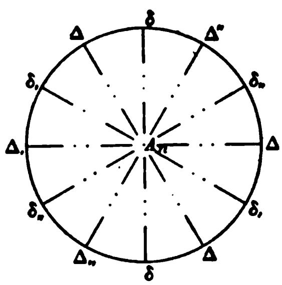

Fig. 102.

The axes R and T are, therefore, inclined to the original axis at an angle less than ϕ. But this contravenes our first assumption, viz. that the pair inclined at the *least* angle was selected. Hence, the nth repetition must give the original diameter reversed in direction. The number of distinct dyad axes parallel to one plane is, therefore, $n = 180° \div \phi = 360° \div 2\phi$.

Again by Euler's theorem, Art. 13, successive rotations about adjacent dyad axes δ and Δ are together equivalent to a single rotation about the line perpendicular to their plane; for the angles $A_n\delta\Delta$, $A_n\Delta\delta$ are both 90°. Furthermore, the angle of rotation about the axis emerging at A_n is double the supplement of the angle $\delta A_n \Delta$. It is, therefore, $2(180° - \phi) = 360° - 2\phi$. But such a rotation about the diameter through A_n brings the sphere and crystal into the same position as a rotation of 2ϕ in the opposite direction. Hence, the least angle of rotation about the axis parallel to the diameter through A_n is 2ϕ, and the degree n is given by $360° \div 2\phi$.

The same proof applies when one of the original axes is a tetrad axis; for two rotations, each of 90°, in the same direction about it are equivalent to a single rotation of 180°. For the purposes of the proposition we need only consider the latter rotation. Hence, the proposition is true when the adjacent pair consist of a dyad and tetrad axis. We shall see later on that it is needless to consider other arrangements; for no crystal can have more than one hexad axis, or have two tetrad axes without a dyad axis midway between them.

19. Prop. 12. To find the possible arrangements of like axes of symmetry inclined to one another at finite angles.

Suppose A_1 and A_2 to be like axes of degree n inclined to one another at the least angle ψ possible between such axes. About any point in A_2 as centre describe a sphere, and let A_2 meet it in a_2, Fig. 103. Through the centre O draw a line parallel to A_1 meeting the sphere in a_1. Then the arc $a_1a_2=\psi$. Rotation about A_2 through $2\pi \div n$ brings A_1 to the position of a like axis A_3 and the radius Oa_1 to a radius Oa_3 parallel to A_3. Hence, $a_3a_2=a_1a_2=\psi$, and the angle $a_1a_2a_3=2\pi\div n$.

A similar rotation about A_3 brings A_2 into a position the direction of which is given by the radius to a_4 on the sphere, where $a_4a_3=a_3a_2$ and the angle $a_4a_3a_2=2\pi\div n$. Proceeding in this manner the point a_1 is ultimately reached again and a closed polygon of p sides is obtained.

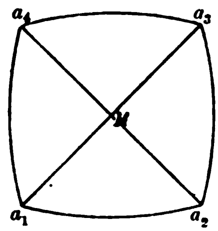

Fig. 103.

The polygon encloses no point at which a similar axis to A_1 or A_2 emerges and has no reentrant angles; for its sides are the least possible between such axes, and on each rotation the interior angle, $2\pi\div n$, is less than 180°, except when $n=2$. In the latter case, the angle $a_1a_2a_3=180°$, and the extremities of all the axes lie in a great circle.

Now the area of the spherical polygon is that of the p triangles formed by joining each of its angular points to a point within it, such as M the middle point. But the area of a spherical triangle ABC is

$$\frac{\pi r^2}{180°}(A+B+C-180°)\text{ ....(15).}$$

Hence, the area of the polygon of p sides is

$$\frac{\pi r^2}{180°}(a_1a_2a_3+a_2a_3a_4+\&c.+360°-p.180°)$$
$$=\frac{\pi r^2}{180°}\left(p.\frac{360°}{n}+360°-p.180°\right)=\frac{4\pi r^2}{4n}\{2n-p(n-2)\}$$
$$=\frac{S}{4n}\{2n-p(n-2)\}\text{ ....(16),}$$

where S is the surface of the sphere.

The area of the polygon is necessarily positive, and, consequently, the values of p are limited by the expression

$$2n - p(n-2) > 0 \qquad \text{(17)}.$$

20. If the vertices a_1, a_2, &c., of the polygon are joined by great circles to the middle point, M, then $a_1M = a_2M = \text{\&c.} = \theta$ (say); and we can readily find θ and ψ. Each of the triangles a_1Ma_2, given separately in Fig. 104, is an isosceles triangle, of which the angles at a_1 and a_2 are $180° \div n$, one-half the angle of rotation about each of the axes, and the angle at M is $360° \div p$. By Euler's theorem, OM is an axis of symmetry, an angle of rotation of $360°(1 - 2 \div p)$ about which is equivalent to successive rotations of $2\pi \div n$ about A_1 and A_2. If, in Fig. 104, a great circle MD is drawn through M at right angles to a_1a_2, it bisects the angle a_1Ma_2 and the arc a_1a_2 at the point D. Hence, by Napier's rules,

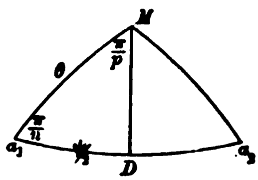

FIG. 104.

$$\cos\theta = \cot\frac{\pi}{n}\cot\frac{\pi}{p} \qquad \text{(18)},$$

$$\cos\frac{\psi}{2} = \cos\frac{\pi}{p}\,\text{cosec}\,\frac{\pi}{n} \qquad \text{(19)}.$$

21. We shall now consider the values of p given by expression (17) for the several possible values of n.

i. *Hexad axis.* If $n=6$, expression (17) becomes $12 - 4p > 0$. Hence, the possible values of p are 1 and 2. The value $p=1$ gives no repetition; the polygon has only one side and is a circle. It is obviously inadmissible as a solution of the problem, for we supposed two axes A_1 and A_2 to occur.

When $p=2$, the polygon is a lune, *i.e.* the segment of a sphere bounded by two great circles. The sides are semicircles, and the apices are the opposite ends of one and the same diameter.

Hence, no crystal can have more than one hexad axis. The value $p=2$, indicates that the distribution of faces about the two ends is similar, but the faces may, or may not, be interchangeable. By (16) the area of the lune is $S \div 6$.

ii. *Tetrad axes.* When $n=4$, expression (17) becomes $8 - 2p > 0$: and $p = 1$, 2, or 3 are possible values.

(*a*) The first gives, as before, no repetition.

(*b*) The value 2 gives, as before, a lune but with area $= S \div 4$. We can, therefore, have a single tetrad axis with a like distribution of faces at the two ends, which may, or may not, be interchangeable.

(*c*) The value $p=3$ gives for the polygon an equilateral triangle the area of which is $S \div 8$. From equation (19) we see that $\psi = 90°$. Hence,

we have three like and interchangeable tetrad axes mutually at right angles to one another.

Also rotation about A_2 in the opposite direction to that first adopted clearly gives a similar quadrantal triangle. Hence, the new tetrad axis is the opposite extremity of that through a_3. The middle point M of the spherical triangle is also the extremity of a triad axis; and, by Euler's theorem, the point D of Fig. 104 is the extremity of a dyad axis. The whole arrangement of axes of symmetry, characteristic of two classes of the cubic system, is shown in the stereogram, Fig. 105. The three tetrad axes are indicated by letters T, the four triad axes by letters ρ, and the six dyad axes by δ.

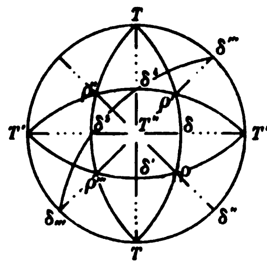

FIG. 105.

iii. *Triad axes.* When $n=3$, expression (17) becomes $6-p>0$, and p may have the values 1, 2, 3, 4 or 5.

(*a*) When $p=1$, there is, as before, no repetition.

(*b*) When $p=2$, the polygon is a lune with area $S\div 3$. A single triad axis is therefore possible with a similar distribution of faces at both ends, which may, or may not, be interchangeable.

(*c*) When $p=3$ the polygon is an equilateral triangle, the area of which is $S\div 4$. The sphere is divided into four equal triangles, the apices of which are at the coigns of the inscribed regular tetrahedron, Fig. 106. The triad axes, joining the coigns ρ in Fig. 106 (coincident with the points a_1, &c. of Fig. 103) to the centre of the tetrahedron, will, when continued, pass through the middle points r of the opposite faces. Each of these continuations corresponds to an axis through the point M of Fig. 103. The radii in opposite directions along the same diameters are, therefore, to be regarded as dissimilar triad axes; and the distribution of faces and edges at the opposite ends will be dissimilar. This character of an axis of symmetry is also described by saying that it is *uniterminal.*

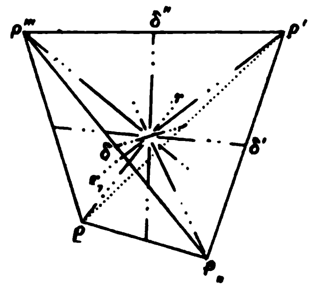

FIG. 106.

The angles between pairs of like triad axes $O\rho'$ and $O\rho_{,,}$ and between adjacent dissimilar axes Or and $O\rho'$, for instance, are obtained from equations (18) and (19), for $\pi\div n=\pi\div p=60°$. Hence $\cos(\theta=\rho'r)=1\div 3$. Therefore $\rho'Or=\rho_{,,}Or=70° 32'$. But, since opposite ends of an axis lie in a diameter, $\rho'O\rho_{,,}=180°-\rho_{,,}Or_{,}=109° 28'$. This is the same result as is obtained from (19) for $\cos\psi/2=\cos 60° \operatorname{cosec} 60°=1\div\sqrt{3}$. Therefore

$\psi/2=54^\circ\,44'$, and $a_1a_2=\rho'O\rho_{,,}=109^\circ\,28'$. The arrangement of axes is characteristic of two merohedral classes of the cubic system.

(*d*) When $p=4$, the polygon is a four-sided one having an area of $S\div 6$. Each of its sides is $70^\circ\,32'$; for from equation (19)

$$\cos\psi/2=\cos 45^\circ \operatorname{cosec} 60^\circ=\sqrt{2}\div\sqrt{3}. \quad \therefore \ \psi\div 2=35^\circ\,16'.$$

The figure inscribed in the sphere is the cube, the diagonals of which give the directions of the four triad axes ρ. We have here to distinguish between two possible cases according as A_1 and A_2 the axes first taken are (α) like and interchangeable, *i.e.* metastrophic, or (β) like but not interchangeable, *i.e.* antistrophic.

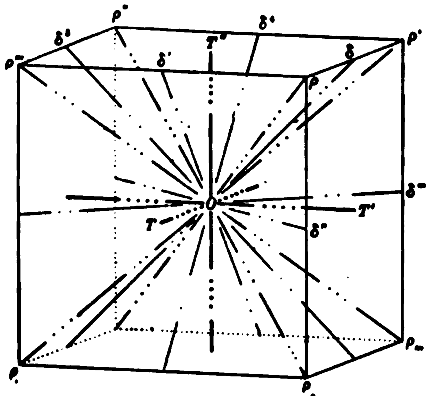

FIG. 107.

α. In this case the complete assemblage of axes is identical with that resulting from (ii *c*), and is shown in Fig. 107. The radii OT, &c., through the middle points M of the four-sided polygons are respectively perpendicular to the faces of the inscribed cube, and are three like and interchangeable tetrad axes, rotation about any one of which through 90° interchanges adjacent angles of the polygon, *i.e.* adjacent coigns of the cube. Furthermore, the radius to the point D of Fig. 104 is a dyad axis, and is parallel to the diagonal of one of the faces of the cube. For, by Euler's theorem, successive rotations of 120° about the triad axis A_2 and of 90° about the tetrad axis emerging at M are equivalent to a single rotation of 180° about the axis emerging at D, the vertex of the triangle a_2MD. The dyad axes are the lines $O\delta$, $O\delta'$, &c., of Fig. 107.

β. From the method of derivation adopted in Art. 19, the pairs of axes A_1 and A_3, A_2 and A_4, are necessarily interchangeable; but the pairs A_1 and A_2, A_2 and A_3, &c., though like axes, may be antistrophic. Again, Euler's theorem requires that M should be at least a dyad axis, for the angles a_1Ma_2, a_2Ma_3, &c., are each 90°. This does not militate with its being a tetrad axis as in case (α), when A_1 and A_2 are metastrophic. Hence, when

A_1 and A_2 are antistrophic, the three lines through the middle points of opposite cubic faces parallel to the edges are only dyad axes; and it can be proved that there are no other axes of symmetry. This arrangement of axes is characteristic of a subdivision of the cubic system of which crystals of pyrites give a good instance.

(*e*) When $p=5$, we obtain a five-sided polygon the area of which is $S \div 12$. The polyhedron inscribed in the sphere is the regular pentagonal dodecahedron, which is inadmissible amongst crystal-forms. For we have seen in Art. 20 that M, the middle point of the polygon, is the extremity of an axis of symmetry. Successive rotations about A_1 and A_2 are equivalent to a single rotation about the radius OM through $2(180° - 72°)$, since, in this particular case, the angle $a_1 M a_2 = 72°$. The polyhedron always presents the same aspect after any number of rotations of this amount. But, two rotations give an angle of $2 \times 360° - 4 \times 72° = 360° + 72°$. The polyhedron is, then, in exactly the same position as if it had been turned once in the same direction through $72° = 360° \div 5$, for complete revolutions of 360° cause no change in the position of the faces. Hence, the least angle of rotation about the axis through M is $360° \div 5$, and the axis is a pentad axis. But this has been shown in Art. 11 to be inadmissible. No class of crystals can, therefore, show the arrangement of axes of this paragraph.

iv. *Dyad axes.* When $n=2$, the expression (17) is independent of p, and the solution is indefinite. The axes all lie in one plane and the area of the polygon is a hemisphere. The possible arrangements of such dyad axes are governed by the relations given in Prop. 11.

It is important to note that, in the case of tetrad axes, a crystal may have one or three, but no other number; similarly, that a crystal may have one, or four, triad axes, but no other number; also that the angles between all the axes of symmetry associated together are fixed and definite.

22. So far the elements of symmetry have been discussed as if crystals were merely polyhedra the faces of which are subject to the law of rational indices. In recent years much attention has been paid to theories as to the internal molecular structure of crystals. It is interesting, therefore, to find that the uniformity of internal structure, which such theories presuppose, imposes the same restrictions on the elements of symmetry as those established in the preceding Articles. Considerations of this nature have also justified the acceptance of a triad axis as an actual zone-axis to which a possible face is perpendicular. We shall, however, only enter into the question to the extent needed to establish two propositions, viz. Props. 13 and 14.

The assumption of the uniformity of internal structure of a crystal is based on the results of observation and experiment, which

have established that a crystal has the same physical properties: (1) at all points on the same straight line, (2) for all parallel directions, and (3) for all homologous directions. Some of the physical characters have a higher symmetry than that manifested by the geometrical relations of the crystal-forms:—as, for instance, the propagation of light. Thus, in a cubic crystal light is propagated with equal velocity in all directions. The cohesion is, however, different in directions which are not homologous; as is indicated by the cleavages being limited to a few directions which are, in many cases, parallel only to homologous faces. Further, within the degree of precision attainable in estimating the perfection of a cleavage and the ease with which it is obtained, it is found that the cleavages in homologous directions are equally perfect and facile, but that, when cleavages occur parallel to faces of different forms, those in non-homologous directions generally show marked differences in their characters. The elasticity is another property which is only the same for directions which are parallel or homologous. The properties of cohesion and elasticity are, however, identical in opposite directions along the same line, though the facial development may be one which excludes symmetry with respect to a centre. Hence, such properties do not enable us to distinguish between related classes which differ, inasmuch as in the one class certain elements of symmetry occur alone, whilst in the other they occur in association with a centre of symmetry.

The physical characteristics of crystals described in the preceding paragraph indicate that the internal structure is the same at all points within a crystal. Hence, the arrangement of the particles about any one of them must be the same as that about any other. This is only partially true of the particles at, or very near to, the surface, but the sphere of action between neighbouring particles is so small that only a very thin layer is affected. The surface-relations, known as surface-tensions, must exert a most important influence on the growth of a crystal, and must, more especially, be the determining cause in the development of the faces. M. P. Curie and Professor Liveing[1] have, independently, discussed the subject, and have endeavoured to estimate the relative values of the surface-

[1] "Sur la formation des cristaux et sur les constantes capillaires de leurs différentes faces." *Bull. Soc. franç. de Min.* VIII, p. 145, 1885.

"On Solution and Crystallization." *Trans. Camb. Phil. Soc.* XIV, p. 370, p. 394, 1889; XV, p. 119, 1894.

tensions of the faces of different forms in the same crystal and thus to account for the predominance of certain forms.

It is, however, unnecessary for our purpose to enter into the questions which such considerations raise, and we need only consider the consequences involved in the regularity of internal structure. We shall suppose the size of the crystal to be indefinitely great in comparison with the distance between adjacent particles, or with the sphere of action of the particles on one another.

Prop. 13. To prove that no crystalline structure, consisting of particles arranged in a regular manner at small but finite distances apart, can have an axis of symmetry of higher degree than six.

The arrangement of the particles about any one of them being the same as that about any other, it follows that, if there is an axis of symmetry A—whether it is an axis of pure, or of screw, rotation—related to one set of particles, then every other similar set of particles must have a similar and parallel axis A similarly related to it. Further, the distance between the nearest parallel axes A cannot be made indefinitely small in comparison with the distance between adjacent particles. Hence, let us assume a pair of similar and parallel axes of symmetry, A_1 and A_2, of degree n to meet a plane perpendicular to them at the points a_1 and a_2, Fig. 108, respectively; and let a_1a_2 be the least distance possible between any pair of such axes. Now, rotation about A_2 through the angle $2\pi \div n$ brings the axis A_1 to the position of a similar parallel axis A_3 which meets the plane at a_3, where $\wedge a_1a_2a_3 = 2\pi \div n$, and $a_1a_2 = a_3a_2$. Draw the perpendicular a_2d on a_1a_3. Then, from the right-angled triangle a_1da_2, we have $a_1d = a_1a_2 \sin \pi \div n$. Hence, $a_1a_3 = 2a_1a_2 \sin \pi \div n$. But, by the selection of A_1 and A_2, a_1a_3 cannot be less than a_1a_2. When a_1a_3 is made equal to a_1a_2, $\sin \pi \div n = 1 \div 2$; and $180° \div n = 30°$. Hence, 6 is the greatest value which can be assigned to n.

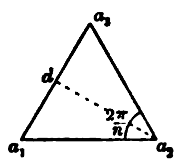

Fig. 108.

23. From the same considerations, we can show that the assumption of a pentad axis is inconsistent with the existence of a finite minimum distance between like axes of symmetry.

Prop. 14. To prove that a pentad axis is inadmissible amongst the axes of symmetry possible in a crystalline structure.

Let A_1, A_2 be two parallel pentad axes having the least distance a_1a_2 between such axes. Let them meet the paper (placed at right angles to the axes) in a_1, a_2, Fig. 109.

If, now, the crystal is rotated about A_2 through $360° \div 5$, the axis A_1 is transferred to A_3, the position of a like pentad axis, and the point a_1 to a_3. Hence $a_2a_3 = a_2a_1$, and the angle $a_1a_2a_3 = 72°$. A similar rotation about A_3 brings the axis A_2 to the position of a like axis A_4 which meets the paper in a_4, where $a_3a_4 = a_3a_2$, and the angle $a_4a_3a_2 = 72°$. It is clear that, if a_2a_1 and a_3a_4 are produced, they will meet, in V (say), for the two angles at a_2 and a_3 are each 72°. The triangle Va_2a_3 is isosceles, and the portions a_1a_2 and a_3a_4 of the equal sides are equal. Hence a_1a_4 is parallel to a_2a_3; and, by Euclid VI. 2, $a_4a_1 : a_3a_2 = Va_1 : Va_2$. Hence, we have two pentad axes A_1 and A_4, the distance between which is less than the minimum distance a_1a_2. Even if a_1a_4 were selected as the initial pair, we can by the same process find pentad axes still nearer to one another, and this can be continued without limit. It is impossible, therefore, to have a number of like parallel pentad axes separated from one another by a finite minimum distance.

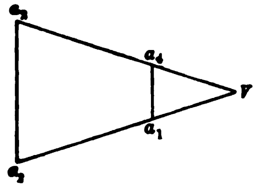

FIG. 109.

CHAPTER X.

THE SYSTEMS; AND SOME OF THE PHYSICAL CHARACTERS ASSOCIATED WITH THEM.

1. THE principles laid down, and the relations established, in the preceding Chapters enable us to classify crystals; and to show that only thirty-two classes are possible, which fall into seven larger groups called *systems*. The names adopted for the systems and classes by various authors differ considerably. We shall generally adopt Miller's names for the systems and shall, likewise, give some of the synonyms employed by other crystallographers. The systems, and the classes included under each system, will be fully developed in the following Chapters. The systems may, however, be briefly defined as follows:—

1. The *anorthic* (*triclinic*) system consists of two classes:— I. Crystals with no symmetry; II. Crystals which have only a centre of symmetry.

2. The *oblique* (*monoclinic*, *monosymmetric*) system includes three classes:—I. Crystals with a single dyad axis; II. Crystals with a single plane of symmetry; III. Crystals in which a plane of symmetry, a dyad axis perpendicular to the plane of symmetry, and a centre of symmetry are associated together.

3. The *prismatic* (*rhombic*, *orthorhombic*, *trimetric*) system includes three classes of crystals, each of which has one set of three dissimilar zone-axes at right angles to one another, which are the most convenient lines to take for axes of reference. In class I the axes are dyad axes, and no other element of symmetry is present. In class II the three axes are, as before, dyad axes, but they are associated with a centre of symmetry and with three planes of symmetry, each perpendicular to one of the dyad axes. In class III one of the axes is a dyad axis, and is the line of intersection of two planes of symmetry at right angles to one another;

the two other axes of reference being the zone-axes normal to the planes of symmetry.

4. The *tetragonal* (*quadratic*, *pyramidal*, *dimetric*) system includes seven classes, and comprises all crystals having each a principal axis (p. 112) which is either a tetrad axis, or a dyad axis of special character. The special character of the dyad axis is due to the fact that like zone-axes at right-angles to it occur in pairs which are at right angles to one another; but they are not interchangeable, as is the case when the principal axis is a tetrad axis.

5. The *cubic* (*octahedral*, *regular*, *isometric*) system includes five classes. All cubic crystals have four triad axes, the directions of which are given by the diagonals of a cube: they have also three like and interchangeable rectangular axes, parallel to the edges of the cube, which are either dyad or tetrad axes.

6. The *rhombohedral* system includes seven classes, the crystals of which are all distinguished by having each a single triad axis, which is a principal axis.

7. The *hexagonal* system includes five classes, the crystals of which have each a single hexad axis, which is also a principal axis.

Miller doubted the correctness of regarding this last system as a separate one; and, as will be shown in Chaps. XVI and XVII, the distribution of faces about the hexad axis is, in particular cases, the same as that of similar forms of the rhombohedral system. In other cases a single form of the hexagonal system can be represented as consisting of two correlative forms of the rhombohedral system, which can be interchanged by a rotation of 180° about the principal axis, and which are therefore connected together by a simple relation between the indices. On the other hand, foreign crystallographers have, until quite recent years, regarded rhombohedral crystals as forming important merohedral classes of the hexagonal system.

The systems were first established by Weiss and Mohs from empirical observations of the development of the forms of crystals and approximate measurement of their angles. They seem to have arrived at the same main subdivisions independently, although Weiss's[1] classification was the first published. The subdivisions of the systems were only partly determined by them; and the question, as to whether oblique and anorthic crystals constituted

[1] De indagando formarum crystallinarum charactere geometrico principali dissertatio. Leipzig, 1809. Über die natürlichen Abtheilungen der Krystallisationssysteme. *Abh. d. Berlin. Akad.*, 1814—1815, p. 289.

independent systems, or were merely hemihedral and tetartohedral subdivisions of the prismatic system, remained a subject of controversy for a long period. Naumann (*Lehrb. d. Kryst.* II, p. 51, 1830) was the first to adopt oblique axes in the representation of oblique and anorthic crystals, which he justified by strong reasons based on the observed differences in the forms commonly found on crystals of these systems and the prismatic.

In the Chapters in which the systems are severally discussed, we shall see that it is not always possible to discriminate the class, or even the system, to which a crystal belongs by the geometry of the facial development. The physical, and more especially the optical, characters of the crystals afford useful tests which generally enable us to discriminate between crystals of different systems, and sometimes between those of different classes in the same system; and it is mainly on the optical characters of their crystals that we now rely in assigning to definite systems several minerals, such as, for instance, the humite group, harmotome, &c. We shall, therefore, give a brief account of the more important of those characters, which are used as tests in discriminating the system or class of a crystal.

Optical characters.

2. We shall assume that the student is familiar with the ordinary facts concerning double refraction and polarised light, an account of which is to be found in text-books[1] on Light. Shortly after the publication by Weiss of his classification of crystals by systems, the labours of Brewster (*Phil. Trans.* CVIII, p. 199, 1818) established that—with certain exceptions, in which anomalous phenomena are observed, which are even now the subject of much discussion—crystals can be divided optically into three groups, and that these three groups are closely related to the systems of Mohs and Weiss. The first group comprises crystals which are isotropic, *i.e.* give only single refraction: these crystals all belong to the cubic system. The second group consists of crystals having one optic axis; and comprises the tetragonal, rhombohedral and hexagonal systems, the crystals of which have a crystallographic principal axis which is coincident in direction with the optic axis: the crystals are

[1] *Physical Optics*, by R. T. Glazebrook. *The Nature of Light*, by E. Lommel. *Light*, by L. Wright. *The Theory of Light*, by T. Preston. *The Optical Indicatrix*, by L. Fletcher.

said to be *uniaxal*. The third group consists of crystals which have two optic axes[1]; and comprises the prismatic, oblique and anorthic systems: the crystals are said to be *biaxal*.

3. In the second group the distinction between the crystal-forms of tetragonal crystals on the one hand, and those of hexagonal and rhombohedral crystals on the other hand, is marked enough. The optical characters do not enable us to distinguish crystals of one of the systems from those of another; and the presence of a single optic axis serves only to establish the fact that the crystals have each a principal axis which is coincident in direction with the optic axis. For discriminating between the hexagonal and rhombohedral systems we have to depend:—(*a*) on the development of the crystal-forms; (*b*) on corrosion-experiments on the faces; (*c*) on the hardness, experiments on which, however, are unreliable; and (*d*) on the general relations of cohesion.

4. So far as Brewster's observations went, no distinction was shown to exist in the optical characters of the three systems of biaxal crystals. In 1821 Fresnel established that the transmission of monochromatic light through biaxal crystals accords with a centro-symmetrical *wave-surface* of two sheets, which is symmetrical with respect to three rectangular planes—*the principal planes*—and, therefore, to the lines of intersection of these planes. These lines are axes of two-fold symmetry of the wave-surface, and are called its *principal*[2] *axes*.

[1] An *optic axis* is a direction along which monochromatic light is propagated with only a single wave-velocity. A beam of such light of appreciable transverse section traversing a crystal in the direction of an optic axis gives only a single beam on emergence, *i.e.* undergoes single refraction whatever may be the inclination of the beam to the face at which the light emerges. This definition does not hold accurately for crystals like quartz; but an optic axis may in all cases be distinguished by the following character. If a beam of plane-polarised light of any colour falls on the crystal so that it is propagated along the optic axis, the light emerges as a plane-polarised beam whatever may be the thickness traversed and the azimuth of the plane of polarization.

[2] The term *principal axis* when used in Optics must not be confused with the similar term when applied to the crystal-development. Optically, it is a direction in the crystal along which the radii to the points of contact of the two wave-fronts with the wave-surface are also the normals to both. Each biaxal crystal has for a definite colour three such directions at right angles to one another. Crystallographically, the term is limited to that axis of symmetry in the tetragonal, rhombohedral and hexagonal systems about which similar edges occur in sets of four, three and six, respectively.

A beam of plane-polarised light, the vibrations of which are parallel to a principal axis of the wave-surface, is transmitted with equal velocity in every direction perpendicular to this axis; and the corresponding index of refraction can be obtained by means of a prism having its refracting edge parallel to the axis. The indices of refraction for waves vibrating parallel to the principal axes are known as the *principal indices* of refraction of the crystal, and are usually denoted by α, β, γ, taken in ascending order of magnitude. Thus, in aragonite, for sodium light $\alpha = 1{\cdot}5301$, $\beta = 1{\cdot}6816$, $\gamma = 1{\cdot}6859$. Along each principal axis two beams can be propagated, the vibrations of which are parallel to the two other principal axes, respectively. Two of the principal indices of refraction of a biaxal crystal can therefore be determined by a prism in which a principal axis is perpendicular to the plane through the refracting edge bisecting the angle between the faces of the prism.

The optic axes lie in the plane containing those two principal axes, vibrations parallel to which are transmitted with the greatest and least velocity, respectively. One of these principal axes bisects the acute angle between the optic axes and is called the *first mean-line*, or the *acute bisectrix* (abbreviated to Bx_a); whilst the second principal axis bisects the obtuse angle between the optic axes, and is called the *second mean-line*, or the *obtuse bisectrix* (Bx_o). In some crystals the acute bisectrix coincides with the principal axis of greatest velocity, and such crystals are said to be optically positive (denoted by +); *e.g.* gypsum. In other crystals the acute bisectrix coincides with the principal axis of least velocity, and these are said to be optically negative (–); *e.g.* mica.

Light being propagated with equal velocity in opposite directions along the same line in a crystal, there can be no optical distinction between the crystals of classes which are related to one another by the fact that the elements of symmetry of the more symmetrical crystals result from the addition of a centre of symmetry to the elements present in the crystals of lower symmetry. In Chaps. XI—XIII it will be seen that in each of the biaxal systems the class of greatest symmetry can be derived from those of lower symmetry in the system by the addition of a centre of symmetry to the elements of symmetry characteristic of the latter. It follows that the three classes of the prismatic system have similar optical characters; the same is true of the three classes of the oblique system as well as of the two classes of the anorthic system; though, as we shall see, there are marked differences in the optical characters of the three systems.

5. It has been found that, in prismatic crystals, the principal axes of the wave-surfaces for all colours coincide with the axes of the crystal, whether these be three dyad axes, or consist of one dyad axis and the pair of axes perpendicular to the planes of symmetry in class III. The planes of symmetry of the crystal, in classes II and III, are also planes of symmetry of the wave-surface. The biaxal figures seen in strongly convergent polarised light, in plates cut perpendicularly to the acute bisectrix, are symmetrical with respect to two lines at right angles to one another; these lines are the traces on the plane of section (*a*) of the plane containing the optic axes, and (*b*) of the plane which bisects the angle between them.

6. But Herschel (*Pogg. Ann.* xxvi, p. 308, 1832), Nörrenberg (*ibidem*), and Neumann (*Pogg. Ann.* xxxv, p. 81, p. 203, 1835), found that the similar biaxal figures of plates of oblique crystals manifested less symmetry when observed in convergent *white light*. In monochromatic light the biaxal figures are symmetrical to two perpendicular lines, as is the case with prismatic crystals. But in white light the figures in oblique crystals are symmetrical: (*1*) to only one line, or (*2*) only to the central point.

1. Thus, in plates of gypsum the colours of the figures are symmetrical only to the trace of the plane of the optic axes. The optic axes lie in the plane of symmetry of the crystal; but they and *the bisectrices are not coincident for all colours.* Such sections are said to manifest *inclined dispersion.*

Again, in sanidine from the Eifel, the colours are symmetrically distributed with respect only to the trace of the plane bisecting the acute angle between the optic axes and perpendicular to their plane. The plane of the optic axes is found in such crystals to be perpendicular to the plane of symmetry of the crystal; and the sections are said to show *horizontal dispersion.*

The optical characters of these two minerals are more fully described in Chap. xii, Arts. 28 and 29. The positions of the optic axes are easily recognised; for dark brushes, curved very like a hyperbola, pass through their extremities, provided the plate is not in an azimuth in which the plane of the optic axes coincides with, or is perpendicular to, the plane of polarization.

2. In sections of crystals of borax, the figure is only symmetrical with respect to the central point. The plane of the optic axes is perpendicular to the plane of symmetry of the crystal;

and the acute bisectrix coincides with the dyad axis, and is perpendicular to the plate. This phenomenon is known by the name of *crossed dispersion*.

7. When similar plates of anorthic crystals are carefully examined in white light, the figures are found to be deficient in symmetry with respect both to a point and line (Neumann, *Pogg. Ann.* xxxv, p. 380, 1835).

8. When thin plates of prismatic and oblique crystals, cut parallel to planes of symmetry or perpendicularly to dyad axes, are observed between crossed Nicols in *parallel light*, the following distinctions are perceived which are useful in the practical discrimination between crystals of the two systems. A wave of light, the vibrations of which are parallel to a principal axis of the wave-surface, is transmitted without double refraction. When, therefore, a plate, cut in the special manner just described, is turned in its plane, until either of the principal axes is in, or perpendicular to, the plane of polarization, the light is transmitted without resolution or alteration, and is consequently extinguished by the analyser. The two perpendicular lines are said to be the directions of extinction.

In prismatic crystals the principal axes of the wave-surface coincide with three zone-axes, or with possible zone-axes which bisect the angles between homologous edges. The directions of extinction in a plate parallel to two crystallographic axes are, then, found to be either parallel, or perpendicular, to important edges of the section, or to bisect the angles between the edges. In descriptive works on minerals and artificial chemical substances, the position of the plane of the optic axes is stated to be parallel to a specified crystallographic axial plane, and the acute bisectrix to a specified crystallographic axis. Thus, in topaz the plane of the optic axes is said to be parallel to (010), and the acute bisectrix to be parallel to OZ. These statements are shortly put as follows:—
O.A. $\parallel$ (010) and $Bx_a \parallel OZ$.

In plates of oblique crystals parallel to the plane XOZ—the plane of symmetry in classes II and III—the lines of extinction are inclined to the edges of the section at arbitrary angles, *i.e.* at angles varying with the substance, and they are not the same for different colours, though the displacement is, in most cases, too slight to be capable of measurement. In descriptive works, the positions of the plane of the optic axes and

of the bisectrices are indicated in the following abbreviated way. The direction of a bisectrix lying in the plane XOZ is given by the angle included between it and OZ. The angle is regarded as positive, if it is to be measured towards positive OX, *i.e.* if the bisectrix lies in the obtuse angle XOZ; it is regarded as negative, if it is to be measured towards $OX_{,}$. Thus, in gypsum, O.A. $\|$ (010) and $Bx_a \wedge OZ = +52{\cdot}5°$ at 9·4° C. (Neumann). In orthoclase, O.A. $\perp$ (010), $Bx_a \wedge OZ = -68°$. In borax, O.A. $\perp$ (010), $Bx_o \wedge OZ = -55° 33'$ (red light), and $Bx_o \wedge OZ = -54° 45'$ (green light).

In anorthic crystals the relations of the three principal axes of the wave-surface to the edges of the crystal differ with the substance; and the directions of the bisectrices and of the plane of the optic axes at a definite temperature have, in each case, to be determined by observation. No systematic method of indicating these directions has yet been adopted; but it would be simplest to give the angles made by the bisectrices with the normals to the axial planes.

9. The student will find it advantageous to examine a crystal in parallel light between crossed Nicols before beginning to measure its angles with a reflecting goniometer. If, on examination, he finds extinction to occur parallel to certain well-marked edges, he may frequently be able to confirm the impression as to the symmetry formed by general observation of the facial development, and thus be fairly well assured as to the system to which the crystal belongs. Thus crystals of the second group give extinction parallel to the triad, tetrad, or hexad axis; a crystal of topaz gives extinction when the prism edges are parallel to the plane of polarization. On the other hand, a crystal of gypsum gives extinction in a direction inclined to the edges of the crystal, save when the cleavage-plane is parallel to the plane of polarization or analysation. Such observations often much lighten the labour of measurement, for they help the student to a decision as to the zones which must be measured in order to completely determine the crystal.

10. Mitscherlich (*Pogg. Ann.* VIII, p. 519, 1826) discovered that in gypsum the angle of the optic axes diminishes when the temperature rises, and that at a definite temperature the angle is reduced to zero; the temperature being still further raised the axes open out in a plane at right angles to their former one, and the angle then increases as the temperature rises. Neumann (*Pogg.*

Ann. xxxv, p. 81, 1835) made the further discovery that the acute bisectrix does not retain the same position during these changes, and that the displacement of the two optic axes is unequally rapid. Des Cloizeaux (*Ann. des Mines*, xi, p. 261, 1857; xiv, p. 339, 1858; *Mém. prés. à l'Acad. d. Sci.*, xviii, p. 511, 1868) has, by an extensive series of investigations on the optical characters of crystals, shown that the principal axes of the wave-surface, lying in the plane XOZ of oblique crystals, do not retain fixed positions when the temperature changes; and, also, that the positions of the principal axes of the wave-surface in anorthic crystals depend on the temperature.

The three systems are therefore optically distinct, and Naumann's view (Art. 1) has been fully confirmed.

11. Herschel (*Trans. Camb. Phil. Soc.* i, p. 43, 1822) succeeded in showing that the rotation of the plane of polarization of a beam of light, traversing a plate of quartz cut perpendicularly to the triad axis, was connected with the facial development of the crystal, as shown by the relative positions of the planes x and s to the other faces on crystals such as that represented in Fig. 11. This peculiarity is, also, shown by crystals of cinnabar; these likewise have a triad axis associated with three dyad axes but have no planes of symmetry. The labours of Marbach (*Pogg. Ann.* xci, p. 482, 1854; xciv, p. 412, 1855; xcix, p. 451, 1856) and others have led us to believe that this is a general character of crystals of all classes which have only axes of symmetry. In such classes two correlated forms are possible, which can be placed so that they are reciprocal reflexions, the one of the other, in a plane perpendicular to one of the axes of symmetry: the correlative forms are said to be *enantiomorphous*. We shall call attention to such cases in dealing with the particular classes in which they are possible.

Pyro- and piezo-electricity.

12. It has long been known that tourmaline crystals, which have dissimilar developments at opposite ends of the triad axis, show different electrifications whilst the temperature is changing. This polar character of the crystals is found to hold in other crystals in certain directions, and is connected with the dissimilar development at the two ends of an axis of symmetry,

which we have designated as one of uniterminal symmetry. The phenomenon is known as *pyro-electricity*. A similar change in the electrical condition—known by the name of *piezo-electricity*—was discovered by MM. Curie (*Bull. Soc. franç. de Min.* III, p. 90, 1880) to be excited in crystals by pressure along axes of uniterminal symmetry.

Corrosion.

13. Another method of investigation, which occasionally aids us in determining the class of symmetry to which a crystal belongs, is that of corroding the faces. When the experiments are carefully carried out with suitable corrosive fluids, the faces are found, in many cases, to become covered with pits of similar outlines, which have well-defined orientations. The *corrosion-figures*, as they are called, produced by different corrosive liquids have frequently different shapes, but they manifest the symmetry of the face etched. Thus, the corrosion-figures, produced by dilute hydrochloric acid, on a cleavage-face of calcite show symmetry with respect to the lines parallel to the diagonal bisecting the obtuse angle of the face, and indicate the presence of a plane of symmetry through the triad axis perpendicular to the face. The corrosion-figures, produced by dilute sulphuric acid, on a face perpendicular to the triad axis of the rhombohedral mineral spangolite, are well displayed in Fig. 110 (after Penfield). The figures are symmetrical and show three-fold symmetry with respect to three intersecting planes of symmetry.

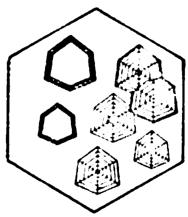

FIG. 110.

CHAPTER XI.

THE ANORTHIC SYSTEM.

I. *Pediad class*; $a\{hkl\}$.

1. Crystals belonging to the first class have no element of symmetry. Such a crystal is, therefore, a polyhedron bounded by a set of faces, none of which are necessarily parallel and which are only connected together by the law of rational indices. Each form consists of a single face, and will be called a *pedion* (πεδίον = a plane, or level piece of land). From this character the class may be called the *pediad class* of the anorthic system. Crystals of calcium thiosulphate ($CaS_2O_3 . 6H_2O$), of strontium hydrogen dextro-tartrate ($Sr(HC_4H_4O_6)_2 . 5H_2O$), and of a few other substances belong to this class.

2. When parallel faces are present, they are to be regarded as belonging to different forms; and their physical characters will be slightly different. By the law of rational indices it is clear that parallel faces may occur on the same crystal; for any face is a possible one which meets the axes at distances capable of representation by exact submultiples of the parameters, whether they are measured on the positive or negative sides of the origin. Hence, (hkl) being present, $(\bar{h}\bar{k}\bar{l})$ is a possible face, and may perhaps be present. But the presence of one does not necessarily involve that of the other. Two forms connected together by this geometrical relation, *i.e.* by parallelism of the faces, will be denoted as *complementary forms* of the pediad class of the anorthic system.

Miller (*Treatise on Cryst.* p. 23) employed a Greek prefix before the symbol of the form to indicate that it consisted of one-half the faces constituting the form of greatest symmetry in the system; the latter, described as the holohedral form, being given by the symbol

$\{hkl\}$ without prefix. Thus, he uses $\kappa\,\{hkl\}$ to represent a hemihedral form (Chap. III, Art. 2) with *inclined* faces, i.e. which has no parallel faces; $\pi\,\{hkl\}$ to represent one which has parallel faces. Although the views on symmetry, according to which the forms of certain classes were regarded as merohedral divisions of a more symmetrical form, have been abandoned by most crystallographers, Greek prefixes will be used in this book to denote forms belonging to the classes of inferior symmetry in a system; for they serve to indicate in a concise manner the geometrical relations of the forms of the several classes of a system: but the prefixes will be dropped in the cases of those particular forms which are geometrically alike, whether they belong to the class of greatest symmetry or to one of inferior symmetry. The symbol $a\,\{hkl\}$, used for the forms of the pediad class, will also be used to indicate the forms of classes in each system, which are enantiomorphous, for they are *without* planes of symmetry. The symbol $\mu\,\{hkl\}$ will be used to denote forms with inclined faces of classes, the crystals of which show one or more pyro-electric axes and are not enantiomorphous: $\kappa\,\{hkl\}$ for forms with inclined faces of classes (like II of the oblique system), the crystals of which have planes of symmetry and do not show characteristic differences in physical phenomena falling under either of the two preceding cases. The symbol $\tau\,\{hkl\}$ will be used when the faces in the form are one-fourth those in the form of greatest symmetry; and some modification of this symbol will be made when the forms of two classes of the system show such a relation.

3. The least number of faces which will completely enclose a limited portion of space is four; none being parallel, and no three being in a zone. In this class such a solid figure is an irregular tetrahedron, Fig. 114.

If two of the faces are parallel, there must be at least five to enclose a finite portion of space, and no pair of the three other faces can be parallel to one another. All three may, however, lie in a zone. Such a possible crystal is shown in Fig. 111. If the three faces, inclined to one another, are not in a zone, the figure will resemble an irregular tetrahedron with one of its coigns modified by a face parallel to the opposite face.

If the possible crystal is bounded by two pairs of parallel faces, there must be at least two other faces which may, or may not, be parallel. The least number of forms is six, of which two pairs, or possibly the three, may consist of complementary forms, *i.e.* of

parallel faces. Possible crystals of the two kinds are shown in Figs.[1] 112 and 113.

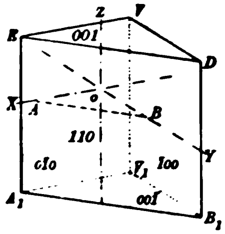

Fig. 111.

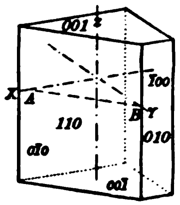

Fig. 112.

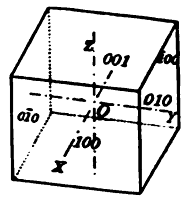

Fig. 113.

When a crystal is, as in the above ideal cases, bounded by a set of faces which are not homologous, it is said to consist of a *combination of forms*, or shortly to be a *combination.*

4. For a crystal consisting of such simple combinations an axial system can be determined, and in some cases the parameters; but the calculations are not easy. The axial planes must be taken to be parallel to three of the faces which meet in a coign, and the coign itself, or any other point, may be taken to be the origin.

Thus, taking the possible four-faced crystal $VA_,B_,C_,$, Fig. 114, the origin O is any point within the crystal, and the axes OX, OY, OZ are drawn parallel to the edges $VA_,$, $VB_,$, $VC_,$, respectively. The fourth face $A_,B_,C_,$ meets the axes in the points A, B, C; and $OA=a$, $OB=b$, $OC=c$, if the face is taken to be (111). The face $VB_,C_,$ is parallel to the axes OY and OZ, and the last two indices are zero: the symbol is therefore (h00), where h may be positive or negative. Care must now be taken to determine whether the face meets the axis of X on the same side of O as ABC (111) or on the other side. It is clear that the two faces meet the axis of X on opposite sides of the origin, for O has been taken within the crystal. Hence h is negative, and the face is ($\bar{1}$00); for there is no advantage in making h any larger integer. Similarly, $VC_,A_,$ is (0$\bar{1}$0), for it meets OY on the negative side of the origin and is

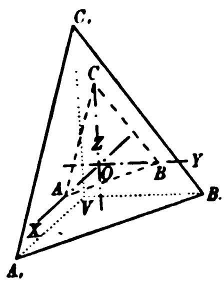

Fig. 114.

[1] In these figures the symbols in smaller type refer to the faces at the back which are shown by dotted lines.

parallel to OX and OZ. Finally, the face $VA_{,}B_{,}$ is $(00\bar{1})$. The crystal is a combination of the four forms of which the symbols have been determined.

5. On such a crystal six angles can be measured; viz. the three $(0\bar{1}0 \wedge 00\bar{1})$, $(00\bar{1} \wedge \bar{1}00)$, $(\bar{1}00 \wedge 0\bar{1}0)$, and the three between (111) and each of the axial faces. Five of these angles suffice to give the angles between the axes as well as the parameters $a : b : c$. Thus we can take any two planes, say two pieces of cardboard, and join them at the angle $(\bar{1}00 \wedge 010)$. If placed vertically, the axis OZ is fixed. Now through any point on OZ, or $VC_{,}$, a third cardboard can be placed at the angle $(0\bar{1}0 \wedge 00\bar{1})$ if the line $VA_{,}$ in $(0\bar{1}0)$ is known; and then the line $VB_{,}$ is also fixed. It is clear, therefore, that, to put the third cardboard in place, we require the plane angle $C_{,}VA_{,}$, an angle which cannot be measured by a reflecting goniometer. By calculation the angles $C_{,}VA_{,}$ and $C_{,}VB_{,}$ can both be found from the three angles between the axial planes, and the lines $VA_{,}$ and $VB_{,}$ determined on the two cardboards through OZ. The third cardboard is then easily placed and a model of the axial planes and axes completed.

6. The three edges $VA_{,}$, $VB_{,}$, $VC_{,}$, and the three axes parallel to them, are connected with the normals to the three faces by the relation known in spherical trigonometry as that of *polar triangles*. The two triangles ABC (that made by the poles of the axial planes) and XYZ (where the axes emerge) are shown on the stereogram, Fig. 115; the axial points being marked by small crosses. The faces of the crystal, which are parallel to the axes, may be supposed to pass through the radii of the sphere emerging at X, Y and Z, and the points A, B, C, are the upper extremities of the normals to the faces. But a normal to a face is at right angles to every line in the face and to every line in any plane parallel to the face. Hence, $AY = AZ = 90°$, also $BZ = BX = 90°$, and $CX = CY = 90°$. From the above we see that $XB = XC = 90°$. Hence, the radius through X is at right angles to the normals through B and C: it is therefore at right angles to the plane containing these normals and is the pole of the great circle $[BC]$. Similarly, Y is the pole of the great circle $[CA]$, and Z of $[AB]$.

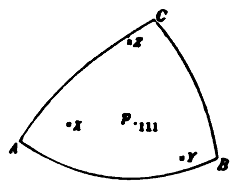

FIG. 115.

Hence, the two triangles are called polar triangles, for the apices of the one are the poles of the great circles forming the sides of the other.

But the angle between two normals is the supplement of the angle between the two planes (Chap. II, Art. 2). Hence,

$$ZXY = 180° - BC,\ XYZ = 180° - CA,\ YZX = 180° - AB.$$

Also $XY = 180° - BCA$, $YZ = 180° - CAB$, $ZX = 180° - ABC$.

7. Now measurement on the reflecting goniometer gives the angles, or arcs, AB, BC, CA; and by well-known formulæ (McLelland and Preston's *Spherical Trig.* I, p. 47) the angles BCA, CAB and ABC can be calculated. The angles XY, YZ, ZX are, then, determined by the relation given above; and the model of the axial planes can be made as suggested in Art. 5, or three rods can be joined together at a point O making with one another the angles XOY, YOZ, ZOX.

In the above projection, the poles A, B, C were all placed above the primitive in order to show the relation with the axial points X, Y, Z more distinctly. It is usual, however, to put two of the poles A and B in the primitive. One of them is placed arbitrarily in this circle, and commonly B is placed at the right extremity of the horizontal diameter. The pole A is then placed on the lower part of the primitive by marking off the arc AB with a protractor. Consequently the axial point Z is at the centre of the primitive. It is not a possible pole. As stated in Chap. VII, Art. 6, small Italics are often used in diagrams to denote the poles and faces (100), (010), (001); and, when so used, care must be taken not to confuse them with the parameters.

8. Having fixed the directions of the axes, the fourth face enables us to find $a : b : c$. The determination of these ratios involves a considerable transformation of the equations (1) of Chap. IV, Art. 15, which will be given in a later section. The introduction of a face $p(\bar{1}\bar{1}\bar{1})$, parallel to $P(111)$, adds nothing to our knowledge, for it gives no new and independent angles. For the angle $Ap = 180° - AP$; and similarly for the other angles.

9. The possible crystal in Fig. 111, bounded by five faces, is not one which enables us to determine $a : b : c$. We can, as before, take the origin at any point O within the crystal, and the

axes parallel to three edges as in the figure. Now the vertical face A_1B_1DE meets the axes OX and OY at A and B; and the ratio, $a : b = OA : OB$, is determined, if we call the face (110). The last index must be zero, for the face is parallel to OZ and meets it at infinity. The faces VDE and $V_1A_1B_1$ are both parallel to the axes OX and OY, and the first two indices are both zero. The distance at which they meet the vertical axis is quite indeterminate, being contingent on the deposition of matter on the parallel faces, and the parameter c cannot be found. But the two faces are given in position, as far as their general directions are concerned, because they are parallel to the two axes. Hence they are represented as (001) and $(00\bar{1})$. The faces VDB_1 and VEA_1 are $(\bar{1}00)$ and $(0\bar{1}0)$, respectively. To determine the parameter c, an additional face meeting OZ and OX, or OZ and OY, or all three axes, at finite distances is needed.

10. In Fig. 113, we have a possible crystal, bounded by three sets of parallel faces, of which none of the parameters can be determined. In this case the three independent faces, meeting at a coign, can be taken to give the axial planes, and the axes are then parallel to the edges of the parallelepiped; but no definite lengths, characteristic of the crystal, can be determined on them. In such a crystal we have only three independent angles, those between the three faces meeting at a coign.

11. If, however, the third pair of faces are not parallel, we have two possible cases. The third pair of faces may be inclined to one another, but lie in a zone with one of the parallel pairs, as shown in Fig. 116. The axes are taken parallel to the edges of the two sets of parallel faces and one of the others. A line drawn, parallel to the sixth face, through any point A on OX and in the plane XOY meets the axis of Y at B. The lengths OA, OB may be taken to be the parameters on the axes of X and Y. The face is then (110). But a line through A in the plane XOZ parallel to the sixth face (110) is also parallel to OZ, for OZ is parallel to the intersection of the sixth face with the faces (010) and (010). Hence no finite length is cut off on OZ, and the parameter c is indeterminate. The reader will notice that this crystal only differs from the five-faced crystal of Art. 9,

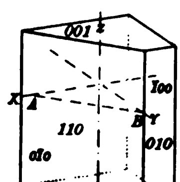

FIG. 116.

Fig. 111, in the fact that in a zone, containing three faces, we have added a fourth parallel to one of them, namely to EVA_1; and this addition gives no new and independent angles.

If, however, the sixth face has any general position, and is not parallel to any edge of the other faces, we can determine $a : b : c$, and all the elements of the crystal. We may clearly take one face from each pair of parallel faces to make with the fifth a cardboard model, as described in Art. 5, of the axial planes of the tetrahedron, Fig. 114. The sixth face meets all the edges of this model at finite distances. Hence such a crystal may be taken to be made up of the forms:—(010), ($0\bar{1}0$), (001), ($00\bar{1}$), ($\bar{1}00$) and (111), of which the first two pairs are complementary.

12. The method of calculating the symbols of the faces and the parameters, $a : b : c$, from the measured angles, is the same in this and the next class, and will be given at a later stage. If, however, a crystal of this, or the next, class has many faces, they will fall into zones from which, by the methods given in Chaps. V and VIII, the indices can be found. The stereogram can, also, be as easily made as those of the simplest crystals. As already stated, the poles A (100) and B (010) are marked on the primitive by a protractor, and at the same time any other poles in their zone. The positions of one or two other poles lying somewhere on the upper hemisphere are, then, determined by the method given in Chap. VII, Arts. 19 and 20. If these are carefully selected, so as to lie in important zones passing through them and poles already placed in the primitive, the rest of the projection can usually be completed without difficulty.

II. *Pinakoidal class*; $\{hkl\}$.

13. In this class the faces occur in pairs which are parallel and physically similar. The crystals have a centre of symmetry, but no other element of symmetry. Each form would, if indefinitely extended, present the appearance of a board, or plank (Greek πίναξ). Hence, it has been called a *pinakoid*. We shall use the same word to denote forms consisting of only two parallel faces, whatever be the class to which the crystal belongs. In other classes very few forms are of this character, whilst in this class every form is a pinakoid. We shall, therefore, call the class the *pinakoidal class* of the anorthic system.

14. The simplest crystal possible in this class is one bounded by three sets of parallel faces, such as is shown in Fig. 117. The distinction between the crystal, regarded as belonging to this class or to the previous one, is this:—that the pairs of parallel faces are exactly similar in their characters and necessarily—save for accidents in the deposition of matter at opposite ends—occur together; whilst in the former class the occurrence of the parallel faces is more or less accidental, and the parallel faces have different characters. The discrimination between the two cases may be attended with great difficulty. To decide such a question a large number of crystals should be examined, and experiments on corrosion and on electrification by change of temperature or pressure should be tried. If parallel faces are almost always present, and if no distinction between the members of each pair of parallel faces can be observed, the crystals are placed in this class.

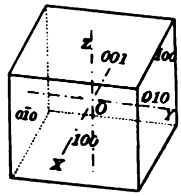

FIG. 117.

The faces of the parallelepiped, Fig. 117, are taken to give the axial planes, and the three forms composing it are: {100} consisting of 100 and $\bar{1}00$, {010} consisting of 010 and $0\bar{1}0$, and {001} consisting of 001 and $00\bar{1}$. The parameters are indeterminate.

15. If other forms are added to such a crystal, the new faces may lie in a zone with two sets of faces of the parallelepiped. In this case, the ratio of two of the parameters may be determined, but the third remains indeterminate.

A crystal of cyanite (Al_2SiO_5), showing such a combination of forms, is represented in Fig. 118 (after Bauer). The axis OZ is taken parallel to the edge $[ab]$; and OX and OY are parallel to $[bc]$ and $[ca]$, respectively. Hence, the faces have the symbols $a\,(100)$, $b\,(010)$, and $c\,(001)$. The two faces M are parallel to OZ, for they are in the zone $[ab]$: the last index is therefore zero. If the faces are extended to cut the axes of X and Y, that to the left front meets OX at A, where OA may be adopted as the parameter a, and $OY_{,}$ at $B_{,}$, where $OB_{,} = b$ may be adopted as the parameter b. The face has the symbol $(1\bar{1}0)$; for it meets OY on the

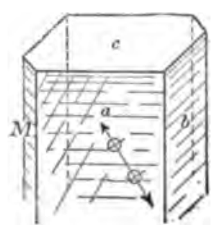

FIG. 118.

negative side of the origin. The parallel face meets OX at $A_{\prime}$, where $OA_{\prime}=a$, and OY at B, where $OB=b$. But the directions, in which the lengths are measured, are reversed. Hence, its symbol is $(\bar{1}10)$. The two faces constitute the pinakoid $\{1\bar{1}0\}$.

If there are, as is sometimes the case, other faces in the zone, and if the angles they make with the faces already determined are known, their symbols can be determined from the anharmonic ratio of four tautozonal faces; or, vice versâ, the angles, which any tautozonal face with known symbol makes with the axial faces, can be computed.

Example. Thus, knowing $ab=73°\ 56'$ and $aM=48°\ 18'$; to find the angle am, where m is (110).

Now the A. R. $\{bmaM\}$ gives

$$\frac{\sin bm}{\sin ba} \div \frac{\sin Mm}{\sin Ma} = \begin{vmatrix} 010 \\ 110 \\ \hline 010 \\ 100 \end{vmatrix} \div \begin{vmatrix} 1\bar{1}0 \\ 110 \\ \hline 1\bar{1}0 \\ 100 \end{vmatrix} = \frac{1}{1} \div \frac{2}{1} = \frac{1}{2}.$$

Hence, by the transformation given in Chap. VIII, Art. 14,

$$\frac{\sin bm}{\sin Mm} = \frac{\sin (ba=73°\ 56')}{2\sin (Ma=48°\ 18')} = \tan\theta \text{ (say)},$$

$$\begin{array}{rlrl} \log 2 = & \cdot 30103 & & \\ L\sin 48°\ 18' = & 9\cdot 87311 & L\sin 73°\ 56' = & 9\cdot 98270 \\ & 10\cdot 17414 & & 10\cdot 17414 \\ & & L\tan(\theta=32°\ 45\cdot 7') = & \bar{9}\cdot 80856. \end{array}$$

$$\therefore\ \theta = 32°\ 45\cdot 7' \text{ and } 45° - \theta = 12°\ 14\cdot 3'.$$

Also $$Mm + bm = ab + aM = 122°\ 14'.$$

$$\therefore\ \tan\tfrac{1}{2}(Mm - bm) = \tan(45° - \theta)\tan\tfrac{1}{2}(Mm + bm) = \tan 12°\ 14\cdot 3' \tan 61°\ 7'.$$

$$\begin{array}{rl} L\tan 61°\ 7' & = 10\cdot 25834 \\ L\tan 12°\ 14\cdot 3' = & \underline{\ 9\cdot 33627} \\ & \overline{9\cdot 59461} = L\tan 21°\ 28'. \end{array}$$

$$\therefore\ Mm - bm = 42°\ 56',$$
$$Mm + bm = 122°\ 14'.$$
$$\therefore\ bm = 39°\ 39', \text{ and } am = 34°\ 17'.$$

The line of arrows in the diagram gives the trace on the face a of the plane of the optic axes.

16. If the forms added to the simple axial parallelepiped meet all the axes at finite distances, one of them is selected to give the parametral plane (111). The indices of the faces of other forms can then be determined by the law of zones; and also, when sets of four, or more, forms occur in a zone, by the anharmonic ratio. Forms,

the faces of which do not fall into zones, must have their symbols determined by the general relations given at the end of the Chapter.

17. *Example.* We shall illustrate the method of determining anorthic crystals by a discussion of those of anorthite shown in Fig. 119 and in Fig. 120 (after vom Rath). They represent two different habits of frequent occurrence amongst crystals of anorthite and the allied plagioclastic felspars. The axes of reference are taken, as shown in Fig. 119, parallel to the edges [*PM*], [*Py*] and [*Ml*] respectively; the faces *P*(001) and

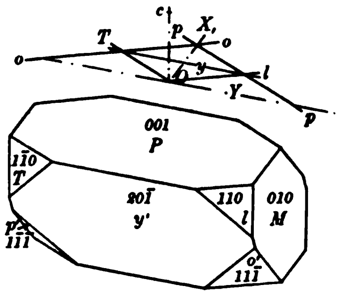

FIG. 119.

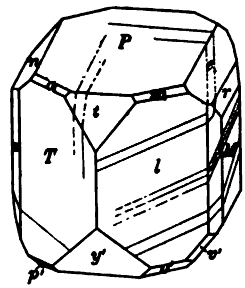

FIG. 120.

M(010) being those of good cleavage in all the felspars. The two pinakoids *l* and *T* are generally well developed, and the symbols {110} and {1$\bar{1}$0} are assigned to them. The faces *h*{100}, parallel to the axes of *Y* and *Z*, are not of frequent occurrence; and the position of *h* is usually calculated from a knowledge of the angles between *M*, *l* and *T*. For, as stated in Chap. IX, Art. 1, the A.R. {*MlhT*} is a harmonic ratio which has the value $1 \div 2$, when *M* is made (010), *l*(110), *h*(100) and *T*(1$\bar{1}$0).

Let us suppose the angles in the principal zones to have been measured, and some of them to be:

i.			ii.			iii.		
	Mf	29° 30′		*Mr*	18° 9′		*l′v*	32° 6′
	Ml	58 4		*Me*	43 11		*vn*	53 4
	lT	59 30		*MP*	85 50		*na*	38 37
	zM′	30 58		*nM′*	47 24		*at*	21 8

Stereogram. From a knowledge of the above angles, and of the zones perceived in Fig. 120, we can now construct Fig. 121. It will summarise the whole of the forms and zonal relations on the two crystals of anorthite; and, approximately enough, those on crystals of other plagioclastic felspars which have the same forms. The zone-axis *OZ* is taken to be the diameter through the eye, so that the poles *M*, *f*, *l*, *T*, &c., lie in the primitive. *M* is placed arbitrarily at the right extremity of the horizontal diameter. Arcs equal to the angles in zone i of the table are then marked off by a protractor, and the poles *f*, *l*, *T*, *z*, &c., are fixed.

The position of *n* (one of the most convenient points) is determined in the way given for anorthite in the example of Chap. VII, Art. 20. This pole having been placed, the zone-circles [*MPnM'*], [*Tnoy*], [*ltanv*] can be drawn. The pole *X'* of [*MPM'*] is then found by the method of Chap. VII, Prob. 4. Arcs on the primitive are now measured off from *M* equal to the angles given in zone ii. The straight lines, joining these points to *X'*, intersect the zone-circle [*MPM'*] in the poles *r*, *e* and *P*.

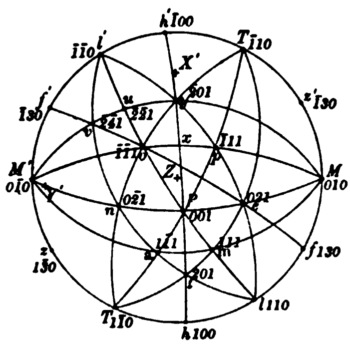

FIG. 121.[1]

The zone-circles [*lPl'*], [*TPT'*], [*lel'*], [*TeT'*] and [*fef'*] are now easily drawn. By their intersections with one another and with the zone-circles already drawn through *n*, they determine the positions of the poles, *t*, *m*, *a*, *p*, *o*, *y* and *v*. Some of the zone-circles have very long radii. The circles are in such cases most easily made by the aid of a *cyclograph*[2] arranged to pass through three known poles. It will now be found that the following circles can be drawn: [*MmaM'*], [*MpxoM'*], [*MyuvM'*] and [*htPxy*]. They establish the facts that the corresponding faces are tautozonal.

The construction of an accurate stereogram involves care in the determination of points, whether they are the projections of poles or the centres of circles, and hence occupies a good deal of time. For working

[1] The pole *r* has been accidentally omitted. The poles *a* and *m* should be in Italics.

[2] This consists of a long strip of narrow and thin wood or steel, bent so as to pass through the three points. The simplest one is a bar, about 6 inches long, compressed by a finger and thumb whilst held in the correct position on the paper. A more convenient one can be made as follows. A strip of stout brass about 7 inches long rests on the paper. At one end is a fixed block with a deep vertical groove; at the other end a similar vertically grooved block which can be moved backwards and forwards by a screw. The spring is placed with its ends in the grooves and any curvature required can be obtained by the action of the screw.

purposes an approximation to a correct one suffices; and after a little practice, the student should be able to construct what may be called a freehand stereogram, showing all the important zones, in a few minutes. The curves, even if circles, being only approximately correct, will not enable him to see when incontiguous faces are tautozonal, although an accurate projection would show their poles to lie on a great circle; but the better his projection, the more likely is such a relation to be suggested. We have supposed the crystal to be measured before the projection is made, and this must necessarily be the case when an accurate projection is required.

The face-symbols. The value of $a:b$ has been fixed by the assumptions made as to P, M, l, and T. We can now assume indices for a face in $[MP]$ which fixes $c:b$; or else, for a face in $[hPy]$ which gives $c:a$. Faces in this latter zone are very generally present and well developed on all felspar crystals, so that they can be easily recognised. We shall, therefore, assume y to be $(\bar{2}01)$. Then y' is $(20\bar{1})$.

The symbols of e and n can be now determined, for they both lie in $[MP]=[100]$. The first index of both faces is therefore zero. The former face is also in $[ly]=[1\bar{1}2]$. Hence, if e is $(0kl)$, we have $2l-k=0$. The symbol of e is therefore (021). Similarly, n is in $[Ty]=[112]$; and its symbol is $(0\bar{2}1)$.

Again, for a pole (hkl) in the zone $[ln]=[\bar{1}12]$, we have $k+2l-h=0$. Hence, t, in which the zone-circles $[ln]$ and $[Py]=[010]$ intersect, is (201). Likewise the pole a, in which $[ln]$ intersects $[PT]=[110]$, is $(1\bar{1}1)$.

Again, from the zones $[Pl]=[1\bar{1}0]$ and $[Te]=[11\bar{2}]$, the pole m is (111). The poles a and m lie, therefore, in a zone-circle passing through M and $M_{,}$.

Similarly, from $[TP]$ and $[ly]$, we find p to be $(\bar{1}11)$; and, from $[lP]$ and $[Ty]$, the pole o is $(\bar{1}\bar{1}1)$. The remaining pole v $(\bar{2}\bar{4}1)$ can be found from the A.R. $\{l'vna\}$; or from $[ltn]$ and $[of]$, after the symbol of f has been determined.

It will be noticed that, by the assumptions made, the inconspicuous faces m, a, p and o, all meet the axes at the distances OA, OB, and OC, in the different octants. We might therefore have begun by assuming m to be the parametral face (111). The student, by paying attention to the zones, shown in Figs. 120 and 121, will have no difficulty in finding the indices of the remaining faces, when those of P, M, h and m, are assumed; and in proving that they are the same as those already given.

The axial points are not often shown on stereograms. In the present case, X' is at the cross which gives the pole of $[MPM']$, Z is at the centre of the circle and Y' is the pole (above the paper) of the zone-circle $[hPy]$.

Zone i. We now proceed to find the angle Mh, and the symbols of j and z, from the angles given in zone i.

From A.R. $\{ThlM\}$ we have

$$\frac{\sin Th}{\sin Tl} \div \frac{\sin Mh}{\sin Ml} = \left|\begin{matrix} 1\bar{1}0 \\ 100 \\ \hline 1\bar{1}0 \\ 110 \end{matrix}\right| \div \left|\begin{matrix} 010 \\ 100 \\ \hline 010 \\ 110 \end{matrix}\right| = \frac{1}{2}.$$

$$\therefore \frac{\sin Th}{\sin Mh} = \frac{\sin (Tl = 59^\circ\ 30')}{2 \sin (Ml = 58^\circ\ 4')} = \tan (\theta = 26^\circ\ 55');$$

$$\therefore \tan \tfrac{1}{2}(Mh - Th) = \tan (45^\circ - 26^\circ\ 55') \tan \tfrac{1}{2}(Mh + Th)$$

$$= \tan 18^\circ\ 5' \tan 58^\circ\ 47'.$$

$$\therefore \tfrac{1}{2}(Mh - Th) = 28^\circ\ 19',$$

$$\tfrac{1}{2}(Mh + Th) = 58^\circ\ 47'.$$

$$\therefore Mh = 87^\circ\ 6' \text{ and } Th = 30^\circ\ 28'.$$

In the above solution, the main steps are alone indicated. The student will find it good practice to test the accuracy of each step, and of the final result, by extracting the logarithms.

Again, from the A.R. $\{MflT\}$, we have

$$\frac{\sin Mf}{\sin Ml} \div \frac{\sin Tf}{\sin Tl} = \left|\begin{matrix} 010 \\ hk0 \\ \hline 010 \\ 110 \end{matrix}\right| \div \left|\begin{matrix} 1\bar{1}0 \\ hk0 \\ \hline 1\bar{1}0 \\ 110 \end{matrix}\right| = \frac{2h}{h+k}.$$

$$\therefore \frac{2h}{h+k} = \frac{\sin 29^\circ\ 30' \sin 59^\circ\ 30'}{\sin 58^\circ\ 4' \sin 88^\circ\ 4'} = (\text{by computation}) \frac{1}{2}.$$

$$\therefore 4h = h + k; \text{ and } h = 1,\ k = 3.$$

The pole f has therefore the symbol (130).

In a like manner, z is found, from the A.R. $\{M'zTl\}$, to be $(1\bar{3}0)$.

Faces r and v. Again, from the A.R. $\{MreP\}$ and the angles given in zone ii, we have

$$\frac{\sin Me}{\sin Mr} \div \frac{\sin Pe}{\sin Pr} = \left|\begin{matrix} 010 \\ 021 \\ \hline 010 \\ 0kl \end{matrix}\right| \div \left|\begin{matrix} 001 \\ 021 \\ \hline 001 \\ 0kl \end{matrix}\right| = \frac{1}{l} \div \frac{-2}{-k} = \frac{k}{2l}.$$

$$\therefore \frac{k}{2l} = \frac{\sin 43^\circ\ 11' \sin 67^\circ\ 41'}{\sin 18^\circ\ 9' \sin 42^\circ\ 39'} = 3 \text{ (by computation)}.$$

$$\therefore k = 6,\ l = 1; \text{ and } r \text{ is } (061).$$

From the angles given in zone iii, the student can, by taking the A.R. $\{l'vna\}$ and following the steps given in the preceding cases, prove v to be $(\bar{2}41)$.

The symbols of all the faces have been now determined. It remains to determine the angles between the axes, and the ratios $a : b$, $c : b$. These being known the crystals can be drawn by the method described for the simpler crystal in Chap. VII, Art. 5.

Formulæ connecting crystal-elements, indices and angles.

18. Let, in Fig. 122, A, B, C be the axial poles 100, 010, 001, and X, Y, Z the corresponding axial points; and let G be the parametral pole (111). Then L, M, N, the intersections of the zone-circles $[AG]$, $[BG]$, $[CG]$ with the opposite axial zones $[BC]$, $[CA]$, $[AB]$, have the symbols (011), (101), (110), respectively. Let us denote the arcs: BL by D, LC by $D_{,}$, CM by E, MA by $E_{,}$, AN by F and NB by $F_{,}$. These angles will be called the *angular elements* of an anorthic crystal. It is required to find the relation between them and the *linear elements* $a : b : c$ and the angles between the axial planes.

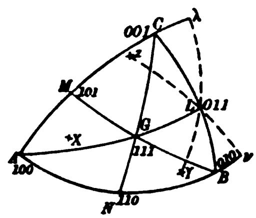

Fig. 122.

The two triangles ABC and XYZ are polar triangles (Art. 6), and $XB = XL = XC = 90°$. Similarly, Y and Z are at 90° from every point in the opposite sides CA and AB, respectively. Computing therefore the angles of the triangle ABC by the formula giving the angles when the sides are known (McL. and P. *Spher. Trig.* I, p. 47), we obtain the angles between the axes.

From the equations of the normal (Chap. IV, Art. 15), we have for L (011)

$$\frac{a \cos XL}{0} = \frac{b \cos YL}{1} = \frac{c \cos ZL}{1}.$$

But $XL = 90°$, $\therefore \cos XL = 0$, and the equations reduce to

$$b \cos YL = c \cos ZL \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Draw the great circles YL, ZL; and let them meet the zone-circles $[CA]$ and $[AB]$ in λ, ν respectively.

Then, $YL = 90° - L\lambda$; $ZL = 90° - L\nu$.

But every great circle through the pole of another meets the latter at right angles. The angles $C\lambda L$, $B\nu L$ are, therefore, both right angles. From Napier's rules for the relations between the sides and angles of a right-angled spherical triangle (McL. and P. *Spher. Trig.* I, p. 87), we have

$$\left.\begin{aligned}\cos YL &= \sin L\lambda = \sin CL \sin \lambda CL = \sin CL \sin BCA \\ \cos ZL &= \sin L\nu = \sin BL \sin \nu BL = \sin BL \sin ABC\end{aligned}\right\} \ldots\ldots (2).$$

Substituting in (1), we have

$$b \sin CL \sin BCA = c \sin BL \sin ABC;$$

$$\therefore \ \frac{b}{c} = \frac{\sin BL}{\sin CL} \frac{\sin ABC}{\sin BCA} = \frac{\sin BL}{\sin CL} \frac{\sin CA}{\sin AB} \ldots\ldots\ldots(3);$$

since $\frac{\sin ABC}{\sin BCA} = \frac{\sin CA}{\sin AB}$ (McL. and P. *Spher. Trig.* I, p. 43).

By drawing great circles through Z and X to M, we can obtain similar right-angled triangles having a common vertex at M; and can transform the equation of the normal to M in a similar manner.

Thus, for M (101), we have

$$a \cos XM = c \cos ZM \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1^*),$$

and

$$\left.\begin{aligned} \cos XM &= \sin CM \sin BCA, \\ \cos ZM &= \sin AM \sin CAB \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2^*).$$

$$\therefore \ \frac{c}{a} = \frac{\sin CM \sin BCA}{\sin AM \sin CAB} = \frac{\sin CM \sin AB}{\sin AM \sin BC} \ldots\ldots\ldots (3^*).$$

And, in an exactly similar manner, from the equation to the normal N (110), we have

$$\frac{a}{b} = \frac{\sin AN}{\sin BN} \frac{\sin BC}{\sin CA} \ldots\ldots\ldots\ldots\ldots\ldots (3^{**}).$$

The above equations enable us to find the parametral ratios when the angular elements $D = BL$, $D_{,} = CL$, &c., are all known.

19. By the aid of the equations established in Art. 18 we can also find D, $D_{,}$, &c., when $a : b : c$ and the angles between the axes, viz. YZ, ZX, and XY, are all known.

It is necessary to find, from the known arcs YZ, ZX and XY, the angles YZX, ZXY, and XYZ (McL. and P. *Spher. Trig.* I, p. 47). Then

$$AB = 180° - YZX, \ BC = 180° - ZXY, \text{ and } CA = 180° - XYZ.$$

Hence, taking (3), for instance, we have

$$\frac{\sin BL}{\sin CL} = \frac{b}{c} \frac{\sin AB}{\sin CA}.$$

All the numbers on the right side being known, the value of the term can be computed, and can, as in Chap. VIII, Art. 14, be expressed by tan θ. If θ is greater than 45°, we invert the equation

before proceeding further. But we may suppose that BL is less than CL, and that θ is, therefore, less than 45°.

$$\therefore \frac{\sin CL - \sin BL}{\sin CL + \sin BL} = \frac{1 - \tan\theta}{1 + \tan\theta} = \tan(45° - \theta).$$

Hence, as in the similarly formed expression of Chap. VIII, Art. 14, we have

$$\tan \tfrac{1}{2}(CL - BL) = \tan(45° - \theta) \tan \tfrac{1}{2}(CL + BL) \text{ ...... (4).}$$

But $CL + BL = CB$ is known.

Hence the right side of (4) can be computed, and $CL - BL$ found. Hence, CL and BL are both determined.

If the same process is applied to (3*) and (3**), we can find the remaining angular elements.

20. Again, if the right sides of (3), (3*), (3**) are multiplied together, and also the left sides, we have

$$\frac{b}{c}\frac{c}{a}\frac{a}{b} = 1 = \frac{\sin BL}{\sin CL}\frac{\sin CM}{\sin AM}\frac{\sin AN}{\sin BN}\frac{\sin CA}{\sin AB}\frac{\sin AB}{\sin BC}\frac{\sin BC}{\sin CA}.$$

Hence, $$\frac{\sin D}{\sin D_{,}}\frac{\sin E}{\sin E_{,}}\frac{\sin F}{\sin F_{,}} = 1 \text{.................. (5).}$$

We have therefore only five independent elements.

21. The chief problems of the crystallographer are: the determination of the face-symbols and elements of the crystal when the angles are measured; or, from a knowledge of the elements of the crystal and the face-indices, to determine the true values of the angles. The angles given in descriptive works are usually those calculated from the angles selected to give the elements. For the solution of these problems the anharmonic ratio of four tautozonal faces is the relation of most general applicability, and is one of great accuracy. In order to apply it to the solution of the first problem, the indices of at least three faces in the zone and all the angles must be known. Then the indices of every other face in the zone can be calculated by taking each in turn with the three known faces.

For the converse problem, we must be satisfied as to the correctness of the symbols of the faces and of at least two angles which are not together equal to two right angles. If the given angles are adjacent, we use the transformation of the A. R. given in Chap. VIII, Art. 14. If the two given angles are not adjacent, we must employ

the less convenient transformation given in Chap. VIII, Art. 18. In crystals, such as that of anorthite, Fig. 120, having numerous faces, the solution can be carried out in a systematic way so as to determine, by the transformation of Chap. VIII, Art. 14, most of the angles from a few known ones. The remaining angles, which have to be computed, can then, generally, be found by the solution of spherical triangles.

22. If, however, a face does not lie in a conspicuous zone, and if the angles, which it makes with two or three known faces, can alone be measured, the determination of the symbol involves somewhat laborious computation, in which oblique-angled spherical triangles enter. We shall only give the case, in which the angles between it and two of the axial planes are known.

The position of a pole $P\,(hkl)$ is fixed, if its arc-distances from two of the axial poles are known; for, by the construction given in Chap. VII, Arts. 19 and 20, we can then place it on the projection. Let us suppose that the arcs BP and CP are known. Produce, in Fig. 123, the zone-circles $[BP]$ and $[CP]$ to meet the opposite axial zones $[CA]$, $[AB]$ in $M_1\,(h0l)$ and $N_1\,(hk0)$ respectively. The symbols of M_1 and N_1 are obtained by Weiss's zone-law, for each of them is the intersection of two zone-circles.

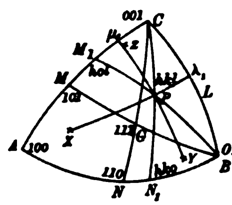

Fig. 123.

Suppose the great circle XP to be drawn and to meet $[BC]$ in λ_1. The great circle XP is not a zone-circle; and λ_1 is not a pole, but is a point at 90° from X useful in the calculation. Hence,

$$\cos XP = \sin P\lambda_1.$$

Similarly, the great circles YP, ZP are drawn to meet $[CA]$ and $[AB]$ in μ_1, ν_1, respectively; of which the former is alone shown in the figure. Then $\cos YP = \sin P\mu_1$, and $\cos ZP = \sin P\nu_1$.

But, since X is the pole of the great circle $[BC]$, the angles at λ_1 are right angles. Hence, by Napier's rules,

$$\left.\begin{array}{r} \cos XP = \sin P\lambda_1 = \sin BP \sin PBC = \sin CP \sin PCB. \\ \text{Similarly, we have right angles at } \mu_1 \text{ and } \nu_1; \\ \text{and } \cos YP = \sin P\mu_1 = \sin CP \sin PCA = \sin AP \sin PAC \\ \cos ZP = \sin P\nu_1 = \sin AP \sin PAB = \sin BP \sin PBA \end{array}\right\} \dots (6).$$

But the equations of the normal $P(hkl)$ are

$$\frac{a\cos XP}{h}=\frac{b\cos YP}{k}=\frac{c\cos ZP}{l}\ldots\ldots\ldots\ldots\ldots(7);$$

substituting from (6) for $\cos XP$ and $\cos YP$, we have

$$\frac{a\sin CP\sin PCB}{h}=\frac{b\sin CP\sin PCA}{k};$$

or cancelling the common factor, $\sin CP$,

$$\left.\begin{aligned}&\frac{a\sin PCB}{h}=\frac{b\sin PCA}{k}.\\ &\text{By similar substitutions from (6), we have}\\ &\frac{b\sin PAC}{k}=\frac{c\sin PAB}{l},\\ &\frac{c\sin PBA}{l}=\frac{a\sin PBC}{h}.\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The rule of these three equations is clear and simple.

If the elements of the crystal are given, either in terms of a, b, c and the axial angles, or as angular elements D, $D_{,}$, &c., the ratios $h:k$, and $k:l$ can be found, provided that four of the angles in the above expressions can be calculated. But $\wedge BP$ and $\wedge CP$ are supposed to be known, and the arc $BC=D+D_{,}$, is also known. Hence in the triangle BPC the three sides are known and the angles PCB, PBC can be calculated (McL. and P. *Spher. Trig.* I, p. 47). But since the elements are known, the whole angles BCA and CBA can be computed by the same formulæ from the arcs BC, CA, and AB; or, if the axial angles are given, the angle

$$BCA=180^\circ-XY,\quad\text{and}\quad CBA=180^\circ-ZX.$$

Hence, the angles PCA and PBA are found, and four of the angles involved in (8) are determined. The problem of finding the ratios of the indices is, therefore, completely solved.

23. By the aid of the relations (3) of Art. 18, the three equations (8) can be thrown into forms more convenient for computation from the angular elements:

$$\left.\begin{aligned}\frac{h}{k}&=\frac{a}{b}\frac{\sin PCB}{\sin PCA}=\frac{\sin AN}{\sin BN}\frac{\sin BC}{\sin CA}\frac{\sin PCB}{\sin PCA};\\ \frac{k}{l}&=\frac{b}{c}\frac{\sin PAC}{\sin PAB}=\frac{\sin BL}{\sin CL}\frac{\sin CA}{\sin AB}\frac{\sin PAC}{\sin PAB};\\ \frac{l}{h}&=\frac{c}{a}\frac{\sin PBA}{\sin PBC}=\frac{\sin CM}{\sin AM}\frac{\sin AB}{\sin BC}\frac{\sin PBA}{\sin PBC}.\end{aligned}\right\}\ldots\ldots(9).$$

Further, we have seen that N_1 is $(hk0)$ and M_1 is $(h0l)$. Hence, from the A. R. $\{ANN_1B\}$ and from the A. R. $\{CMM_1A\}$, we can get expressions for the ratios $h \div k$, and $l \div h$ which involve the arcs on the axial zones. Since the ratios of the indices are the same, the expressions will be equivalent to those given in (9).

From the A. R. $\{ANN_1B\}$, we have

$$\left.\begin{aligned} &\frac{\sin AN}{\sin AN_1} \div \frac{\sin BN}{\sin BN_1} = \frac{\begin{vmatrix}100\\110\end{vmatrix}}{\begin{vmatrix}100\\hk0\end{vmatrix}} \div \frac{\begin{vmatrix}010\\110\end{vmatrix}}{\begin{vmatrix}010\\hk0\end{vmatrix}} = \frac{h}{k}\,; \\ &\text{from A. R. } \{BLL_1C\}, \quad \frac{\sin BL}{\sin BL_1} \div \frac{\sin CL}{\sin CL_1} = \frac{k}{l}\,; \\ &\text{and from A. R. } \{CMM_1A\}, \quad \frac{\sin CM}{\sin CM_1} \div \frac{\sin AM}{\sin AM_1} = \frac{l}{h}\,. \end{aligned}\right\} \ldots\ldots(10).$$

24. We can prove directly that each of the expressions (10) is equal to the corresponding expression in (9). For, in the triangle BN_1C, we have

$$\frac{\sin BN_1}{\sin BC} = \frac{\sin N_1CB}{\sin BN_1C} = \frac{\sin PCB}{\sin BN_1C}\,;$$

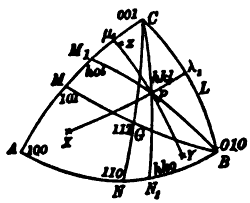

FIG. 123.

and in the triangle AN_1C, we have

$$\frac{\sin AN_1}{\sin CA} = \frac{\sin N_1CA}{\sin AN_1C} = \frac{\sin PCA}{\sin BN_1C}\,;$$

since $AN_1C = 180° - BN_1C$.

Hence, dividing the first by the second, we have

$$\frac{\sin BN_1}{\sin AN_1} = \frac{\sin BC}{\sin CA}\frac{\sin PCB}{\sin PCA}.$$

Therefore

$$\frac{h}{k} = \frac{\sin AN}{\sin BN}\frac{\sin BN_1}{\sin AN_1} = \frac{\sin AN}{\sin BN}\frac{\sin BC}{\sin CA}\frac{\sin PCB}{\sin PCA},$$

which is the expression given in (9). The other expressions can be proved to be identical in an exactly similar manner.

25. The converse problem of finding AP, BP and CP, when the symbol (hkl) of P is known, is solved by determining the arcs BL_1, CM_1, AN_1, &c., and the angles PBC, PBA, &c.

From equations (9) we have

$$\frac{\sin PCB}{\sin PCA} = \frac{h}{k}\frac{b}{a}, \text{ or } = \frac{h}{k}\frac{\sin BN}{\sin AN}\frac{\sin CA}{\sin BC} = \tan\theta.$$

By hypothesis we know either a, b, c, or the equivalent angles AN, BN, &c. Hence, the right side can be computed, and the auxiliary angle θ is determined.

Then, by the transformations employed in adapting a ratio of sines to logarithmic computation (Chap. VIII, Art. 14), we have

$$\tan\tfrac{1}{2}(PCA - PCB) = \tan(45^\circ - \theta)\tan\tfrac{1}{2}(PCA + PCB).$$

But $PCA + PCB = BCA$ is known, or can be calculated from the elements.

$$\left.\begin{aligned} \text{Hence,}\quad & \tan\tfrac{1}{2}(PCA - PCB) = \tan(45^\circ - \theta)\tan\tfrac{1}{2}BCA. \\ \text{Similarly,}\quad & \tan\tfrac{1}{2}(PAB - PAC) = \tan(45^\circ - \phi)\tan\tfrac{1}{2}CAB, \\ & \tan\tfrac{1}{2}(PBC - PBA) = \tan(45^\circ - \psi)\tan\tfrac{1}{2}ABC. \end{aligned}\right\}\ldots(11).$$

Where
$$\tan\phi = \frac{k}{l}\frac{c}{b} = \frac{k}{l}\frac{\sin CL}{\sin BL}\frac{\sin AB}{\sin CA},$$

$$\tan\psi = \frac{l}{h}\frac{a}{c} = \frac{l}{h}\frac{\sin AM}{\sin CM}\frac{\sin BC}{\sin AB}.$$

The angles θ, ϕ, and ψ must be arranged so that they are, each of them, less than 45°. The angles PBC, PBA, &c., can then be all computed.

The arcs AN_1, BN_1 and the two other similar pairs can be computed by a like process. For, from equations (10), we have

$$\frac{\sin BN_1}{\sin AN_1} = \frac{h}{k}\frac{\sin BN}{\sin AN} = \tan\theta_1,$$

$$\left.\begin{aligned} \therefore\ & \tan\tfrac{1}{2}(AN_1 - BN_1) = \tan(45^\circ - \theta_1)\tan\tfrac{1}{2}AB; \\ \text{similarly,}\quad & \\ & \tan\tfrac{1}{2}(BL_1 - CL_1) = \tan(45^\circ - \phi_1)\tan\tfrac{1}{2}BC; \\ & \tan\tfrac{1}{2}(CM_1 - AM_1) = \tan(45^\circ - \psi_1)\tan\tfrac{1}{2}CA. \end{aligned}\right\}\ldots(12).$$

Care must be taken that θ_1, ϕ_1 and ψ_1, obtained from the similar expressions to that given in full for θ_1, are each less than 45°. Since AN, BL, CM, &c., are all known, the angles θ_1, ϕ_1, ψ_1, can be readily computed for any values of h, k, l, which may be taken. Hence, the arcs AN_1, BN_1, BL_1, &c., can be all computed.

26. The values of AP, BP and CP, are now found as follows. From the triangle ACN_1, we have

$$\frac{\sin CN_1}{\sin AN_1} = \frac{\sin CAB}{\sin PCA};$$

and we obtain similar relations from each of the six triangles into which ABC is divided by AL_1, BM_1 and CN_1. Hence,

$$\left.\begin{aligned} \sin AL_1 &= \sin BL_1 \frac{\sin ABC}{\sin PAB} = \sin CL_1 \frac{\sin BCA}{\sin PAC} \\ \sin BM_1 &= \sin CM_1 \frac{\sin BCA}{\sin PBC} = \sin AM_1 \frac{\sin CAB}{\sin PBA} \\ \sin CN_1 &= \sin AN_1 \frac{\sin CAB}{\sin PCA} = \sin BN_1 \frac{\sin ABC}{\sin PCB} \end{aligned}\right\} \ldots (13).$$

Hence, AL_1, BM_1, and CN_1 can be computed.

Again, from the triangles APC and PN_1A, we have

$$\frac{\sin CP}{\sin CA} = \frac{\sin PAC}{\sin APC} \ldots\ldots\ldots\ldots (14),$$

and

$$\frac{\sin PN_1}{\sin AN_1} = \frac{\sin PAB}{\sin APN_1} \ldots\ldots\ldots\ldots (15).$$

But

$$APC + APN_1 = 180°,$$

$$\therefore \sin APC = \sin APN_1.$$

Hence, dividing (14) by (15),

Similarly,

$$\left.\begin{aligned} \frac{\sin CP}{\sin PN_1} &= \frac{\sin CA}{\sin AN_1}\frac{\sin PAC}{\sin PAB}. \\ \frac{\sin AP}{\sin PL_1} &= \frac{\sin AB}{\sin BL_1}\frac{\sin PBA}{\sin PBC}, \\ \frac{\sin BP}{\sin PM_1} &= \frac{\sin BC}{\sin CM_1}\frac{\sin PCB}{\sin PCA}. \end{aligned}\right\} \ldots\ldots\ldots\ldots (16).$$

The values on the right side can be computed and made equal to $\tan \Theta$, $\tan \Phi$, $\tan \Psi$, respectively.

Hence,

$$\left.\begin{aligned} \tan \tfrac{1}{2}(PN_1 - CP) &= \tan(45° - \Theta) \tan \tfrac{1}{2} CN_1; \\ \tan \tfrac{1}{2}(PL_1 - AP) &= \tan(45° - \Phi) \tan \tfrac{1}{2} AL_1; \\ \tan \tfrac{1}{2}(PM_1 - BP) &= \tan(45° - \Psi) \tan \tfrac{1}{2} BM_1. \end{aligned}\right\} \ldots\ldots (17).$$

Hence, AP, BP, CP, PL_1, &c., can be computed.

27. *Example.* To illustrate the application of some of the preceding formulæ, we take the crystal of oligoclase described by vom Rath (*Pogg. Ann.* CXXXVIII, p. 464, 1869). The forms observed were: $P\{001\}$, $M\{010\}$, $f\{130\}$, $l\{110\}$, $h\{100\}$, $T\{1\bar{1}0\}$, $z\{1\bar{3}0\}$, $y\{\bar{2}01\}$, $r\{\bar{4}03\}$, $x\{\bar{1}01\}$, $e\{021\}$, $n\{0\bar{2}1\}$, $p\{\bar{1}11\}$, $g\{\bar{2}21\}$, $o\{\bar{1}\bar{1}1\}$, $u\{\bar{2}\bar{2}1\}$. The zonal relations of these forms are shown in the plan, Fig. 124, and in the stereogram, Fig. 125, already employed in the discussion of the crystals of anorthite.

Apparently, the most trustworthy angles measured were:—$MP = 86° 32'$, $M'T = 61° 40'$, $PT = 68° 48'$, $Pu = 84° 57'$ and $M'u = 58° 13'$. It is required to determine from these five angles the parametral ratios and the angles between the axes.

The equations of the normal $u\,(\bar{2}\bar{2}1)$ are:

$$\frac{a\cos Xu}{-2} = \frac{b\cos Yu}{-2} = \frac{c\cos Zu}{1};$$

or taking X' and Y', the axial points opposite to X and Y:

$$\frac{a\cos X'u}{2} = \frac{b\cos Y'u}{2} = \frac{c\cos Zu}{1} \quad\ldots\ldots\ldots\ldots\ldots(18).$$

But from equations (6) of Art. 22,

$$\left.\begin{aligned} \cos X'u &= \sin M'u \sin PM'u = \sin Pu \sin M'Pu, \\ \cos Y'u &= \sin Pu \sin h'Pu = \sin h'u \sin Ph'u, \\ \cos Zu &= \sin h'u \sin M'h'u = \sin M'u \sin h'M'u. \end{aligned}\right\} \quad\ldots\ldots\ldots(19).$$

All the sides of the two triangles $M'Pu$, $M'PT$ being known, we can compute all the angles by the well-known formula (McL. and P. *Spher. Trig.* I, p. 47). The

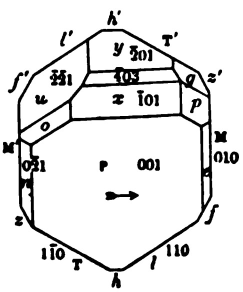

Fig. 124.

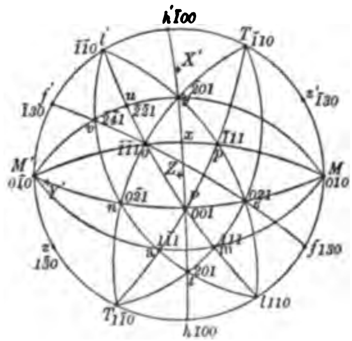

Fig. 125.

angles are:—$M'Pu = 57° 39'$, $PM'u = 81° 52{\cdot}7'$, $M'uP = 97° 15'$; $TPM' = 57° 45{\cdot}4'$, $M'TP = 106° 25{\cdot}6'$, $TM'P = 63° 37{\cdot}4' = 180° - \beta$.

$\therefore\ hTP = 180° - M'TP = 73° 34{\cdot}4'$, $h'M'u = 180° - PM'u - TM'P = 34° 30'$, $TPu = 115° 24{\cdot}4'$.

Hence, from (18) and (19),

$$\frac{a}{c} = \frac{2\cos Zu}{\cos X'u} = \frac{2\sin M'u \sin h'M'u}{\sin M'u \sin PM'u} = \frac{2\sin(h'M'u = 34° 30')}{\sin(PM'u = 81° 52{\cdot}7')} = 1{\cdot}1443\ldots(20).$$

This expression could have been at once obtained by employing equations (8) of Art. 22.

But other angles must be determined before either of the other parametral ratios can be found. Now one of the useful properties of the anharmonic ratio of four poles in a zone-circle is the fact, that the four arcs on the circle can be replaced by the four angles included between the great circles which join the four poles to any other pole outside the zone (Chap. VIII, Art. 19). Hence the A. R. $\{hTM'l'\}$ is equal to that of the four angles between the zone-circles joining each of the poles to P. This may be expressed symbolically thus:— A. R. $\{hTM'l'\}$ = A. R. $\{P . hTM'l'\}$.

But we know the symbols of the four poles; hence

$$\text{A. R. } \{hTM'l'\} = \begin{vmatrix} 100 \\ 1\bar{1}0 \\ \hline 100 \\ 0\bar{1}0 \end{vmatrix} \div \begin{vmatrix} \bar{1}\bar{1}0 \\ 1\bar{1}0 \\ \hline \bar{1}\bar{1}0 \\ 0\bar{1}0 \end{vmatrix} = \frac{1}{2}.$$

$$\therefore \text{ A. R. } \{P . hTM'l'\} = \frac{\sin hPT}{\sin hPM'} \div \frac{\sin l'PT}{\sin l'PM'} = \frac{1}{2} \quad \text{............(21).}$$

But $\quad l'PM' = M'Pu = 57° \, 39', \; l'PT = TPu = 115° \, 24{\cdot}4' = 180° - 64° \, 35{\cdot}6'$;

and $\quad hPM' - hPT = TPM' = 57° \, 45{\cdot}4'$.

Hence,

$$\frac{\sin hPT}{\sin hPM'} = \frac{\sin l'PT}{2 \sin l'PM'} = \frac{\sin 64° \, 35{\cdot}6'}{2 \sin 57° \, 39'} = \tan (\theta = 28° \, 7{\cdot}75'), \text{ by computation.}$$

$$\therefore \tan \tfrac{1}{2}(hPM' - hPT) = \tan 16° \, 52{\cdot}25' \tan \tfrac{1}{2}(hPM' + hPT);$$

$$\therefore \tan \tfrac{1}{2}(hPM' + hPT) = \tan 28° \, 52{\cdot}7' \div \tan 16° \, 52{\cdot}25'.$$

$$\therefore \tfrac{1}{2}(hPM' + hPT) = 61° \, 11{\cdot}8',$$

$$\tfrac{1}{2}(hPM' - hPT) = 28° \, 52{\cdot}7',$$

$$\therefore \gamma = hPM' = 90° \, 4{\cdot}5'.$$

Also, $\quad hPT = 32° \, 19{\cdot}1'$, and $h'Pu = 180° - hPT - TPu = 32° \, 16{\cdot}5'$.

Hence, $\quad \dfrac{a}{b} = \dfrac{\cos Y'u}{\cos X'u} = \text{(from (19))} \; \dfrac{\sin h'Pu}{\sin M'Pu} = \dfrac{\sin 32° \, 16{\cdot}5'}{\sin 57° \, 39'}$;

and $\quad \dfrac{c}{b} = \dfrac{c}{a}\dfrac{a}{b} = \dfrac{\sin 81° \, 52{\cdot}7'}{2 \sin 34° \, 30'} \, \dfrac{\sin 32° \, 16{\cdot}5'}{\sin 57° \, 39'}$.

If $b = 1$, $\therefore a = {\cdot}6321$, and $c = {\cdot}5524$.

It remains to calculate $PhM' = \alpha$, and $hP = (100 \wedge 001)$.

A great circle is drawn from P perpendicular to the zone-circle $[M'Tl]$ to meet it, say, at Q. The right-angled triangle TPQ gives, by Napier's rules, $TQ = 36° \, 5{\cdot}7'$, $PQ = 63° \, 24{\cdot}9'$, and $TPQ = 39° \, 11{\cdot}3'$.

From these and the known angles hPT and hTP, we find, from the right-angled triangle hPQ, $hQ = 6° \, 9'$, $hP = 63° \, 34{\cdot}75'$, $hPQ = 6° \, 52{\cdot}25'$, and $PhT = 93° \, 4' = \alpha$. Also, $hT = TQ - hQ = 29° \, 56{\cdot}7'$; and $hM' = hT + TM' = 91° \, 36{\cdot}7'$.

The elements of the crystal are therefore:—

$$\alpha = YOZ = 93° \, 4', \; \beta = ZOX = 116° \, 22{\cdot}6', \; \gamma = XOY = 90° \, 4{\cdot}5';$$

$$a : b : c = {\cdot}6321 : 1 : {\cdot}5524.$$

We also know the [illegible] $\lambda M = 86° 28{\cdot}5'$, $MP = 6[illegible]$ [illegible] formulæ (3) and (4) to [illegible] the angles Pc, $P[illegible]$ [illegible] from the right-angled [illegible] known. For [illegible] known.

Instead of [illegible] which is given [illegible] Napier's [illegible] PA [illegible] results [illegible]

CHAPTER XII.

THE OBLIQUE SYSTEM.

1. THIS system includes three classes:

I. That in which a dyad axis occurs alone;

II. That in which a plane of symmetry occurs alone;

III. That in which a dyad axis, a centre of symmetry, and a plane of symmetry perpendicular to the dyad axis are associated together. In Chap. IX, Prop. 4, it was established that a centro-symmetrical crystal, having a dyad axis or a plane of symmetry, must have the third element which also occurs in class III.

From Chap. IX, Props. 1, 2 and 3, it follows that, in crystals of all three classes, there is one plane which is parallel to a possible face and perpendicular to a possible zone-axis. In classes II and III this plane is a plane of symmetry, the normal to which is a possible zone-axis. In classes I and III the zone-axis is a dyad axis, and is, therefore, perpendicular to a possible face.

2. Hence, in the possible face just mentioned there are a number of possible edges which are all perpendicular to the zone-axis, but are not necessarily, and indeed only very exceptionally, at 90° to one another. The prominent zone-axis, which is either a dyad axis or the normal to a plane of symmetry, will in all cases be taken as the axis of Y. As the axes of X and Z, it is most convenient to take lines parallel to a pair of edges at right angles to OY. Any pair will do, but for the sake of simplicity in the symbols, the most conspicuous zone-axes are usually taken; and the positive directions are taken to include the obtuse angle. The choice being arbitrary, the angle between these axes is an element of the crystal which must be specified. The acute angle $X_{,}OZ$ is denoted by β in

Dana's *Mineralogy* and in Hintze's *Mineralogie*, whilst the same letter is used in Groth's *Zeitschr. f. Kryst. u. Min.* to denote the obtuse angle XOZ. We shall take β to be the acute angle $X_{,}OZ$.

Any face of the crystal meeting the axes at finite distances from the origin may be taken to determine the parameters $a : b : c$. Hence, the elements of the crystal are:

$$XOY = YOZ = 90°, \ X_{,}OZ = \beta; \text{ and } a : b : c.$$

But, in general, only those elements which vary with the substance are given; the constant elements being known from the system. Thus, in oblique crystals the angle β and the parameters are those which have to be specially given in the description.

3. The parameters $a : b : c$ may have any values, and are not, as a rule, in any simple ratios to one another. Their values may, by the aid of the equations of the normal (Chap. IV, Art. 15), be determined from any face meeting all three axes at finite distances from the origin. It may, occasionally, happen that different crystallographers adopt different edges for X and Z, and also a different parametral face. A difference in the stated value of β, or in the indices ascribed to faces inclined to one another at the same angles, will show the reader whether the same axes and parameters are used or not.

I. *Hemimorphic class; a* $\{hkl\}$.

4. Crystals of this class possess a single dyad axis and no other element of symmetry. The faces occur in pairs which change places on rotation through 180° about the dyad axis, except in the single case in which the face is perpendicular to the axis. We know that the dyad axis is perpendicular to a possible face (Chap. IX, Prop. 3), the position of which remains the same when the crystal is rotated about the axis. This face constitutes the form called a pedion.

The dyad axis being a possible zone-axis, faces parallel to it are possible. The pair of such interchangeable faces are parallel and constitute a pinakoid. Such forms appear to show a centre of symmetry, which will not be the case with other forms.

5. We shall throughout, in each class and system, use the term *special forms* to indicate forms, the faces of which are parallel to planes or axes of symmetry, or are perpendicular to them, or

are related to the elements of symmetry in such a way as to give rise to a peculiarity of figure distinguishing the forms from those of the general case. We shall speak of the *general form*, when we wish to emphasize the fact that the faces have no exceptional relation to the elements of symmetry, but occupy any general position. The general form may be said to have the characteristic configuration belonging to the class.

6. The general form of this class consists of two inclined faces which intersect in an edge perpendicular to the dyad axis. Such a form is represented by the two faces of Fig. 126, in which, however, the dyad axis is placed vertically. By transposing the figure, we may suppose the faces to be brought into a position represented by Fig. 127, in which the dyad axis—now taken to be OY—is perpendicular to the paper, and meets the edge EF (represented by a continuous line) at a distance $b \div k$ from the origin: the axes of X and Z lie in the paper, and meet the faces at the points H, $H_{,}$ and L, $L_{,}$, respectively—the traces of the faces being given by the discontinuous lines. Hence, if $OH = -OH_{,} = a \div h$, and $OL = -OL_{,} = c \div l$, the two faces have the symbols (hkl), $(\bar{h}k\bar{l})$, respectively. For it is clear that a rotation through 180° about the perpendicular to the paper, OY, interchanges equal lengths on $XX_{,}$, measured on opposite sides of O; and, likewise, equal positive and negative lengths on $ZZ_{,}$. The form $a\{hkl\}$ consists, therefore, of the faces hkl, $\bar{h}k\bar{l}$.

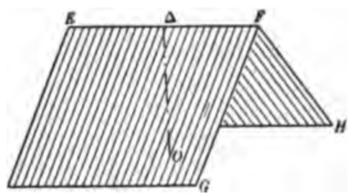

Fig. 126.

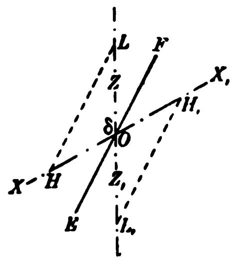

Fig. 127.

7. The special forms are: *(1)* pedions, *(2)* pinakoids.

1. The pedion is either (010) or (0$\bar{1}$0), according as it meets the axis of Y on the positive, or negative, side of the origin. Both are possible, for they are parallel to two zone-axes OX and OZ; but the presence of one does not involve that of the other. Owing to the geometrical relation between them, they are called complementary pedions.

2. The pinakoid $\{h0l\}$ consists of the two faces $h0l$ and $\bar{h}0\bar{l}$; these faces being parallel to one another and to the dyad axis. Particular cases of the pinakoid, such as $\{100\}$ and $\{001\}$, differ in no essential respect from $\{h0l\}$: they only indicate the pairs which have been selected to give the axes of Z and X.

3. The crystals have a different facial development at opposite ends of the dyad axis, which is one of uniterminal symmetry. Such axes are also called *polar* or *hemimorphic*. We shall use the latter word to denote this class of the system (Groth's *sphenoidal* class).

The crystals belonging to this class, which have been examined, have been found to manifest, during change of temperature, opposite electrifications at the two ends of the dyad axis, which is then called a pyro-electric axis; and this character may be expected to distinguish all crystals of the class. It is found that, at one end, positive electrification is manifested with rise of temperature and negative electrification with fall of temperature. This end is known as the *analogous pole*. The opposite end is called the *antilogous pole;* and is that, at which positive electrification is manifested whilst the temperature is falling, and negative electrification as the temperature rises. By carefully dusting the crystal with a mixture of red lead and flowers of sulphur, forced through a very fine gauze, the different electrification is rendered very manifest. In passing through the gauze, the red lead becomes positively electrified, the sulphur negatively electrified. Hence, as the mixture falls on the electrified crystal, the red lead is attracted to the one pole, and the sulphur to the opposite pole. If, as is generally the case, the electrification is examined whilst the crystal is cooling, the red lead is attracted to the neighbourhood of the analogous pole whilst the pale yellow sulphur is attracted to the antilogous pole.

Solutions of crystals of organic substances of this class, also, rotate the plane of polarization of a beam of light; and it is found that solutions of opposite rotation can be obtained from crystals formed of the same chemical constituents combined in the same proportions. Thus, crystals of tartaric acid ($C_4H_6O_6$), Fig. 128, an be obtained, showing the forms: $c\{001\}$, $r\{101\}$, $a\{100\}$, $x\{10\bar{1}\}$, $m = a\{110\}$, $q = a\{011\}$, and $m_{,} = a\{1\bar{1}0\}$. In these crystals the analogous pole is at the end of the horizontal dyad axis to the left at which the pair of faces $m_{,}$ occur alone, and the antilogous pole at the end where both the forms m and q occur on the right. A solution of such crystals rotates the

plane of polarization to the right, *i.e.* with the hands of a watch to an observer receiving the light. The crystals may be described as those of dextro-tartaric acid.

Crystals of tartaric acid having the same general appearance but in which the development at the ends of the dyad axis is reversed, can be obtained. Their solution rotates the plane of polarization to the left, *i.e.* against the hands of a watch to an observer receiving the light; and, for equal paths traversed in solutions of equal strength, the amount of rotation produced is the same for crystals of both kinds. The forms c, r, a and x, are developed as before, but on the right the faces $m = a\{110\}$ occur alone; and this end of the dyad axis is the analogous pole. The antilogous pole is on the left, where the faces $m_{,} = a\{1\bar{1}0\}$ and $q_{,} = a\{0\bar{1}1\}$ occur together. Were a crystal of this latter type placed with the faces c, r, a, x all parallel to those of the one shown in Fig. 128, it would be seen to be its reciprocal reflexion. Hence, the crystals of dextro- and of lævo-tartaric acid are enantiomorphous. These two bodies and pairs of similarly correlated organic substances, which give dextro- and lævo-gyral solutions, are called *stereo-isomers.*

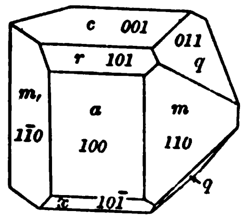

FIG. 128.

The elements, and optical characters, of the crystals are the same whether their solutions rotate the plane of polarisation to the right or to the left. The elements are:

$$\beta = 79^\circ 43'; \; a : b : c = 1{\cdot}2747 : 1 : 1{\cdot}0266.$$

The plane of the optic axes contains the dyad axis, which coincides with the obtuse bisectrix. The acute bisectrix lies in the acute angle $ZOX_{,}$, and makes with OZ an angle of 71° 18′ for red light, and of 72° 10′ for blue light.

9. Fig. 129 represents a crystal of cane-sugar ($C_{12}H_{22}O_{11}$) having the forms: $a\{100\}$, $c\{001\}$, $r\{10\bar{1}\}$, $m = a\{110\}$, $m_{,} = a\{1\bar{1}0\}$, $q_{,} = a\{0\bar{1}1\}$, $o = a\{1\bar{1}1\}$.

$\beta = 76^\circ 31'$; $a : b : c = 1{\cdot}259 : 1 : {\cdot}878$.

The corrosion-figures on m and $m_{,}$ are different; and the antilogous pole is situated at the end of the dyad axis at which the faces o, $q_{,}$, and $m_{,}$, are developed.

O.A. ‖ XOZ; $Bx_a \wedge OZ = +67{\cdot}75^\circ$. The angle of the optic axes varies considerably

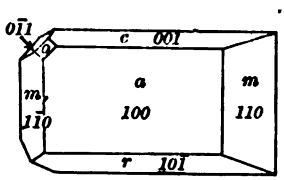

FIG. 129.

with change of temperature; and, likewise, the positions of the bisectrices. Des Cloizeaux found that with red light Bx_a was displaced through an apparent angle in air of 1° 35′ between 17° and 121° C. The solution rotates the plane of polarization to the right.

Fig. 130 represents a crystal of quercite ($C_6H_{12}O_5$) with the forms: $c\{001\}$, $r\{10\bar{1}\}$, $m=a\{110\}$, $m_{,}=a\{1\bar{1}0\}$, $q=a\{011\}$.

$$\beta=69° \ 50'; \ a:b:c=\cdot7935:1:\cdot7533.$$

O.A. $\parallel XOZ$; $Bx_a \wedge OZ = -11° \ 49'$ (lithium flame),
$= -11° \ 22'$ (thallium flame).

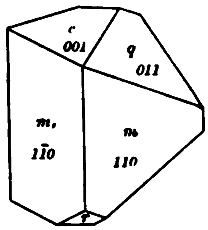

FIG. 130.

Des Cloizeaux gives $2E = 55° \ 30'$ (red light), 58° 25′ (blue light), at 19° C. With rise of temperature the angle diminishes rapidly; and at 121° C. $2E=37° \ 12'$ (red light). The acute bisectrix undergoes an apparent displacement of 1° 26′ between the same temperatures.

II. *Gonioid class*; $\kappa\{hkl\}$.

10. The crystals of this class have a single plane of symmetry Σ and no other element of symmetry. The plane Σ must be parallel to possible faces, which form a pinakoid. It is also perpendicular to a possible zone-axis which is taken to be the axis OY. Hence, we get a series of possible faces all perpendicular to Σ, which are necessarily divided symmetrically by it, and constitute special forms which are pedions. Two of these pedions are selected to give the axial planes XOY, ZOY; and have therefore the symbols $\kappa\{001\}$, $\kappa\{100\}$, and the angle between them β is the angle $X_{,}OZ$. Any other pedion will be $\kappa\{h0l\}$, and consists of the single face $(h0l)$.

The special form $\{010\}$ is a pinakoid consisting of the faces (010), $(0\bar{1}0)$ parallel to Σ.

11. Any other face (hkl) is repeated in a like face over Σ in such a manner that, if both are equally distant from the origin—a point in Σ—the line of intersection lies in Σ. The plane of symmetry also bisects the angle between them. The faces, therefore, meet the axis of Y at equal distances on opposite sides of the origin. Hence the form $\kappa\{hkl\}$ consists of the faces (hkl), $(h\bar{k}l)$; and is a hemihedral form with inclined faces. The class will therefore be called the *gonioid* class, from γωνία, an angle: it is also known as the *hemihedral*, *clinohedral*, or *domatic*, class.

12. Crystals of potassium tetrathionate ($K_2S_4O_6$), Fig. 131, belong to this class. The elements are: $\beta = 75° 44'$;

$$a : b : c = \cdot 93 : 1 : 1\cdot 26.$$

The forms are: $a = \kappa\{100\}$, $a_{,} = \kappa\{\bar{1}00\}$, $c = \kappa\{001\}$, $m = \kappa\{110\}$, $m' = \kappa\{\bar{1}10\}$, $q = \kappa\{011\}$, $o = \kappa\{11\bar{1}\}$, $v = \kappa\{13\bar{3}\}$, $p = \kappa\{\bar{1}1\bar{1}\}$.

The optic axes lie in the plane Σ, and one optic axis is inclined at a small angle to the normal (100).

The faces of the complementary forms m and m' should show differences in the corrosion-figures and also in the electrification during change of temperature.

Hydrogen trisodic hypophosphate ($HNa_3P_2O_6 . 9H_2O$) has, also, been placed in this class (see Art. 31).

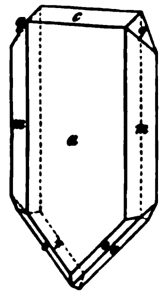

FIG. 131.

The crystals of clinohedrite, (ZnOH) (CaOH) SiO_3, have been shown by Messrs. Penfield and Foote (*Am. Jour. of Sci.* [iv] v, p. 289, 1898) to belong to this class; and to have the elements: $\beta = 76° 4'$; $a : b : c = \cdot 6826 : 1 : \cdot 3226$. The crystals show numerous forms; and have a perfect cleavage parallel to (010). O.A. $\perp$ 010, $Bx_a \wedge OZ = -62°$. The crystals are pyro-electric; and as they cool, the portions in the neighbourhood of $\kappa\{111\}$, $\kappa\{101\}$ are positively electrified, whilst those in the neighbourhood of $\kappa\{\bar{1}3\bar{1}\}$, $\kappa\{\bar{1}2\bar{1}\}$ and $\kappa\{\bar{1}0\bar{1}\}$ are negatively electrified.

III. *Plinthoid class;* $\{hkl\}$.

13. In crystals of this class the three elements of symmetry, which occurred singly in the two preceding classes and in the pinakoidal class of the anorthic system, are associated together; viz. a centre of symmetry, a plane of symmetry Σ, and a dyad axis δ perpendicular to Σ. The class will be called the *plinthoid* (*prismatic, holohedral*) class, from πλίνθος, a brick.

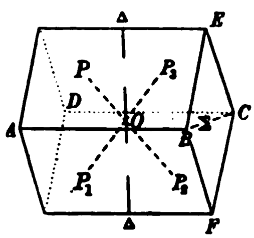

FIG. 132.

14. The general form consists of four tautozonal faces, as shown by the faces of Fig. 132, which are perpendicular to POP_2 and P_1OP_3. The four faces are divided into pairs in three ways:—(1) pairs of parallel faces, (2) pairs which are metastrophic (p. 18), *i.e.* interchangeable on rotation through 180° about δ, (3) pairs which are antistrophic (p. 21), and are reciprocal reflexions with respect to Σ. The edges of intersection are all parallel to Σ, and perpendicular to δ.

15. The special forms are pinakoids; (*1*) the faces of one pinakoid being parallel to Σ; (*2*) those of the others being perpendicular to Σ and parallel to the dyad axis. Of these latter pinakoids there may be several on one and the same crystal.

16. The axes are related to one another as in the two preceding classes; viz. OY coincident with the dyad axis, and perpendicular to the axes OX and OZ which lie in Σ at an angle β to one another. This relation is not affected by the fact that δ and Σ are now both elements of symmetry. The symbols of the pairs of faces interchangeable by rotation about the axis OY are connected by the same rule as was established in Art. 6, for the pairs of metastrophic faces of class I. Hence, if (hkl) is one face, $(\bar{h}k\bar{l})$ is the symbol of the associated face. Since the crystal is centro-symmetrical, this pair is associated with a pair of faces, $(\bar{h}\bar{k}\bar{l})$ and $(h\bar{k}l)$, each parallel to one of the first pair. This second pair obeys the rule for the change of signs of a pair of faces metastrophic about OY. The four faces can be divided into two pairs—(hkl), $(h\bar{k}l)$ and $(\bar{h}k\bar{l})$, $(\bar{h}\bar{k}\bar{l})$—which obey the rule connecting pairs of faces symmetrical with respect to the plane XOZ. Hence, no other faces are associated together; and the form $\{hkl\}$ consists of the faces: hkl, $\bar{h}k\bar{l}$, $h\bar{k}l$, $\bar{h}\bar{k}\bar{l}$. It constitutes what is called an *open* form, for it does not by itself enclose a finite portion of space. No completely developed crystal of the oblique system can, therefore, consist of a single form.

17. If the face through HL of Fig. 127 is parallel to OY, it is perpendicular to the paper and the two faces through HL, on opposite sides of the plane of symmetry, become coincident. Hence, we have only the two parallel faces $(h0l)$ and $(\bar{h}0\bar{l})$ through HL and $H_{,}L_{,}$. They constitute the pinakoid $\{h0l\}$. The particular case, obtained by drawing planes through H and $H_{,}$ parallel to OZ, gives the pinakoid $\{100\}$ with faces (100) and $(\bar{1}00)$. Similarly, by drawing faces through L and $L_{,}$ parallel to OX we obtain the pinakoid $\{001\}$ with faces (001) and $(00\bar{1})$. These particular cases only differ from the pinakoid $\{h0l\}$, inasmuch as they are selected to give the directions of Z and X, respectively. Usually the most conspicuous faces in the zone [010] are taken for the purpose; but this selection is often modified by other characters of particular substances. Thus, in amphibole, and also in orthoclase, the face (100) is absent or inconspicuous. But in each of these substances a very

conspicuous form occurs to which crystallographers assign the symbol {110}; and the faces of the forms thus placed in a vertical position, and called *prisms*, are inclined to one another at angles of 55° 49′ and 61° 13′, respectively.

The remaining pinakoid {010} has its faces parallel to Σ, and includes the faces (010), ($0\bar{1}0$).

18. The general relations of the axes and parameters are the same for all three classes of the system, and in each of them the angle β and the parametral ratios $a : b : c$ vary with the substance. The form $\alpha\{hkl\}$ of class I is geometrically connected with the form $\{hkl\}$ of class III, inasmuch as the latter may be regarded as made up of the pair of faces in the former and the pair of faces parallel to them. Thus, $\alpha\{hkl\}$ consists of the pair of faces (hkl) and $(\bar{h}k\bar{l})$ which lie on one side of the plane XOZ. Taking with them the parallel faces $(\bar{h}\bar{k}\bar{l})$ $(h\bar{k}l)$ a four-faced figure is obtained which is geometrically the same as $\{hkl\}$ of class III. But, when the crystal belongs to class I, the second pair of faces constitute a form $\alpha\{\bar{h}\bar{k}\bar{l}\}$, the faces of which have a different physical character from those of $\alpha\{hkl\}$; and the two forms do not necessarily occur together. The two forms $\alpha\{hkl\}$, $\alpha\{\bar{h}\bar{k}\bar{l}\}$, are known as *complementary forms*; but, in using this term, the student must understand that it implies no more than the geometrical relation given above.

Similarly, the form $\kappa\{hkl\}$ of class II consists of the pair of faces (hkl), $(h\bar{k}l)$ symmetrically situated with respect to the plane XOZ, which is now a plane of symmetry. By taking the parallel faces, we get a complementary form $\kappa\{\bar{h}\bar{k}\bar{l}\}$ which consists of $(\bar{h}\bar{k}\bar{l})$, $(\bar{h}k\bar{l})$. These two complementary forms make up a four-faced figure geometrically identical with $\{hkl\}$ of class III.

Similar geometrical relations are found to connect forms of different classes in each of the other systems. The form of the class of greatest symmetry was said to be holohedral; those of the other classes being stated to be hemihedral when the number of faces was one-half of that in the holohedral class, and tetartohedral when the number was one-fourth. As this mode of expression is still used in descriptive works, we shall in all cases indicate the classes which were regarded as holohedral, and give the corresponding name for the classes having merohedral forms.

19. The reader should notice that the special form $\{h0l\}$ has the same pair of faces, and the same development, whether it

belongs to a crystal of class I or class III; and similarly, that {010} has the same development, whether the crystal belongs to class II or class III. Hence it may not always be possible to distinguish from the development of the crystal-forms the class to which a crystal really belongs, and physical considerations, such as those of pyro-electricity, the corrosion of faces by solvents, &c., may be needed to determine the class and even the system.

20. This class includes the crystals of many of the most common constituents of rocks:

Pyroxene, (Ca, Mg, Fe) SiO_3, and (Ca, Mg, Fe) $SiO_3 + xAl_2O_3$. The forms shown in Fig. 133 are: *a* {100}, *b* {010}, *c* {001}, *m* {110}, *p* {$10\bar{1}$}, *u* {111} and *o* {$22\bar{1}$}. The elements are: $\beta = 74° 10'$; $a : b : c = 1{\cdot}0921 : 1 : {\cdot}5893$.

Amphibole, with composition similar to that of pyroxene, Fig. 134. Elements: $\beta = 75° 2'$; $a : b : c = {\cdot}5318 : 1 : {\cdot}2936$.

Orthoclase, $KAlSi_3O_8$, see Art. 29.

Epidote, $HCa_2(Al, Fe)_3Si_3O_{13}$, see Art. 30.

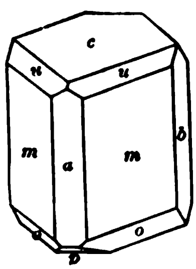

FIG. 133.

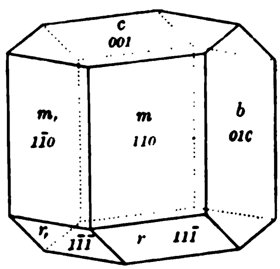

FIG. 134.

Mica, including the species biotite, muscovite, &c., and also the allied clintonite and chlorite groups. Several zeolites:—heulandite, stilbite, harmotome, &c.

Gypsum, $CaSO_4 . 2H_2O$; see Art. 28.

The class also includes a very large number of artificial salts: *e.g.* borax, $Na_2B_4O_7 . 10H_2O$; ferrous sulphate, $FeSO_4 . 7H_2O$; sodium carbonate—the soda of commerce—$Na_2CO_3 . 10H_2O$.

Methods of calculation.

21. Since a similar arrangement of axes is taken in the three classes, and since the general and special forms of classes I and II only differ from those of class III, inasmuch as those of the two former, for the most part, include only one-half of the faces in the latter,

it follows that the analytical formulæ involved in the determination of the face-symbols, parameters, and angles of crystals of all three classes will be the same. They may, indeed, be easily deduced from the corresponding formulæ given in the preceding Chapter by making $XOY = YOZ = 90°$. The angular elements employed in the anorthic system would be modified accordingly, since $BL = D = 90° - CL$, and $AN = F = 90° - BN$. Further, $E + E_{,} = \beta = X_{,}OZ$.

We can, however, easily obtain formulæ which connect β and the ratios $a : b : c$ with the angles between the axial and other faces without reference to those of the anorthic system.

22. In a stereogram of an oblique crystal, such as Fig. 135, the plane of symmetry will always be taken as that of the primitive, and the axis OY is therefore perpendicular to the paper. The axial point Y and the pole B(010) are both projected in the centre, and all zone-circles through B will be projected in diameters of the primitive. OZ is the vertical line in the paper, and is the zone-axis of the zone-circle which is projected in the horizontal diameter $[AmB]$. An arc β is measured on the primitive to the right from Z and gives the negative extremity $X_{,}$ of the axis $XOX_{,}$. This axis is perpendicular to the zone-circle projected in the diameter $[CLB]$. Then $AC = 90° - CZ = ZX_{,} = \beta$. The pole A is (100), and C is (001).

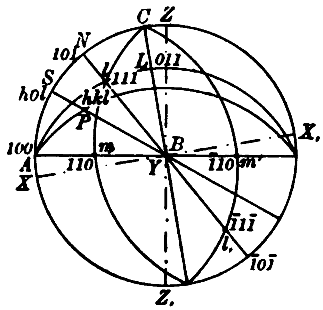

Fig. 135.

To select any other way of projecting the axial points and poles gives much unnecessary trouble. If, for instance, the prism-zone $[AmB]$ is placed in the primitive and B at the right extremity of the horizontal diameter, the centre is the axial point Z and is not the pole of a possible face. We then have to determine the positions of poles by the more cumbrous methods employed in projecting anorthic crystals; and several of the most important zone-circles have radii of inconvenient length.

In order to compare a stereogram, like Fig. 135, with the drawing of a crystal made in the usual way, Fig. 140, we need only suppose the drawing to be turned through 90° about the vertical axis, until the plane XOZ of the drawing comes into the paper and

positive OX is directed towards the left. The positive direction of OY is then directed towards the front.

23. Suppose l of Fig. 135 to be the pole of the parametral face (111)—for the sake of brevity, such a pole may be called the *parametral pole* (111)—and let the zone-circles [AlL], [BlN], [Clm] be drawn to meet the axial zones [BC], [CA] and [AB] in the poles L(011), N(101) and m(110), respectively.

Let P be the pole (hkl), and draw the zone-circle [BP] to meet [CA] in S($h0l$). Then, in the equations of the normal,

$$\frac{a \cos XP}{h} = \frac{b \cos YP}{k} = \frac{c \cos ZP}{l},$$

the angle YP is BP, for Y and B(010) coincide. But the angles XP and ZP, involving the axial points X and Z which are not poles, cannot be obtained by measurement in the ordinary way. We must, therefore, replace $\cos XP$ and $\cos ZP$ by a similar artifice to that used in Chap. XI, Art. 22. Since the zone-circle [BP] meets [CA] in S at right angles, XPS and ZPS are right-angled triangles; and, by Napier's rules, we have:

$$\left.\begin{aligned}\cos XP = \cos XS \cos SP = \sin CS \sin BP,\\ \cos ZP = \cos ZS \cos SP = \sin AS \sin BP.\end{aligned}\right\} \quad \ldots\ldots (1).$$

Hence, the equations of the normal become

$$\frac{a \sin CS \sin BP}{h} = \frac{b \cos BP}{k} = \frac{c \sin AS \sin BP}{l};$$

and, dividing throughout by $\sin BP$, we have

$$\frac{a \sin CS}{h} = \frac{b \cot BP}{k} = \frac{c \sin AS}{l} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Similarly, for the pole l(111) we have, since [Bl] meets CA in N(101),

$$\frac{a \sin CN}{1} = \frac{b \cot Bl}{1} = \frac{c \sin AN}{1} \quad \ldots\ldots\ldots\ldots\ldots (3).$$

Hence $a : b : c = \dfrac{1}{\sin CN} : \dfrac{1}{\cot Bl} : \dfrac{1}{\sin AN} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$

For this reason Miller used the three angles CN, Bl and AN, as the *angular elements* of the crystal. It is clear they also give β. For $AN + CN = AC = \beta$.

24. There is no advantage in giving a series of formulæ of rare application. The above formulæ (2) and (3) enable us to connect the angles with the parameters and face-indices in all cases. It is, however, frequently more advantageous to select a face, such as m, to be (110). When β is known, a knowledge of the angles which the homologous faces m make with one another suffices to fix the ratio $a : b$, but leaves c indefinite. To determine c we can assume some other pole to have known indices consistent with m being (110). The most convenient pole to take is one lying in one or other of the zones $[BC]$, $[CA]$, or $[Cm]$. The formulæ, applicable in these cases, are easily deduced from the general equations (2) and (3), or they can be determined directly by elementary geometry.

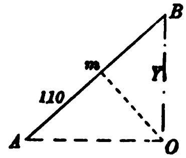

FIG. 136.

Thus, m being (110), we can take the section of the crystal in the plane AOB, when we get the relation shown in Fig. 136. Here AOB is a right angle, OB is the axis of Y, and OA is a normal and *not* an axis. Now the angle $BAO = 90° - OBm = BOm$, an angle supposed to be known.

Hence, $$OA \div OB = \cot BAO = \cot Bm.$$

But, if a section, Fig. 137, is made in the plane XOZ, we have the lines showing the axes and the line AA', parallel to OZ, in which the face m meets the plane of section. Now $\wedge AA'O = \beta$; and

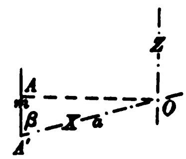

FIG. 137.

$$OA = OA' \sin AA'O = a \sin \beta, \text{ since } OA' = a.$$

$$\therefore \frac{a}{b} = \frac{OA'}{OB} = \frac{OA}{OB}\frac{1}{\sin\beta} = \frac{\cot Bm}{\sin\beta} \ldots\ldots (5).$$

This same expression may be also obtained from equations (2) by making $h = k = 1$ and $l = 0$; and remembering that the pole S becomes coincident with A when P coincides with m.

Equations (2) then reduce to

$$a \sin CA = b \cot Bm; \therefore \frac{a}{b} = \frac{\cot Bm}{\sin\beta} \ldots\ldots\ldots\ldots (5).$$

Similar figures to Figs. 136 and 137, showing the lines in which the face $L(011)$ meets the planes COB and XOZ, give the lengths on the axes OY and OZ; and establish an equation between the parameters b and c similar to (5). It is

$$\frac{c}{b} = \frac{\cot BL}{\sin\beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5^*).$$

This can also be obtained from equations (2) by making P coincide with L, and S with C.

If P of equations (2) is made to coincide with $N(101)$, we have

$$a \sin CN = c \sin AN \text{..................} (5^{**}).$$

This last equation is easily established by Fig. 138, which represents a section of the crystal by the plane XOZ. The face (101) meets the axes of X and Z in the points A', C', where $OA' = a$ and $OC' = c$. The lines OA and OC are the normals (100) and (001), and are at right angles to OC' and OA' respectively. We shall, whenever it is necessary, distinguish between the poles A, C, and the points on the axes at distances a and c from the origin by denoting the latter points by A', C', if measured in the positive directions; and by $A_{,}$, $C_{,}$, if measured in the negative directions.

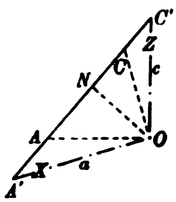

FIG. 138.

Hence

$$\frac{a}{c} = \frac{\sin A'C'O}{\sin C'A'O} = \frac{\sin(90° - C'AO)}{\sin(90° - A'CO)} = \frac{\sin AON}{\sin CON};$$

since the angles at N are 90°.

25. To find the position of P, when the symbol (hkl) and the elements of the crystal are known, we proceed as follows.

Eliminating a, b, c from (2) and (3) we have

$$\frac{\sin CS}{h \sin CN} = \frac{\cot BP}{k \cot Bl} = \frac{\sin AS}{l \sin AN} \text{...... .. ..} (6).$$

$$\therefore \frac{\sin CS}{\sin AS} = \frac{h \sin CN}{l \sin AN} \text{ (known numbers)} = \tan\theta \text{ (say)......} (7).$$

Hence, CS and AS can be computed by the process of transformation given in Chap. VIII, Art 14. Assuming $CS < AS$,

$$\therefore \tan \tfrac{1}{2}(AS - CS) = \tan(45° - \theta) \tan \tfrac{1}{2}(AS + CS) \text{......} (8).$$

Also $AS + CS = \beta$, a known angle.

The arcs AS and CS having been found, BP can be found from equations (6); and the position of P is fully determined.

The expression (7) can be readily deduced from the A.R. $\{CSNA\}$.

26. The arc-distance of any two poles $P(hkl)$ and $Q(pqr)$ may be found from the anharmonic ratio of four poles in their zone, the poles associated with them being either fully known, or being

poles easily determined from our knowledge of the crystal. As a rule this method gives most of the angles which are needed.

When the above method fails, or is too laborious, we determine by the method of Art. 25, the arcs AS and BP for the pole P, and likewise the arcs AS_1 and BQ for the pole Q. Then from the expression, $\cos PQ = \cos BP \cos BQ + \sin BP \sin BQ \cos (AS_1 - AS)$, the angle PQ can be computed.

27. If the two poles P and Q lie in a zone containing $B\,(010)$, we can find a simple relation which enables us to determine the position of the one pole when that of the other is known.

Equations (2) holding for $P\,(hkl)$, we have for $Q\,(pqr)$ lying in the zone $[BPS]$

$$\frac{a \sin CS}{p} = \frac{b \cot BQ}{q} = \frac{c \sin AS}{r} \quad \text{..............} \, (9).$$

Dividing each of the terms of (2) by the corresponding term of (9), we have

$$\frac{p}{h} = \frac{q \cot BP}{k \cot BQ} = \frac{r}{l} \quad \text{........................} \, (10).$$

Hence, when either BP or BQ is known, the other can be computed. Expressions (10) are nothing more than those of the A.R. $\{BPQS\}$, and can be obtained from it.

Examples.

28. *Gypsum.* Fig. 140 shows a simple crystal of gypsum such as can be very easily obtained. In one direction, that parallel to $b\,(010)$, there is a very perfect cleavage. The faces m and $m_,$ are brighter and more even than those marked l and l', and the face-angle $mm_,$ is more acute than ll'. Approximate measurements can be made without difficulty. From Dana's *Mineralogy* we take the following angles:

bm	55° 45′	bl	71° 54′	lm	49° 9·5′.
$mm_,$	68 30	ll'	36 12		
$m_,b_,$	55 45	$l'b_,$	71 54		

The angles between the edges $[bm]$ and $[bl]$—127° 34·5′ and 52° 25·5′—can be computed from the triangle blm by the expression given in Art. 26. It is the same as the angle between the normals to possible faces $A\,(100)$ and $N\,(101)$ which are occasionally found to truncate the edges $[mm_,]$ and $[ll']$. From the figure and the table of angles, it is clear that the face b is parallel to Σ and perpendicular to the dyad axis.

The stereogram, Fig. 139, is quickly made. Describe a circle with any convenient radius and draw a horizontal diameter $[AmBm']$. On the primitive the arc $AN = 52^\circ\ 25{\cdot}5'$ is marked off by a protractor, and the diameter $[NlBl_{,}]$ drawn. On these two diameters the poles m and l are determined by the method given in Chap. VII, Prob. 1, so that $Bm = 55^\circ\ 45'$, and $Bl = 71^\circ\ 54'$. The homologous poles are inserted by the aid of a pair of compasses. The zone-circle $[ml]$ is then drawn by either of the methods of Chap. VII, Art. 18, and the homologous circle $[m'l_{,}]$ is drawn at the same time.

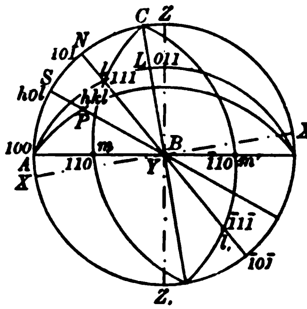

Fig. 139.

The face b is (010) of which B is the pole: the axis of Z is taken parallel to the edge $[mm_{,}]$ and is the vertical line in the stereogram. The pole A at 90° to Z is the possible pole (100), and m can be taken to be (110). We are at liberty to take the axis of X parallel to any other possible edge in the plane Σ parallel to (010), and might, were we determining a new substance, select the zone-axis $[ll']$ for this purpose. As, however, gypsum has been fully described, we follow other crystallographers and select for axis of X the line parallel to a face of somewhat rare occurrence, the pole of which is the point C in which the zone-circle $[lm]$ meets the primitive. The position of $XX_{,}$ is given by the arcs $CX = CX_{,} = 90^\circ$. Now the pole l lies in the zone $[Cm] = [001,\ 110] = [1\bar{1}0]$; and its symbol is (hhk). We are free to give any value we please to the third index k, and the value selected determines the numerical value of the parameter c. Let $k = h$; then all the indices are equal, and the simplest symbol is (111). The possible pole N is then (101).

Hence, the angular elements are: AN, Bl, and NC, of which AN and NC are determined by computation.

To find the pole C, we have by Napier's rules, from the right-angled triangles ACm and NCl,

$$\sin AC = \tan Am \cot ACm = \cot Bm \cot ACm,$$

$$\sin NC = \tan Nl \cot ACl = \cot Bl \cot ACl.$$

But $\wedge ACm = \wedge ACl$; $\therefore$ dividing the latter equation by the former, we have

$$\frac{\sin NC}{\sin AC} = \frac{\cot (Bl = 71^\circ\ 54')}{\cot (Bm = 55^\circ\ 45')} = \tan (\theta = 25^\circ\ 38{\cdot}5'), \text{ by computation;}$$

$$\therefore \frac{\sin AC - \sin NC}{\sin AC + \sin NC} = \frac{\tan \frac{1}{2}(AC - NC)}{\tan \frac{1}{2}(AC + NC)} = \frac{1 - \tan\theta}{1 + \tan\theta} = \tan (45^\circ - \theta).$$

But $\quad \frac{1}{2}(AC - NC) = \frac{1}{2}AN = 26^\circ\ 12{\cdot}75'$, and $45^\circ - \theta = 19^\circ\ 21{\cdot}5'$;

$$\therefore \tan \tfrac{1}{2}(AC + NC) = \tan 26^\circ\ 12{\cdot}75' \div \tan 19^\circ\ 21{\cdot}5'.$$

Therefore, $\quad AC + NC = 108^\circ\ 58{\cdot}6'$.

Hence $\quad AC = \beta = 80^\circ\ 42'$, and $NC = 28^\circ\ 16{\cdot}5'$.

To obtain the linear elements, we make $b=1$ in equations (3);

$$\therefore\ a \sin 28^\circ 16{\cdot}5' = \cot 71^\circ 54' = c \sin 52^\circ 25{\cdot}5'.$$

$$L \cot 71^\circ 54' \ = 9{\cdot}51435 \ldots\ldots\ldots\ldots\ldots\ldots\ 9{\cdot}51435$$

$$L \sin 28^\circ 16{\cdot}5' = \underline{9{\cdot}67550};\ L \sin 52^\circ 25{\cdot}5' = 9{\cdot}89903$$

$$\log a = \bar{1}{\cdot}83885;\qquad \log c = \bar{1}{\cdot}61532:$$

$$\therefore\ a = {\cdot}690 \text{ and } c = {\cdot}4124.$$

The crystal is, therefore, fully determined. It consists of the forms: {010}, {110}, {111}; and has the elements: $\beta = 80^\circ 42'$; $a : b : c = {\cdot}690 : 1 : {\cdot}4124$. The indices of the faces and poles are inscribed in the diagrams. The form m {110} includes the four faces 110, $1\bar{1}0$, $\bar{1}\bar{1}0$, $\bar{1}10$; and l {111} includes 111, $\bar{1}1\bar{1}$, $\bar{1}\bar{1}\bar{1}$, $1\bar{1}1$.

To draw the crystal the axes are projected in the manner described in Chap. VI, Art. 16. The points B, $B_,$ are left at unit distance on the horizontal axis $YY_,$; C' and $C_,$ (see Art. 24) are found on the vertical at distances $OA'' \times {\cdot}412$ from the origin. Similarly, the points A', $A_,$ are determined on the inclined axis $XX_,$ at distances $OX \times {\cdot}69$ from the origin. The edge [ll'] in the drawing is the line $C'A'$. The edges [lm], [$l'm_,$] are the lines $A'B$, $A'B_,$, respectively. From them equal lengths are cut off by proportional compasses corresponding to the width to be given to the faces m and b; or a point being taken on [lm] a line parallel to $YY_,$ gives the corresponding point on [$l'm_,$]. Through these points and A' vertical lines are drawn giving the edges [bm], [$b_,m_,$], [$mm_,$]; and through the first two points the edges [bl], [$b_,l'$] are drawn parallel to $C'A'$.

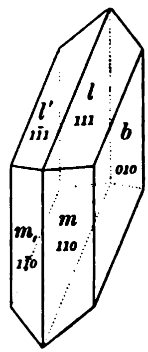

Fig. 140.

The edges [lm'], [$l'\bar{m}$] at the top and the parallel edges at the bottom have now to be drawn. Their zone-symbols are [$11\bar{2}$] and [$\bar{1}12$] respectively; and their directions can be found by the ordinary construction given in Chap. V. In such simple crystals it is easier to find them as follows. A line OD is drawn through O parallel to [$l'm_,$] *i.e.* to $A'B_,$ to meet $A'B$ in D: the line $C'D$ is the upper edge [lm']. Similarly, the line $C_,D_,$ is the direction of the lower edge [$ml_,$]; $D_,$ being the point in which $OD_,$, parallel to $A'B$, meets $A_,B$. The parallelogram giving the face b (010) can be now completed. The two continuous edges [$l'\bar{m}$] and [$m_,\bar{l}$] are parallel to $C_,D_,$ and $C'D$ respectively. The dotted lines are then easily introduced by the method given in Chap. VI, Art. 26.

Optic characters. The plane of the optic axes coincides with the plane of symmetry, and Bx_a lies in the obtuse angle XOZ, and at 9·4° C. makes an angle of 52° 27′[1] with OZ (the line $\grave{c}$), as shown in Fig. 141. This figure represents a thin cleavage plate, bounded by the poor cleavages which are distinguished by having a fibrous and conchoidal character,

[1] When the nature of the light is not specified, day- (or white) light is indicated. Red light is that transmitted by glass coloured by oxide of copper, blue that by glass coloured by cobalt or that given by an ammoniacal solution of copper sulphate.

respectively. The fibrous cleavages are parallel to $n\{11\bar{1}\}$ and give the edges truncated by the faces $\{10\bar{1}\}$, the cleavages with conchoidal surface are parallel to $\{100\}$. The axis OX is that marked $\grave{a}$.

When a plate, cut perpendicularly to Bx_a, is inserted in convergent white light between crossed Nicols, two series of oval rings (often called *eyes*) are seen which are each traversed by a dark brush. When the plane of the optic axes is at 45° to that of polarization, the brushes form a rectangular hyperbola having its vertices in the beams transmitted along the optic axes. These vertices may be called the *centres* of the eyes. The hyperbolic brush, which has its vertex in an optic axis inclined to the normal (100) at an angle of 8·5°

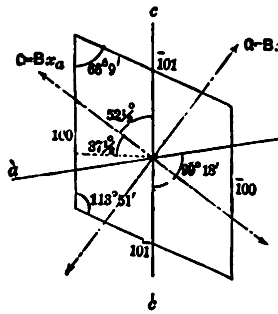

Fig. 141.

(nearly), has a bright violet fringe on the concave side, and will be denoted as the violet brush: the other brush has dull fringes, and will be called the dark brush. The rings traversed by the violet brush which can be easily seen are more numerous than those forming the eye traversed by the dark brush; and the shapes of the rings forming the eyes are different, those traversed by the violet brush being more oval than those of the other set. The phenomena are symmetrical only to the trace of the plane of symmetry.

The apparent angle of the optic axes in air is always denoted by $2E$, and is measured by bringing each hyperbolic brush in turn to cover the cross-wire of the instrument. The difference of readings of the graduated circle, to which the axis carrying the crystal-plate is attached, gives the angle required. The angle of the optic axes in the crystal is denoted by $2V$, and can be computed from the principal indices of refraction when these are known; or, if $2E$ and β (the mean principal index) are known, we find V by the equation—$\sin E = \beta \sin V$. In many substances V is so large that light, transmitted in the crystal along an optic axis, suffers total internal reflexion when the crystal is in contact with air: in these cases $2H$, the apparent angle in oil of known refractive index, is often given.

In gypsum the angle $2E$ is for red light 95° 14′, and for blue light 94° 24′, at 20° C. With increase of temperature the angle $2E$ diminishes rapidly, and becomes zero for red light at 116° C., whilst the optic axes for blue and intermediate colours (having previously closed in) have opened out in a plane perpendicular to Σ. At 120° C. and at slightly higher temperatures, the plane of the optic axes is perpendicular to Σ for all colours. If the temperature is not raised above 125° C., the phenomena are reversed as the section cools; and the original state is recovered when the original temperature is reached. Further, the hyperbolic brushes close in towards

the centre at unequal rates, that having the bright violet fringe undergoing the greater displacement. Thus between 20° and 95·5° C., the brush with violet fringe is displaced through 33° 35′, that with dull fringes through 22° 38′ (Des Cloizeaux). The acute bisectrix has, therefore, been displaced in the same direction as the brighter brush through an apparent angle of 5° 29′—one-half the difference of the displacements. If the mean refractive index β is taken to be 1·519 at 95·5° C., and the plate was accurately perpendicular to Bx_a at 20° C., the displacement of Bx_a in the crystal is very nearly 3·5°; and $Bx_a \wedge OZ = 49°$ at 95·5° C. The above variation of the angle in air can be easily confirmed by the student. Neumann determined the displacement of Bx_a in the crystal by similar observations made whilst the plate was immersed in rape-seed oil, which has a refractive index slightly lower than β of gypsum. He found the displacement to be 3° 50′ between 19·1° C. and the temperature at which the axial angle is reduced to zero, the latter temperature being probably about 115° C. This angle is smaller than that deduced from Des Cloizeaux' observation, if we assume the displacements for equal increments of temperature to be nearly equal.

By means of a thin cleavage-plate of a twin-crystal, the displacement of the acute bisectrix can be measured directly. If the twin-face is (100), the acute bisectrices in the two portions include at about 10° C. an angle of 104° 54′ and 75° 6′. The least angle between the extinctions when the compound plate is inserted in parallel light between crossed Nicols is 14° 54′ = 90° − 75° 6′. If the twin-face is (101) the extinctions make with one another a minimum angle of 29° 45′. As the temperature rises the least angle between the extinctions diminishes in both cases. Between 20° and 96° C. the diminution in the angle in a twin-plate, with twin-face (101), was determined by students in the Cambridge laboratory to be approximately 6·5°. The acute bisectrix is, therefore, displaced through 3·25°; and its inclination to the vertical axis is at 96° C. approximately 49·25°.

29. *Orthoclase.* In the crystal of orthoclase, Fig. 143, the following angles are known :—

$$\text{i. } \left[\begin{array}{ll} bm & 59°\ 23{\cdot}5' \\ mm_{,} & 61\ \ 13. \end{array}\right. \qquad \text{ii. } \left[\begin{array}{ll} bn & 45°\ 3{\cdot}5' \\ nc & 44\ \ 56{\cdot}5 \\ nn' & 89\ \ 53. \end{array}\right. \qquad \text{iii. } \left[\begin{array}{ll} cx & 50°\ 16{\cdot}5' \\ xy & 30\ \ 1{\cdot}5. \end{array}\right.$$

The faces b and c are selected as the axial planes (010) and (001), respectively. They are directions of perfect cleavage. The faces m are invariably taken as those of the prism {110}; and the possible face (100) therefore truncates the edge [$mm_{,}$]. Its angular distance from (001) was, in Chap. VIII, Art. 15, determined to be 63° 57′ (neglecting fractions of 1′) from the angles given in zone iii.

In the stereogram, Fig. 142, the poles A, C, x, y, etc., are marked on the primitive by means of a protractor, placing A (100) at the left hand of the horizontal diameter. The axis of Z parallel to the edge [$mm_{,}$] is the vertical

line, and the points Z and X are both marked by crosses, the angle $ZX_{,}=AC$ being 63° 57′. The zones $[BmA]$, $[BnC]$, $[Box]$, are projected in diameters through A, C, and $x_{,}$ and the poles m and n upon them can be easily placed by the construction of Chap. VII, Prob. 1. Having placed m or n, the zone-circles $[y'mno]$, $[y'o_{,}n_{,}m']$ can be drawn, and fix the positions of the remaining poles.

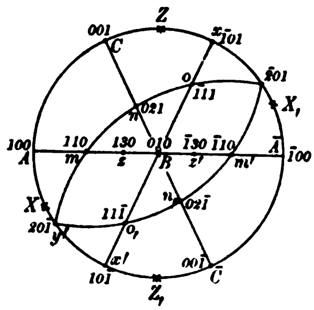

Fig. 142.

The symbol (110) having been assigned to m, the parameter a is found from formula (5), in which b is made $=1$.

$$\therefore\ a=\frac{\cot(Bm=59°\ 23'\cdot 5)}{\sin(\beta=63°\ 57')}=\cdot 6585.$$

The parameter c is still left undetermined, and we are at liberty to assign known indices to one or other of the poles n, o, x or y. Let us take x to be $(\bar{1}01)$. Then a plane parallel to x drawn through $A_{,}$ on OX at distance a from the origin meets the axis OZ at C', where $OC'=c$. By a diagram similar to that employed in Art. 24, but in which $OA_{,}$ takes the place of OA', we have

$$\frac{c}{a}=\frac{OC'}{OA_{,}}=\frac{\sin C'A_{,}O}{\sin A_{,}C'O}=\frac{\sin Cx}{\sin \bar{A}x}.$$

The same expression is obtained from the equations of the normal, by taking P to be x and (hkl) to be $(\bar{1}01)$. The equations become $a\cos X_{,}x=c\cos Zx$.

But $$X_{,}x=90°-Cx,\ \text{and}\ Zx=90°-\bar{A}x.$$

$$\therefore\ a\sin Cx=c\sin\bar{A}x.$$

Introducing the angles and the value of a, we have

$$\log c=\log\cdot 6585+L\sin 50°\ 16\cdot 5'-L\sin 65°\ 46\cdot 5'=\bar{1}\cdot 74458.$$

$$\therefore\ c=\cdot 55536.$$

In the zone $[Cxy]$ we know the angles and the symbols of $\bar{A}$, C, and x. Hence the symbol of y can be found by the A. R. $\{\bar{A}xyC\}$.

$$\frac{\sin(\bar{A}x=65°\ 46\cdot 5')}{\sin(\bar{A}y=35°\ 45')}\div\frac{\sin(Cx=50°\ 16\cdot 5')}{\sin(Cy=80°\ 18')}=\begin{vmatrix}\bar{1}00\\ \bar{1}01\\ \hline \bar{1}00\\ \bar{h}0l\end{vmatrix}\div\begin{vmatrix}001\\ \bar{1}01\\ \hline 001\\ \bar{h}0l\end{vmatrix}=\frac{h}{l}.$$

By computation, $h=2l$; $\therefore$ the symbol of y is $(\bar{2}01)$.

Having found y, the zone-symbol of $[ym]$ is found from Chap. v, table 10, to be $[1\bar{1}2]$. Hence, the poles n and o are found, by Weiss's zone-law, to be (021) and $(\bar{1}11)$, respectively; for n lies also in $[BC]=[100]$, and o lies in $[Bx]=[101]$.

If it is desired to prove that n lies in the zone $[ym]$, we can do so practically by adjusting the zone on the reflecting goniometer, when it will be seen that n and o both lie in the zone. It can, also, by the help of the stereogram be

proved from the measured angles. Thus, suppose R to be the pole in which $[ym]$ meets $[BC]$. Then, from the right-angled triangles Aym and CyR, we have

$$\sin Ay = \tan Am \cot Aym,$$
$$\sin Cy = \tan CR \cot CyR.$$

But Aym and CyR are the same angle. Hence, dividing the second equation by the first,

$$\tan CR = \tan Am \sin Cy \div \sin Ay.$$

$$\begin{array}{lr} L \tan (Am = 30^\circ\, 36{\cdot}5') = & 9{\cdot}77202 \\ L \sin (Cy = 80^\circ\, 18') = & \underline{9{\cdot}99375} \\ & 19{\cdot}76577 \\ L \sin (Ay = \pi - 35^\circ\, 45') = & \underline{9{\cdot}76660} \\ & 9{\cdot}99917. \end{array}$$

$\therefore\ CR = 44^\circ\, 56{\cdot}75' =$ the observed angle Cn; and R coincides with the pole n.

By means of equations (3) the angular elements of Miller can be found and the parametral pole $l\,(111)$ introduced in the stereogram. The angles are: $AN = 35^\circ\, 0{\cdot}5'$, $Bl = 72^\circ\, 19{\cdot}5'$, $CN = 28^\circ{\cdot}56{\cdot}5'$, using the same letters to denote the poles as were used in equations (3). The parametral face (111) is a very rare one, and the axial face (100) is not very common.

The face o might at once have been assumed as the parametral face ($\bar{1}11$), giving the lengths $-a$, b and c on the axes and therefore necessarily the length a on the positive side of the axis of X. Being given $Bo = 63^\circ\, 8'$, the reader can, by the aid of formulæ (2) and (3), find the parameters and the symbols of all the faces; and prove that the results are the same as those given above.

The method of constructing the axes, by which Fig. 143 is made, was given in Chap. VI, Art. 16. A linear projection is then made in the plane XOY, similar to that used in Chap. VII, Art. 5, and the edges are given by joining C' or $C_{,}$ on OZ to the zonal-points. The face c (001) is first drawn by joining A' to B and $B_{,}$. On the edges $[cm]$, $[cm_{,}]$ given by $A'B$ and $A'B_{,}$, equal lengths are cut off by proportional compasses, and through the points so obtained and A' $[bm]$, $[b_{,}m_{,}]$ and $[mm_{,}]$, are drawn parallel to OZ. Through any point on $[mm_{,}]$ the edges $[my]$, $[m_{,}y]$ are drawn. Through a point on $[my]$ the edges $[yx]$, $[mo_{,}]$, $[xo_{,}]$, are then drawn. The face b can be now completely drawn; and the rest of the figure presents no difficulty.

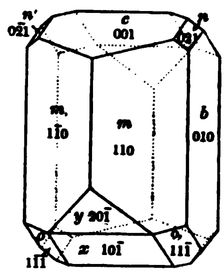

FIG. 143.

The optic characters are extremely variable. Crystals from St Gotthard have a very wide angle of the optic axes, those from the Eifel a very small one. In both cases the first mean line lies in the obtuse angle XOZ and makes an angle of approximately 5° with OX. Considering now the "glassy" variety from the Eifel, the plane of the optic axes is, at ordinary temperatures, very generally perpendicular to the plane of symmetry. As the temperature is raised the angles of the optic axes for different colours diminish, the optic axes for blue light closing in first and those for red last. When the temperature is further raised, the axes for all colours

separate *in* the plane of symmetry; the angle between the optic axes for blue light being greater than that for red light. If the temperature is raised not beyond 500° C., and then allowed to fall, the axes go through the opposite series of changes, and ultimately, at the initial temperature, recover their original positions. If, however, the temperature is raised to 600° C. or higher, the axes never recover their original positions when the plate is allowed to cool, although the angle between them is much diminished. The acute bisectrix is but slightly displaced during these changes. In an unstrained crystal the dispersion is, therefore, horizontal, whilst in a strained one it is inclined. The following table gives a few of the changes observed by DesCloizeaux in a plate examined in red light, θ being the temperature, and $2E$ (the angle in air) being regarded as negative when measured in a plane perpendicular to the plane of symmetry, and as positive when in Σ.

θ	$2E$	θ	$2E$
18·7° C.	− 16°	150° C.	+ 40°
42·5°	0°	200°	+ 46° 23′
50°	+ 12°	250°	+ 55°
75°	+ 24°	300°	+ 59° 46′
100°	+ 30°	343°	+ 64°.

30. *Epidote.* This mineral is usually found as dark green attached crystals, elongated in the direction of the dyad axis. The crystals are commonly in the form of irregular six-faced prisms—*a* {100}, *c* {001}, *r* {$\bar{1}$01}—terminated by a pair of faces *n* {$\bar{1}$11}; and the faces *a*, *c* and *r*, are striated parallel to the edges in which they meet. We shall show how to calculate the angles in the following table from a knowledge of the symbols of the faces and of the angles marked by asterisks. The tables of angles given in descriptive works are the results of similar computations. In the plan, Fig. 144, the dyad axis is perpendicular, and the face *b* (010) is parallel, to the paper. Selecting *a*, *b*, *c* as axial planes and *n* as ($\bar{1}$11), the symbols of the faces, with the exception of a few of those in the zone [*ac*], can be found by Weiss's zone-law. We assume the symbols of all the forms to have been determined; they are: *a* {100}, *b* {010}, *c* {001}, *e* {101}, *i* {$\bar{1}$02}, *r* {$\bar{1}$01}, *l* {$\bar{2}$01}, *f* {$\bar{3}$01}, *u* {210}, *z* {110}, *k* {012}, *o* {011}, *d* {111}, *n* {$\bar{1}$11}, *q* {22$\bar{1}$}, *y* {21$\bar{1}$}.

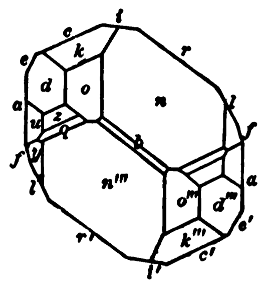

Fig. 144.

The face *c* (001) is parallel to a perfect cleavage, *a* (100) to an imperfect cleavage.

By the aid of the stereogram, Fig. 145, the student can follow the analysis; the poles *A* (100), *e*, *C* (001), *i*, *r*, *l*, *f* being in the primitive.

ae	29°	54′	*cd*	52°	19·3′	*ad*	49°	52′
**ac*	64	37	*cz*	75	45	*ao*	77	2·6
ci	34	21	*cq*	90	18	*an*	110	57·6
cr	63	41·5	**cn‴*	104	49	*ay′*	134	53·5
cl	89	25·7	**nn‴*	70	29	*au*	35	30
cf	98	37	*bn*	35	14·5	*az*	54	58
fa,	16	46				*ab*	90	0

i. *Zone* $[AC]$. The angle Cr is first found from the right-angled spherical triangle Crn, for $rn = 90° - Bn$, and

$$Cn = 180° - Cn''' = 75°\ 11'.$$

Hence,

$$\cos Cr = \cos(Cn = 75°\,11') \div \cos(rn = 54°\,45{\cdot}5').$$

$$L \cos 75°\ 11' \quad = 9{\cdot}40778$$

$$L \cos(54°\ 45{\cdot}5') = 9{\cdot}76120$$

$$L \cos(63°\ 41{\cdot}5') = \overline{9{\cdot}64658}.$$

$\therefore\ Cr = 63°\ 41{\cdot}5'$, and

$$\bar{A}r = 180° - AC - Cr = 51°\ 41{\cdot}5'.$$

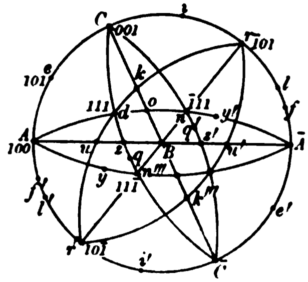

Fig. 145.

The angles between the other faces in the zone can now be found by the A. R. of four faces. Let T be $(\bar{h}0l)$.

$\therefore$ A. R. $\{\bar{A}TrC\}$ gives

$$\frac{\sin \bar{A}T}{\sin \bar{A}r} \div \frac{\sin CT}{\sin Cr} = \left|\frac{\begin{array}{c}\bar{1}00\\ \bar{h}0l\end{array}}{\begin{array}{c}\bar{1}00\\ \bar{1}01\end{array}}\right| : \left|\frac{\begin{array}{c}001\\ \bar{h}0l\end{array}}{\begin{array}{c}001\\ \bar{1}01\end{array}}\right| = \frac{l}{h}.$$

$$\therefore\ \frac{\sin \bar{A}T}{\sin CT} = \frac{l \sin(\bar{A}r = 51°\ 41{\cdot}5')}{h \sin(Cr = 63°\ 41{\cdot}5')} = \tan\theta \ \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

$$\therefore\ \tan\tfrac{1}{2}(CT - \bar{A}T) = \tan\tfrac{1}{2}(CT + \bar{A}T)\tan(45° - \theta) = \tan 57°\ 41{\cdot}5' \tan(45° - \theta).$$

For $f\,(\bar{3}01)$,

$$\tan\theta = \frac{\sin 51°\ 41{\cdot}5'}{3 \sin 63°\ 41{\cdot}5'}, \quad \therefore\ \theta = 16°\ 16' \text{ and } 45° - \theta = 28°\ 44'.$$

$$\therefore\ \tan\tfrac{1}{2}(Cf - \bar{A}f) = \tan 57°\ 41{\cdot}5' \tan 28°\ 44'.$$

$$\therefore\ Cf - \bar{A}f = 81°\ 10{\cdot}8', \text{ and } Cf = 98°\ 36{\cdot}9',\ \bar{A}f = 16°\ 46{\cdot}1'.$$

For $l\,(\bar{2}01)$,

$$\tan\theta = \frac{\sin 51°\ 41{\cdot}5'}{2 \sin 63°\ 41{\cdot}5'}; \quad \therefore\ \theta = 23°\ 38{\cdot}3',\ 45° - \theta = 21°\ 21{\cdot}7'.$$

$$\therefore\ \tan\tfrac{1}{2}(Cl - \bar{A}l) = \tan 57°\ 41{\cdot}5' \tan 21°\ 21{\cdot}7'.$$

$$\therefore\ Cl - \bar{A}l = 63°\ 28{\cdot}4', \text{ and } Cl = 89°\ 25{\cdot}7'.$$

If T is $i\,(\bar{1}02)$, equation (a) must be inverted,

$$\therefore\ \frac{\sin Ci}{\sin \bar{A}i} = \frac{\sin 63°\ 41{\cdot}5'}{2 \sin 51°\ 41{\cdot}5'} = \tan(\theta_{,} = 29°\ 44');$$

$$\therefore\ \tan\tfrac{1}{2}(\bar{A}i - Ci) = \tan 57°\ 41{\cdot}5' \tan 15°\ 16'.$$

$$\therefore\ \bar{A}i - Ci = 46°\ 41{\cdot}4', \text{ and } Ci = 34°\ 20{\cdot}8'.$$

The angle Ae is found in a similar manner from A. R. $\{AeCr\}$.

ii. The parameters can now be obtained from equations (2) by taking P to be $n\,(\bar{1}11)$.

$\therefore\ a \sin Cr = b \cot Bn = c \sin \bar{A}r$; and,

$$\frac{a}{b} = \frac{\cot 35^\circ\, 14{\cdot}5'}{\sin 63^\circ\, 41{\cdot}5'} = 1{\cdot}57896,$$

$$\frac{c}{b} = \frac{\cot 35^\circ\, 14{\cdot}5'}{\sin 51^\circ\, 41{\cdot}5'} = 1{\cdot}80375.$$

iii. *Zones* $[Bz]$ and $[Bo]$. By formulæ (5) and (5*) the angles Bz and Bo can now be found; and then by the A. R. the angles Bu and Bk.

Hence
$$\cot Bz = \frac{a \sin (\beta = 64^\circ\, 37')}{b} = \frac{\cot 35^\circ\, 14{\cdot}5' \sin 64^\circ\, 37'}{\sin 63^\circ\, 41{\cdot}5'};$$

$$\cot Bo = \frac{c}{b} \sin (\beta = 64^\circ\, 37') = \frac{\cot 35^\circ\, 14{\cdot}5' \sin 64^\circ\, 37'}{\sin 51^\circ\, 41{\cdot}5'}$$

$$\therefore\ Bz = 35^\circ\, 2', \text{ and } Bo = 31^\circ\, 32'.$$

Also,
$$\tan Bu \div \tan Bz = h \div k = 2;$$

$$\tan Bk \div \tan Bo = l \div k = 2.$$

$$\therefore\ Bu = 54^\circ\, 30', \text{ and } Bk = 50^\circ\, 49{\cdot}5'.$$

iv. *Zone* $[Aon]$. The angles Ao and $\bar{A}n$ can now be found by the right-angled spherical triangles AoC and $\bar{A}nr$; and then Ad and Ay' can be obtained from the A. R. of four poles. The formulæ are:

$$\cos Ao = \cos (AC = 64^\circ\, 37') \sin (Bo = 31^\circ\, 32'), \quad \therefore\ Ao = 77^\circ\, 2{\cdot}6'.$$

$$\cos \bar{A}n = \cos (\bar{A}r = 51^\circ\, 41{\cdot}5') \sin (Bn = 35^\circ\, 14{\cdot}5'), \quad \therefore\ \bar{A}n = 69^\circ\, 2{\cdot}4'.$$

From A.R. $\{Adon\}$, we have

$$\frac{\sin Ad}{\sin nd} = \frac{\sin (Ao = 77^\circ\, 2{\cdot}6')}{2 \sin (no = 33^\circ\, 55')} = \tan (\theta = 41^\circ\, 7{\cdot}8').$$

$\therefore$ by computation, $nd - Ad = 11^\circ\, 13{\cdot}8'$, and $Ad = 49^\circ\, 52'$.

Similarly, $\bar{A}y'$ is found from

$$\frac{\sin \bar{A}y'}{\sin oy'} = \frac{\sin (\bar{A}n = 69^\circ\, 2{\cdot}4')}{2 \sin (no = 33^\circ\, 55')}.$$

v. *Zone* $[Czq]$. The angle Cz is found from the right-angled triangle CAz;

$$\cos Cz = \cos (AC = 64^\circ\, 37') \sin (Bz = 35^\circ\, 2'). \quad \therefore\ Cz = 75^\circ\, 45{\cdot}25'.$$

If now the A. R. $\{Cdzn'''\}$ is taken to find Cd, an auxiliary angle of less than 4′ enters into the computation. It is, therefore, best to compute Cd from the right-angled triangles Cde and CzA, in the way adopted in a similar calculation for gypsum; thus

$$\cos eCd = \tan eC \cot Cd,$$

$$\cos ACz = \tan AC \cot Cz.$$

$$\therefore\ \tan Cd = \cot (AC = 64^\circ\, 37') \tan (Cz = 75^\circ\, 45{\cdot}25') \tan (eC = 34^\circ\, 43').$$

$$\therefore\ Cd = 52^\circ\, 19{\cdot}3'.$$

The angle Cq is then found to be 90° 18′ from the A. R. $\{Cdzq\}$.

31. Crystals of *hydrogen trisodic hypophosphate*, $HNa_2P_2O_6 . 9H_2O$, are described by M. H. Dufet (*Bull. Soc. franç. de Min.* IX, p. 205, 1886) as having only a plane of symmetry; and belong, therefore, to class II. He determines the elements of the crystal from the following three angles: $Nx = 101 \wedge \bar{1}01 = 87° 52'$, $Cm = 001 \wedge 110 = 83° 25'$, $Co = 001 \wedge \bar{1}11 = 65° 21'$. To obtain the elements, we proceed as follows:—

A freehand stereogram of the above poles, of l (111), and of the poles of the complementary forms is first made.

i. From A. R. $\{Clmo,\}$, we find $Cl = 55° 24{\cdot}5'$.

ii. From the right-angled triangle CNl, $\cos NCl = \tan CN \cot Cl$;

" " " " " Cox, $\cos (Cox = NCl) = \tan Cx \cot Co$.

$$\therefore \frac{\tan CN}{\tan Cx} = \frac{\tan (Cl = 55° 24{\cdot}5')}{\tan (Co = 65° 21')} = \tan (\theta = 33° 38{\cdot}4').$$

But
$$\frac{\tan CN}{\tan Cx} = \frac{\sin CN \cos Cx}{\sin Cx \cos CN};$$

$$\therefore \frac{\sin Cx \cos CN - \sin CN \cos Cx}{\sin Cx \cos CN + \sin CN \cos Cx} = \frac{1 - \tan\theta}{1 + \tan\theta} = \tan (45° - \theta).$$

$$\therefore \sin (Cx - CN) = \tan 11° 21{\cdot}6' \sin (Cx + CN) = \tan 11° 21{\cdot}6' \sin 87° 52'.$$

Hence, by computation, $Cx - CN = 11° 35'$; and $Cx = 49° 43{\cdot}5'$, $CN = 38° 8{\cdot}5'$.

iii. From the several right-angled triangles, we have:

$$\cos (Cox = NCl) = \tan (Cx = 49° 43{\cdot}5') \cot (Co = 65° 21'),$$
$$\cos NCl = \tan CA \cot (Cm = 83° 25'),$$
$$\sin Bl = \cos Nl = \cos Cl \div \cos CN,$$
$$\sin Bo = \cos ox = \cos Co \div \cos Cx,$$
$$\sin Am = \sin Cm \sin NCl.$$

Hence,

$NCl = 57° 12{\cdot}5'$, $AC = \beta = 77° 58{\cdot}2'$, $Bl = 46° 12{\cdot}5'$, $Bo = 40° 10{\cdot}\dot{6}'$, $Am = 56° 37{\cdot}\dot{6}'$.

The parameters can now be found by equations (2); and are

$$a : b : c = 1{\cdot}552 : 1 : 1{\cdot}497.$$

CHAPTER XIII.

THE PRISMATIC SYSTEM.

1. The prismatic system includes all crystals having one set of three dissimilar zone-axes which are, at all temperatures, perpendicular to one another, and are the directions of the principal axes of the wave-surface for light of all colours. The system includes the three following classes:

I. The *sphenoidal* (*bisphenoidal* (Groth), *hemihedral*) class, the crystals of which have three dissimilar dyad axes at right angles to one another, but have no other element of symmetry.

II. The *bipyramidal* (*holohedral*) class, in which the three dyad axes of class I are associated with a centre of symmetry and with three dissimilar planes of symmetry, each perpendicular to one of the dyad axes.

III. The *acleistous*[1] *pyramidal* (*pyramidal*, *hemimorphic*) class, the crystals of which have only one dyad axis associated with two dissimilar planes of symmetry intersecting at right angles in the axis: the normals to these planes are zone-axes, but are not axes of symmetry.

2. The three rectangular zone-axes of the several classes can be selected as the axes of X, Y and Z; and no other set of zone-axes is equally advantageous. The angles, 90°, between the three axes of reference are now fixed by the symmetry, and will remain constant as long as the crystal is subjected only to strains, such as that due to the expansion caused by change of temperature, which affect the substance in a manner consistent with the retention of the

[1] From ἄκλειστος = not closed.

crystalline structure. The parameters, $a : b : c$, are therefore the only elements which can vary with the substance in crystals of this system.

In classes I and II it is immaterial which of the three dyad axes is selected as axis of Z and placed vertically, but generally that axis is taken which brings out the habit most distinctly in drawings, or which gives prominent positions to any cleavage-faces which may exist. In class III, however, it is best to place the single dyad axis in the vertical position. The parameter a is arranged to be less than b, but c may be either greater or less than b. Hence the axis of X is called the *brachy-axis* or *brachy-diagonal*, the axis of Y the *makro-axis* or *makro-diagonal*.

3. In drawings of this and the two following systems, discussed in Chaps. XIV and XV, in which rectangular axes of reference are also adopted, the axis of Z is placed in the vertical plane and nearly, or quite, in the paper as was described in Chap. VI; the axis of Y is placed right-and-left, the positive direction being to the right and the negative to the left; the axis of X is directed back-and-fore with the positive direction to the front.

In stereograms the axis of Z is usually perpendicular to the plane of the primitive, and is projected in the central point; the axes of X and Y are in the paper. We shall usually place the axis of Y horizontal and the positive direction to the right. The axis of X is therefore the vertical line in the diagram and its positive direction is taken downwards. Hence, to compare a drawing of a crystal with its stereogram, the drawing has to be turned about the horizontal axis OY until the axis of Z is perpendicular to the paper. Stereograms correspond exactly with, and have the same orientation as, plans on the plane XOY; as was illustrated by the plan and stereogram of barytes discussed in Chap. VII, Arts. 3, 21 and 22.

I. *Sphenoidal class*; $\alpha\{hkl\}$.

4. If a crystal has two dyad axes, the line at right angles to them is also an axis of symmetry, and the degree, or angle of rotation, is determined by the least angle between the dyad axes in the plane of the first pair (Chap. IX, Prop. 11). The lowest degree such an axis can have is 2, and the angle of rotation about it, 180°, is twice that between the original pair of axes. Hence, the simplest case is that in which three dissimilar dyad axes occur together, all

at right angles to one another. Axes of symmetry of the same degree in the same crystal, which are neither interchangeable nor reciprocal reflexions in a plane of symmetry, are said to be dissimilar. The angles between pairs of homologous faces, interchangeable by rotation about dissimilar axes, are unequal.

The class having three dissimilar dyad axes at right angles to one another and no other element of symmetry, we shall call the *sphenoidal* class of the prismatic system. We shall denote the general form, Fig. 146, which has four faces bounded by scalene triangles and is known as a *sphenoid*, by the symbol $a\{hkl\}$. The Greek prefix *a*, meaning *without*, serves to indicate that the crystal has no planes of symmetry.

5. The two pairs of faces in Fig. 146, interchangeable by rotation through 180° about $ZZ_,$, meet in edges LM and NQ, both perpendicular to $ZZ_,$. The edge LM is parallel to the line XY in which the face (hkl) meets the axial plane XOY; and the edge NQ to the line $XY_,$ in the same axial plane. Similarly, the pairs interchangeable by rotation about $XX_,$ meet in lines LN and MQ, both perpendicular to $XX_,$ and parallel, respectively, to the lines ZY and $ZY_,$. The faces are similarly arranged in pairs meeting in the edges MN and LQ perpendicular to $YY_,$ and parallel, respectively, to XZ and $XZ_,$. By rotation about one or other of the axes each face can be brought into the position of any of the others. The homologous faces are, therefore, all exactly alike and are triangles with unequal sides and angles. The dihedral angles over the three edges of each face are unequal, but those over pairs of opposite edges, such as LM and NQ, perpendicular to each dyad axis are equal. Such a form clearly encloses a finite portion of space, and a crystal of this class may exist which shows only a single form. The form is called a sphenoid, from σφήν, a wedge.

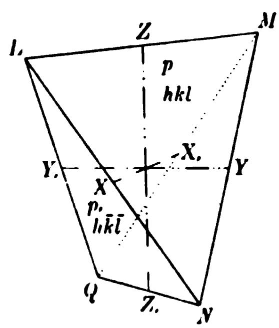

FIG. 146.

6. The face of any sphenoid can be selected to give the parameters $a : b : c$; and we may for the present suppose them to be known.

Let p, in Fig. 146, be (hkl), then $OX = a \div h$, $OY = b \div k$, $OZ = c \div l$. Rotation about OX interchanges p with $p_,$, and at the same time

equal positive and negative lengths on each of the axes OY and OZ perpendicular to OX. Hence, $OY_{,} = -OY = b \div \bar{k}$, $OZ_{,} = -OZ = c \div \bar{l}$. The point X is common to both faces. Hence $p_{,}$, passing through $XY_{,}Z_{,}$, has the symbol $(h\bar{k}\bar{l})$. Similarly, by rotation about $ZZ_{,}$, the face p changes places with the face which passes through LM and the parallel line $X_{,}Y_{,}$. The intercepts are: $OX_{,} = a \div \bar{h}$, $OY_{,} = b \div \bar{k}$, $OZ = c \div l$; and the symbol of the face LMQ is $(\bar{h}\bar{k}l)$. Again, by rotation about YY', the face p is brought into a position in which it meets the axes at $OX_{,} = a \div \bar{h}$, $OY = b \div k$, $OZ_{,} = c \div \bar{l}$. The symbol of the fourth face MNQ is $(\bar{h}k\bar{l})$. These four faces complete the sphenoid $a\{hkl\}$; which, therefore, consists of:

hkl, $h\bar{k}\bar{l}$, $\bar{h}k\bar{l}$, $\bar{h}\bar{k}l$. ....(a).

In a stereogram of the sphenoid $a\{hkl\}$, two poles above the paper occupy opposite quadrants, and two poles below the paper, represented by circlets, occupy the remaining pair of opposite quadrants. Such a stereogram is shown in Fig. 152, Art. 13.

7. Similarly, the sphenoid, Fig. 147, which consists of the faces parallel to those in the first, and which is consequently called the complementary form, has the symbol $a\{hk\bar{l}\}$, or $a\{\bar{h}\bar{k}\bar{l}\}$. It consists of the faces:

$hk\bar{l}$, $h\bar{k}l$, $\bar{h}kl$, $\bar{h}\bar{k}\bar{l}$ ....(b).

This latter sphenoid is clearly possible when $a\{hkl\}$ exists, for the symbols of its faces conform to the law of rational indices, and to the rule connecting the faces interchangeable by rotation through 180° about the axes of reference. Since each of the faces of the one sphenoid is parallel to a corresponding face of the second, the angles between the pairs of faces on the one are equal to those between the pairs of parallel faces of the other. The sphenoids cannot, however, be similarly orientated, or, as we may express it, superposed; for they are reciprocal reflexions of one another in planes parallel to any pair of the axes. The complementary sphenoids are therefore enantiomorphous.

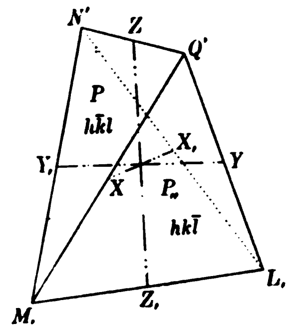

Fig. 147.

8. The rule, established above, connecting the symbols of pairs of faces interchangeable by rotation about a dyad axis perpendicular to two axes of reference is perfectly general. For any line

perpendicular to a dyad axis remains parallel to itself after rotation through 180°; and, if it meets the axis, equal lengths on opposite sides of the axis are interchanged. Hence, a pair of homologous faces makes equal intercepts on the pair of axes of reference meeting them on opposite sides of the origin. The corresponding indices differ therefore only in sign.

9. The special forms (Chap. XII, Art. 5) are of two kinds. *1.* When a face is parallel to one of the dyad axes, it is clearly brought into a parallel position by rotation through 180° about that axis; but into new positions inclined to its original position when it is turned about each of the axes which it meets. The two faces obtained by rotations about these latter axes will necessarily be parallel to each other and to the first axis. We, thus, have a set of four tautozonal faces intersecting in edges parallel to one of the dyad axes; and such that the angles between pairs of the faces, or their normals, are bisected by each of the two other axes. If the faces are parallel to $ZZ_{\prime}$, the vertical axis, the form $\{hk0\}$ is called *a prism*. If parallel to the makro-axis $YY_{\prime}$, the form $\{h0l\}$ is called a *makrodome*; and if to the brachy-axis $XX_{\prime}$, the form $\{0kl\}$ is a *brachydome*. Figs. 148—150 show three such forms, which, not enclosing a finite portion of space, cannot occur alone. They are in the figures cut short by pinakoids, the faces of which are perpendicular to the zone-axes of the four-faced forms.

The prism, Fig. 148, with vertical faces may be represented as $\alpha\{hk0\}$, or $\{hk0\}$; for, as we shall see later on, there is in this case no advantage in keeping the Greek prefix before the brackets. It includes the faces:

$$hk0,\ \bar{h}k0,\ \bar{h}\bar{k}0,\ h\bar{k}0 \ldots\ldots.(c).$$

The alternate faces are parallel; whilst the first two, and the last two, interchange places by rotation about $YY_{\prime}$; the middle pair, and also the extreme faces, interchange places by rotation about $XX_{\prime}$. The faces of the particular case $\{110\}$ we shall in this and the tetragonal system always denote by the letter *m*.

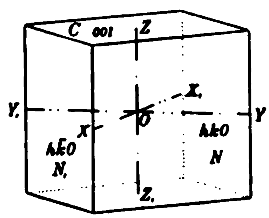

FIG. 148.

Similarly, the makrodome $\{h0l\}$, Fig. 149, includes the faces:

$$h0l,\ h0\bar{l},\ \bar{h}0\bar{l},\ \bar{h}0l \ldots\ldots\ldots.(d).$$

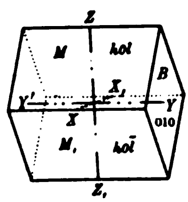

FIG. 149.

The student can easily arrange them

in pairs interchangeable about each of the axes, as was done in the case of the prism.

The brachydome $\{0kl\}$, Fig. 150, includes the faces:

$0kl$, $0\bar{k}l$, $0\bar{k}\bar{l}$, $0k\bar{l}$. ....(e).

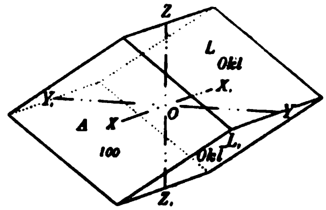

FIG. 150.

Although, in the above special forms, one index alone changes sign in the deduction of the symbol of a face from that of an adjacent face, we have not contravened the rule given for the general form $a\,\{hkl\}$. For, if one of these latter indices becomes zero, there can be no distinction between its positive and negative value.

2. Lastly, we have the three special forms, in each of which the faces are perpendicular respectively to one dyad axis, and are therefore parallel to the two other axes. In such a case the form is a pinakoid consisting of a pair of parallel faces. Thus we have the *makro-pinakoid* $\{100\}$ including (100) and $(\bar{1}00)$, both of which are perpendicular to $XX_{,}$ and parallel to $YY_{,}$ and $ZZ_{,}$. Similarly, the *brachy-pinakoid* $\{010\}$ includes (010) and $(0\bar{1}0)$. The pinakoid $\{001\}$ has the horizontal faces (001) and $(00\bar{1})$, and is often called the *basal pinakoid*, and one or other of its faces is called the *base*. One, two, or all three, pinakoids may be present on any one crystal of the class. The faces of the different pinakoids will as a rule have different physical characters, such as striæ, &c., so that it is often easy to distinguish one from another.

10. Crystals of any particular substance may present *combinations*, *i.e.* have several forms of the same or of different kinds associated together; and must be combinations if any special forms are present. We may, for instance, have the three pinakoids alone, forming a figure distinguishable from a cube only by the facts that the three sets of parallel faces have different physical characters and that the crystals are optically biaxal. We may also have crystals, like those shown in Figs. 148—150, and also combinations of the prism with one or more domes. Crystals on which special forms alone occur cannot be distinguished geometrically from similar crystals belonging to the next class.

11. Crystals of the following substances belong to this class:

Magnesium sulphate (*epsomite*), $MgSO_4 . 7H_2O$.

$$a : b : c = \cdot 990 : 1 : \cdot 571.$$

The crystals are usually prisms {110}, terminated by α {111} and occasionally also by {101}, {011}. O.A. ‖ (001); Bx_a ‖ OY (Chap. X, Art. 8).

Zinc sulphate (*goslarite*), $ZnSO_4 . 7H_2O$, isomorphous with the preceding salt. $a : b : c = \cdot 980 : 1 : \cdot 563$. O.A. ‖ (001); Bx_a ‖ OY.

Sodium potassium dextro-tartrate (*seignette salt*), $NaKC_4H_4O_6 . 4H_2O$.
$a : b : c = \cdot 8317 : 1 : \cdot 4296$.

The crystals are usually combinations of several prisms: {110}, {210}, {120}, with the pinakoids {100}, {010}, {001}. The horizontal edges made by these forms are sometimes modified by small faces: $o = \alpha$ {11$\bar{1}$}, $v = \alpha$ {211}, q {021} and r {101}. O.A. ‖ (010); Bx_a ‖ OX.

The isomorphous sodium ammonium salt forms similar crystals. O.A. ‖ (100); Bx_a ‖ OZ.

Hydrogen potassium dextro-tartrate, $HKC_4H_4O_6$, and the isomorphous ammonium salt, form crystals in which special forms and sphenoids occur together. In the first salt, $a : b : c = \cdot 7148 : 1 : \cdot 7314$; in the second, $a : b : c = \cdot 6931 : 1 : \cdot 7100$. In both O.A. ‖ (001); Bx_a ‖ OY.

12. Potassium-antimonyl dextro-tartrate (*tartar-emetic*),
$K(SbO)C_4H_4O_6$. $a : b : c = \cdot 9556 : 1 : 1 \cdot 1054$.

The forms, Fig. 151, are: $o = \alpha$ {111}, $\omega = \alpha$ {1$\bar{1}$1}, m {110}, c {001}. O.A. ‖ (001); Bx_a ‖ OY.

Similar crystals of the lævo-tartrate have been obtained, in which the parameters are the same, but in which the faces $\omega = \alpha$ {1$\bar{1}$1} give the predominant form; and the crystals are enantiomorphous to those of the dextro-tartrate.

The drawing is made as follows. On the projected cubic axes (Chap. VI, Art. 11), lengths $OA = \cdot 9556$, and $OC = 1 \cdot 1054$ are cut off. The sphenoid α {111} is then drawn with edges passing through A, B, C, &c., parallel to BC, CA, &c., in the manner described in Art. 5. From the middle points of the opposite edges through A and A_1, equal lengths are cut off by proportional compasses; and through the points so obtained the edges [mo], [m_1o_1], &c., are drawn parallel to the horizontal edges of the sphenoid. From the same edges of the sphenoid equal lengths are now cut off measured from the coigns, and through these points the edges [co], [c_1o_1], &c., are drawn. The faces ω must now be introduced. They are most easily inserted by cutting off equal lengths on the horizontal edges of the sphenoid o. Through each of the points on these edges, lines are drawn parallel to the opposite edges of the two faces o which pass through the point. The points at which they meet the edges [co], &c., give when joined in pairs the edges [$c\omega$], &c. These edges should be parallel to [mo], [m_1o_1], &c.

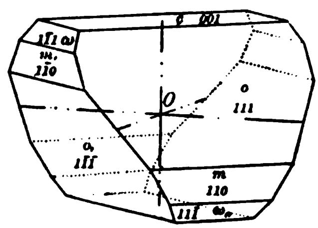

FIG. 151.

The edges $[om_{,}]$, $[o_{,}m]$, &c., have now to be found. The zone-symbol of $[om_{,}]$ is $[11\bar{2}]$. But, instead of forming the parallelepiped required by this symbol, it is easier to make a partial linear projection in the projected plane XOY. The face $m_{,}$ gives a trace through O parallel to $AB_{,}$, and m a trace through O parallel to AB. Hence, the direction of $[om_{,}]$ is given by joining C to the point of intersection of the trace $m_{,}$ with AB. The edge is to be drawn through the coign, already fixed, in which o, $o_{,}$ and $m_{,}$ meet. It meets the edge $[o\omega]$ at a coign through which the edge $[m_{,}\omega]$ has to be drawn. The rest of the drawing presents no special difficulty.

Solutions of the above tartrates rotate the plane of polarization. The crystals have, however, the ordinary optical properties characteristic of biaxal crystals, and there is no difference between those of a right- and left-handed tartrate of the same base.

13. *Asparagine*, $C_4H_8N_2O_3 . H_2O$, usually forms crystals such as are shown in Fig. 153; $a : b : c = \cdot4737 : 1 : \cdot8327$. The forms are: $m\{110\}$, $q\{021\}$, $c\{001\}$ and $o = a\{11\bar{1}\}$. O.A. $\parallel (010)$; and $Bx_a \parallel OZ$. The solution of these crystals rotates the plane of polarization to the left.

A dextro-asparagine has been made synthetically, and shows the enantiomorphous form $a\{111\}$ instead of $a\{11\bar{1}\}$; its solution rotates the plane of polarization to the right. The optical characters of the crystals are the same as those of ordinary asparagine.

The most important angles are: $mm_{,} = 50° \, 42'$, $cq = 59° \, 1'$, $qq_{,} = 61° \, 58'$. Hence, by the method given in Chap. VI, Art. 8,

$$a \div b = \tan \tfrac{1}{2} mm_{,} = \tan 25° \, 21' = \cdot4737, \; c \div b = \tfrac{1}{2} \tan cq = \cdot8327.$$

As shown in Fig. 152, $mom_{,}$ and $m''q'B_{,}$ are right-angled triangles. From the latter, $\sin B_{,}m'' = \tan B_{,}q' \cot B_{,}m''q'$; and from the former, $\sin mm_{,} = \tan om_{,} \cot m_{,}mo$. The first equation enables us to find the angle $B_{,}m''q'$ which is the same as $m_{,}mo$; or, if $om_{,}$ is alone needed, we can divide the one equation by the other and avoid one part of the computation. Thus,

$$\tan om_{,} = \tan B_{,}q' \sin mm_{,} \div \sin B_{,}m''.$$

$$\begin{aligned} L \tan (B_{,}q' = 30° \, 59') &= 9\cdot77849 \\ L \sin (mm_{,} = 50° \, 42') &= \underline{9\cdot88865} \\ & 19\cdot66714 \\ L \sin (B_{,}m'' = 64° \, 39') &= \underline{9\cdot95603} \\ L \tan (om_{,} = 27° \, 12\cdot\dot{6}') &= \overline{9\cdot71111} \\ \therefore \; co = 90° - om_{,} &= 62° \, 47'. \end{aligned}$$

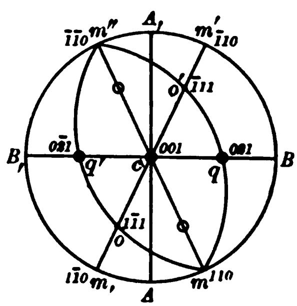

FIG. 152.

The stereogram is made by marking on the primitive points A and B at 90° from one another and the points m at distances 25° 21′ from A and $A_{,}$. The diameters through these points are then drawn and give the zone-circles $[cq]$, $[cm]$, &c. On $[cB]$ a point q at arc-distance 59° 1′ from c is found by the method given in Chap. VII, Prop. 1; and then the circles $[moq']$, $[mqo']$ are described.

For the drawing, the projected cubic axes of X and Z are multiplied by a and c to give OA and OC respectively; and OB is bisected. The lines AB, $AB_{,}$, $A_{,}B$, $A_{,}B_{,}$ are drawn; as also parallels to OX through the points of bisection of OB and $OB_{,}$: the points of intersection give the coigns $qmq_{,}$, &c., Fig. 153. Parallels to OZ are drawn through A and $A_{,}$, and also parallels to OX through C and $C_{,}$: the points of intersection of these lines are then joined to the adjacent coigns $qmq_{,}$, &c., and give the edges $[mq]$, &c. Lengths in any desired ratio to those of the edges are cut off by proportional compasses on the alternate edges $[mq]$; and through them the horizontal edges $[mc]$, $[qc]$ are drawn. Other equal lengths are then cut off from the remaining edges $[m_{,}q']$; and through the points so obtained the edges $[m_{,}o]$, $[oq']$ are drawn. Each face of the sphenoid is a parallelogram, two of the edges being parallel to $[cm_{,}]$, or to $[c_{,}m]$; the other edges being parallel to $[mq_{,}]$, &c. The rest of the construction is easy.

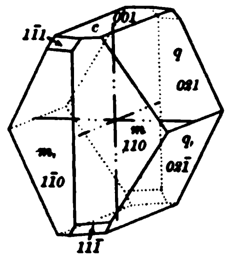

Fig. 153.

II. *Bipyramidal class*; $\{hkl\}$.

14. If to the three dyad axes of the preceding class a centre of symmetry is added, then, by Chap. IX, Prop. 4, the planes Σ perpendicular to the dyad axes are planes of symmetry, and the crystals have the following elements of symmetry: δ, $\delta_{,}$, $\delta_{,,}$, C, Σ, $\Sigma_{,}$, $\Sigma_{,,}$. Crystals of this class have hitherto been described as holohedral. The class will be called the *bipyramidal* class of the prismatic system.

15. The general form $\{hkl\}$, Fig. 154, consists of a diplohedral pyramid, or *bipyramid*, on a rhombic base, and has eight faces, each of which is a similar scalene triangle. A pyramid, such as Fig. 160, having faces meeting at only one apex on an axis of symmetry and thus having no parallel faces may be denoted as *acleistous* (Art. 1). One having parallel faces meeting at opposite apices on the axis may be denoted as *diplohedral*. In the latter pyramid the dyad axes join opposite coigns, and four like edges lie in each of the planes of symmetry; these like edges being interchangeable by rotation about the dyad axes, the angles over them are equal. But the edges in different planes Σ are dissimilar, and the angles over the edges in one plane cannot be equal to those over the edges in either of the other planes of symmetry except at particular temperatures in very special cases.

The face of any bipyramid may be taken as parametral plane (111), and determines the ratios $a : b : c$. Or we may take the face

of any prism to be (110), which determines the ratio $a:b$ in the manner given in Chap. VI, Art. 8. The ratio $c:b$ is then determined in a similar way by a second arbitrary choice of a dome $\{011\}$. The face common to the two zones [001, 110], and [100, 011] is, by Chap. V, Art. 11, the parametral plane (111). Occasionally it may be convenient to take a face of a makrodome $\{101\}$ to give the ratio $c:a$. When a has been found from the prism-face (110), c is found in terms of the same unit of length.

16. The symbols of the faces of $\{hkl\}$, Fig. 154, can now be found; for the faces consist of the sets which formed the two complementary sphenoids of the last class. The symbols are therefore:

$$\left.\begin{matrix} hkl, & \bar{h}kl, & \bar{h}\bar{k}l, & h\bar{k}l \\ hk\bar{l}, & \bar{h}k\bar{l}, & \bar{h}\bar{k}\bar{l}, & h\bar{k}\bar{l}. \end{matrix}\right\}\text{......(f).}$$

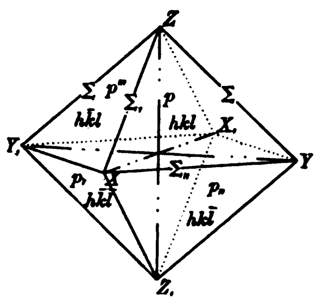

Fig. 154.

It is clear that (hkl) and $(hk\bar{l})$ meet the axes of X and Y in the same points, and that they meet the axis of Z at equal distances $c \div l$ on opposite sides of the origin. They are therefore symmetrically placed with respect to the plane of symmetry $\Sigma_{\prime\prime}$ containing the axes of X and Y and perpendicular to $\delta_{\prime\prime}$. The same is true of each of the pairs in the vertical columns. The face-symbols can be similarly arranged in pairs symmetrical with respect to Σ and $\Sigma_{\prime}$.

The rule for the deduction of the symbols of the faces from that of one of them is clear: the indices retain the same order, but the signs are changed in every possible way.

17. The special forms are identical with those of the preceding class. For since the faces of these forms are parallel to one dyad axis at least, they have parallel faces, and therefore the introduction of a centre of symmetry introduces no new faces. The plane of symmetry $\Sigma_{\prime\prime}$ intersects at right angles the faces of the prism $\{hk0\}$, and the faces of the pinakoids $\{100\}$ and $\{010\}$; similarly, each of the other planes of symmetry intersects at right angles the faces of a dome and of two pinakoids.

Hence, the prism $\{hk0\}$ has the four faces: $hk0$, $\bar{h}k0$, $\bar{h}\bar{k}0$, $h\bar{k}0$.

The brachydome $\{0kl\}$ has the faces: $0kl$, $0\bar{k}l$, $0\bar{k}\bar{l}$, $0k\bar{l}$.

And the makrodome $\{h0l\}$ has the faces: $h0l$, $\bar{h}0l$, $\bar{h}0\bar{l}$, $h0\bar{l}$.

The pinakoids are: {100} consisting of (100) and ($\bar{1}$00); {010} of (010) and (0$\bar{1}$0); and {001} of (001) and (00$\bar{1}$).

Hence, in crystals showing only special forms, *i.e.* such that the faces are parallel to one, or two, of the dyad axes, it is impossible to distinguish between the two classes by geometric development.

18. The following are some of the substances which form crystals belonging to this class:

Sulphur, S. The crystals from the mines of Sicily and Spain are developed in bipyramids {111} associated with the brachydome {011} and the basal pinakoid {001}. $a : b : c = \cdot 813 : 1 : 1\cdot 903$. O.A. $\|$ (010); $Bx_a \| OZ$.

Mispickel (arsenopyrite), FeAsS. The crystals rarely show general forms, and are usually combinations of the prism {110} with one or more brachydomes {011}, {012}, {014}; the latter forms being striated parallel to the axis OX. $a : b : c = \cdot 67726 : 1 : 1\cdot 18817$.

Bournonite, $(Pb,Cu_2)_3Sb_2S_6$, is usually found in much twinned crystals in which the base c {001} and faces parallel to OZ are alone shown. Fig. 155 (after Miers) represents a crystal from Cornwall. The forms are: a {100}, b {010}, c {001} m {110}, n {011}, e {210}, l {320}, f {120}, o {101}, x {102}, s {212}, y {111}, v {121}, u {112}.

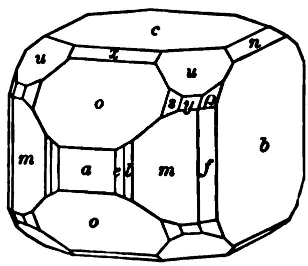

FIG. 155.

The elements are:

$$D = 010 \wedge 011 = 48° \; 6\cdot 75';$$
$$E = 001 \wedge 101 = 43° \; 43';$$
$$F = 100 \wedge 110 = 43° \; 10'$$

(see Art. 26). Hence,

$$a : b : c = \cdot 93797 : 1 : \cdot 89686.$$

Brookite, TiO_2, has been found at Fronolen, near Tremadoc, in deep red crystals of thin tabular habit, the pinakoid a {100} being largely developed. The edges of the tablets are modified by numerous faces belonging to pyramids and to special forms. Fig. 156 represents with fair approximation the habit of a crystal from this locality. The forms are: a {100}, l {210}, m {110}, b {010}, t {021}, c {001}, y {104}, x {102}, o {111}, z {112}, e {122}, n {121}.

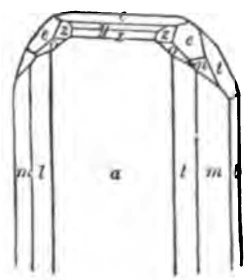

FIG. 156.

The elements given by Miller are:

$$D = 010 \wedge 011 = 46° \; 38\cdot 3';$$
$$E = 001 \wedge 101 = 48° \; 17\cdot 7';$$
$$F = 100 \wedge 110 = 40° \; 5'.$$

Hence $a : b : c = \cdot 8416 : 1 : \cdot 9444$.

O. A. for red and yellow light || (001), for green and blue light || (010). Bx_a in all cases || OX.

The mineral has been found at Magnet Cove, Arkansas, in black crystals, Fig. 169, which closely resemble hexagonal bipyramids, for they have twelve almost similar triangular faces. This is due to the facts that the faces of the two forms $m\{110\}$, $e\{122\}$ are nearly of the same size, and that the angles between the adjacent faces of the combination approximate closely to those of a hexagonal pyramid. Thus, the angle $mm_{,} = 80^\circ 10'$, approximates to $ee''' = e_{,}e_{,,} = 78^\circ 57'$; and $me = me_{,,} = 45^\circ 42'$ is nearly the same as $ee' = 44^\circ 23'$. These crystals are discussed in Art. 38.

Aragonite, $CaCO_3$, and the isomorphous carbonates of barium, strontium, and lead, give crystals which are usually very much twinned and will be discussed in Chap. XVIII. For aragonite, O. A. || (100); Bx_a || OZ.

Barytes and the isomorphous sulphates of strontium and lead afford good instances of crystals of this class. The habit of the crystals varies much. Fig. 157 represents a simple combination, which serves to show the habit of barytes from the Auvergne, and of celestine ($SrSO_4$) from

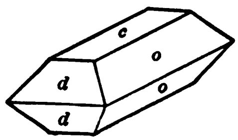

FIG. 157.

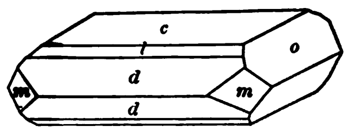

FIG. 158.

Sicily. Fig. 158 represents a habit found in crystals of barytes from Dufton, and in crystals of celestine from Yate in Gloucestershire and from Lake Erie; the forms being:—$c\{001\}$, $o\{011\}$, $m\{110\}$, $d\{102\}$, $l\{104\}$. The parameters are: for barytes, ·8152 : 1 : 1·3136; and for celestine, ·7789 : 1 : 1·2800. For all three minerals, O. A. || (010); Bx_a || OX.

Topaz, $\{Al(F, OH)\}_2SiO_4$, is usually found in well developed crystals. The common habit is that of an elongated prism of eight faces {110} and {210}, terminated by the base {001} and by domes and pyramids.

There is a very facile cleavage parallel to the base; and complete crystals are somewhat rare. Occasionally the crystals have different pyramid faces at the two ends of the vertical axis, but the physical characters seem to show that the crystals belong to this class, and not to the next. O. A. || (010); Bx_a || OZ. The crystals are discussed in detail in Art. 39.

Cordierite, $H_2(Mg, Fe)_4Al_8Si_{10}O_{37}$, is found in large stout crystals at Bodenmais, Bavaria. The crystals are easily altered, giving rise to numerous pseudomorphs. The crystal shown in Fig. 159 has

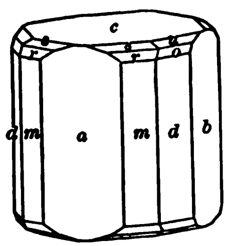

FIG. 159.

the forms: *a* {100}, *b* {010}, *c* {001}, *m* {110}, *d* {130}, *r* {111}, *s* {112}, *o* {131}, *u* {134}. O. A. ‖ (100); Bx. ‖ *OZ*.

The crystals are remarkable for their pleochroism. Thus, if polarised white light is transmitted by vibrations parallel to *OX*, the colour is a pale blue; if by vibrations parallel to *OY*, a dark blue; and if by vibrations parallel to *OZ*, a yellowish white.

III. *Acleistous pyramidal class*; $\mu\,\{hkl\}$.

19. If a crystal has a plane of symmetry Σ and a dyad axis δ lying in it, then, by Chap. IX, Prop. 9, the crystal must have a second plane of symmetry $\Sigma_{,}$ at right angles to Σ and intersecting it in the axis. Further, if any two of the above elements occur together, the third must also be present in the crystal. The class of crystals having this symmetry may be called the *acleistous pyramidal*, or *hemimorphic*, class of the prismatic system: the dyad axis is uniterminal, and the development of the crystals at the two ends is sometimes markedly different.

20. Since planes of symmetry are perpendicular to possible zone-axes (Chap. IX, Prop. 1), there must be a pair of zone-axes at right angles to one another and to the dyad axis. These three lines are the most convenient axes of reference. Further, since the relations between δ, Σ, and $\Sigma_{,}$, are not altered by change of temperature, or by similar physical changes which give rise to homogeneous strains, it follows that the axes will remain at right angles to one another so long as the structure of the crystal is not destroyed. They differ from those taken as axes of reference in the preceding classes, inasmuch as the two perpendiculars to Σ and $\Sigma_{,}$ are not dyad axes and their plane is not one of symmetry.

The face of any pyramid may be taken to be (111), and will give the ratio $a : b : c$.

21. It is easy to see that the general form $\mu\,\{hkl\}$, Fig. 160, consists of an acleistous pyramid of four faces meeting the dyad axis at the same point, and is similar in all respects to one half of the bipyramid of the last class. For the four faces of the latter form meeting at a point on a dyad axis are symmetrically placed with respect to the dyad axis and to the two planes of symmetry intersecting in the axis.

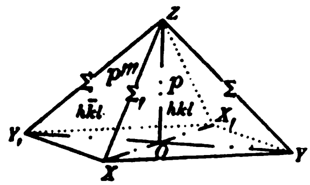

FIG. 160.

There is nothing to limit the position of the dyad axis, though on general principles it is desirable to place it vertically. It is therefore advisable to indicate by a dot, or other conventional mark, the index which refers to the dyad axis. Thus, $\mu\{hk\underset{\cdot}{l}\}$ indicates that the dyad axis is vertical, and that the index l does not change sign; the four faces in the form are therefore:

$$hkl,\ \bar{h}kl,\ \bar{h}\bar{k}l,\ h\bar{k}l \ldots\ldots\ldots\ldots\ldots\ldots\ldots(g).$$

The four faces parallel to those given above satisfy the law of rational indices, and conform to the symmetry of crystals of this class; and they constitute the complementary form $\mu\{hk\underset{\cdot}{\bar{l}}\}$. The complementary forms—though geometrically *tautomorphous*[1], for the one can be placed in the position of the other by a rotation of 180° about either of the normals to the planes of symmetry—do not necessarily occur together; and when they do, their faces often differ in physical characters.

22. The special forms are of four kinds:

1. Pedions, $\mu\{001\}$ and $\mu\{00\bar{1}\}$; if, as we shall throughout suppose, the dyad axis is vertical.

2. Pinakoids having their faces parallel to Σ and $\Sigma_{,}$ respectively. The first is $\{100\}$, consisting of (100) and $(\bar{1}00)$; the second $\{010\}$ has the faces (010) and $(0\bar{1}0)$.

3. Prisms $\{hk0\}$, which are identical geometrically with those of the preceding classes.

4. Gonioids (see Chap. XII, Art. 11), or *hemi-domes*, $\mu\{0k\underset{\cdot}{l}\}$, and $\mu\{h0\underset{\cdot}{l}\}$, each of which consists of two faces meeting in an edge bisected at right angles by the dyad axis:—$\mu\{0k\underset{\cdot}{l}\}$ consists of $(0kl)$, $(0\bar{k}l)$; $\mu\{h0\underset{\cdot}{l}\}$ of $(h0l)$, $(\bar{h}0l)$. The faces of the first are symmetrical to $\Sigma_{,}$, those of the second to Σ.

23. It has been found that in simple, *i.e.* untwinned, crystals of this class the physical properties at the opposite ends of the uniterminal dyad axis are different. This is shown by the different electrifications excited at the opposite ends by change either of temperature or pressure. When crystals of this class have shown the same electrifications at opposite ends of the dyad axis, corrosion of the prism- and pinakoid-faces has revealed a structure which can only be explained by twinning (see Chap. XVIII).

[1] From ταὐτό, the same, and μορφή, shape.

24. Crystals of struvite, $Mg(NH_4)PO_4 . 6H_2O$, Fig. 161; of smithsonite, $Zn_2(HO)_2SiO_3$, Fig. 162; and of pirssonite, $CaCO_3 . Na_2CO_3 . 2H_2O$, belong to this class.

In *struvite*, $a : b : c = \cdot5667 : 1 : \cdot9121$; and the forms shown in the figure are: $B\{010\}$, $k = \mu\{04\underline{1}\}$, $q = \mu\{01\underline{1}\}$, $r = \mu\{10\underline{1}\}$, $\mu = \mu\{10\bar{3}\}$ and $c = \mu\{00\bar{1}\}$. The lower end, where $(00\bar{1})$ is shown, is the analogous pole,

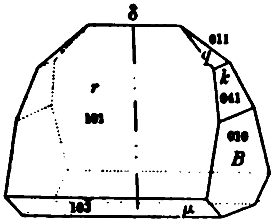

Fig. 161.

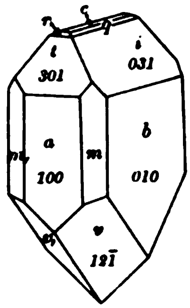

Fig. 162.

i.e. that which becomes positively electrified by rise of temperature and negatively electrified by fall of temperature; the upper end, where the faces r and q meet, is the antilogous pole, *i.e.* that which becomes negatively electrified by rise of temperature and positively electrified by fall of temperature. O.A. || (001); Bx_a || OY.

In *smithsonite*, $a : b : c = \cdot7835 : 1 : \cdot4778$. Fig. 162 shows the forms: $a\{100\}$, $m\{110\}$, $b\{010\}$, $i = \mu\{03\underline{1}\}$, $q = \mu\{01\underline{1}\}$, $c = \mu\{00\underline{1}\}$, $r = \mu\{10\underline{1}\}$, $t = \mu\{30\underline{1}\}$ and $v = \mu\{12\bar{1}\}$. The upper end showing base and domes is the analogous pole; the lower end, where the pyramid v appears, is the antilogous pole. O.A. || (100); Bx_a || OZ.

The crystals of *pirssonite*, described by Mr J. H. Pratt (*Am. Jour. of Sci.* [iv], II, p. 126, 1896), are sometimes of prismatic habit, $m\{110\}$ being large; and sometimes of tabular habit, $b\{010\}$ being large: they are generally terminated by different pyramids at opposite ends of the dyad axis. The crystals are strongly pyro-electric; at the analogous pole $\mu\{11\underline{1}\}$ and $\mu\{13\underline{1}\}$ are developed, at the antilogous pole $\mu\{11\bar{1}\}$. The measured angles, $mm_{,} = 59° 2'$, $pp'' = 111 \wedge \bar{1}\bar{1}1 = 63° 0'$, give $a : b : c = \cdot5661 : 1 : \cdot3019$. O.A. || (001); Bx_a || OY.

Formulæ and methods of calculation.

25. In the preceding sections we have seen that crystals of all the three classes can be referred to rectangular axes parallel to three dissimilar edges, and that some of the special forms—the prisms and a pair of pinakoids—are common to all classes. We

we see that the [illegible] of the second class can be obtained from a [illegible], or from a [illegible], by the introduction of a centre of symmetry, i.e. of faces parallel to those included in the latter forms. It follows, therefore, that the formulæ required for the determination of the parameters and of the face-symbols from the angles, or of the angles from the symbols and parameters, will be the same for all three classes.

26. Let D, E and F represent the angles [illegible], [illegible], and [illegible], respectively. They will be called the angular elements of a prismatic crystal. Let a, b, c be the parameters, or linear elements.

From equations (1), Chap. IV, we have for the pole P (hkl), Fig. 163,

$$\frac{a \cos XP}{h} = \frac{b \cos YP}{k} = \frac{c \cos ZP}{l}.$$

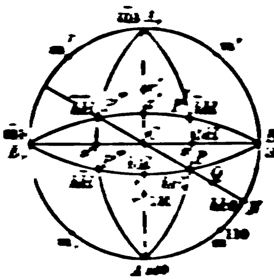

Fig. 163.

And, since the axes are rectangular, the axial points X, Y, Z coincide with the poles A (100), B (010) and C (001), respectively. Also (McL. and P. *Spher. Trig.* II, p. 71)

$$\cos^2 AP + \cos^2 BP + \cos^2 CP = 1 \ldots . (1).$$

The above equations can, therefore, be written:

$$\frac{a \cos AP}{h} = \frac{b \cos BP}{k} = \frac{c \cos CP}{l} = \frac{1}{\sqrt{\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}}} \ldots\ldots (2).$$

The last term is got from the others as follows. Let the value of each term be r. Then $\frac{hr}{a} = \cos AP$, $\frac{kr}{b} = \cos BP$, $\frac{lr}{c} = \cos CP$. Add together the squares of each equation. Then

$$r^2 \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}\right) = \cos^2 AP + \cos^2 BP + \cos^2 CP = 1.$$

Hence, r has the value given in the last term of (2).

Equations (2) are true of every pole of the crystal. Hence, if the values of h, k, l are introduced and a, b, c are known, the angles AP, BP and CP can be computed. Or, vice versâ, if the angles and parameters are known, the indices h, k, l can be found. But the direct process just indicated is laborious, and we proceed to obtain simpler formulæ better adapted to logarithmic computation.

27. Suppose any two of the forms $m\{110\}$, $q\{011\}$, $r\{101\}$ to be present on the crystal. Then a pair of the angular elements, D, E, F, can be determined by direct measurement with the goniometer. For $D = Bq = \frac{1}{2}(011 \wedge 01\bar{1}) = 90° - \frac{1}{2}(011 \wedge 0\bar{1}1)$; $E = Cr = \frac{1}{2}(101 \wedge \bar{1}01)$; $F = \frac{1}{2}(110 \wedge 1\bar{1}0)$.

But if, in equations (2), the pole P is made to coincide with each of the poles q, r and m, in turn, we have:

$$\left.\begin{aligned} &\text{for } q, \quad b\cos Bq = c\cos Cq, \\ &\;\;,, \quad r, \quad c\cos Cr = a\cos Ar, \\ &\;\;,, \quad m, \quad a\cos Am = b\cos Bm. \end{aligned}\right\} \quad \text{(3)}.$$

But $\cos Cq = \sin Bq$, $\cos Ar = \sin Cr$, and $\cos Bm = \sin Am$; since the angle $BC = CA = AB = 90°$.

Hence,
$$\left.\begin{aligned} \frac{b}{c} &= \frac{\sin Bq}{\cos Bq} = \tan Bq = \tan D, \\ \frac{c}{a} &= \frac{\sin Cr}{\cos Cr} = \tan Cr = \tan E, \\ \frac{a}{b} &= \frac{\sin Am}{\cos Am} = \tan Am = \tan F. \end{aligned}\right\} \quad \text{(4)}.$$

Hence, if a point at a finite distance from the origin is taken on one of the axes, through which one face of two of the forms m, r and q is to be drawn, the two other parameters are obtained by the above equations in terms of the length taken on the first axis. It is most common in descriptive works to take b on OY for unit length. The parameter a is then less than unity, and c may be either greater or less than unity: they are given by the formulæ;

$$a = \tan F, \quad c = \cot D \quad \text{(4*)}.$$

28. If the right sides of the three equations (4) are multiplied together, as also the left sides, we have

$$\tan D \tan E \tan F = \frac{b}{c}\frac{c}{a}\frac{a}{b} = 1 \quad \text{(5)}.$$

This relation shows that only two of the angles D, E, F, are needed. For, if D and F are determined either by measurement or calculation, then E can be found from equation (5). Consequently, in a prismatic crystal belonging to a given substance there are two independent constants, and two only: they vary with the substance.

29. The important equations (4) connecting the angular with the linear elements can be obtained directly from the elementary geometry of the axial system.

Let, in Fig. 164, AB be the line of intersection of the prism-face m with the axial plane XOY; and let OM be the face-normal which necessarily lies in the plane XOY. Then $AOM = F$. But the angle $AOM = 90° - BOM = OBM$.

Hence, $\tan F = \tan OBM = OA \div OB = a \div b$.

And, when $b = 1$, $a = \tan F$.

By taking the lines of intersection of each of the domes q and r with the axial plane to which their edges are perpendicular, and the

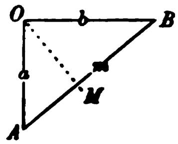

FIG. 164.

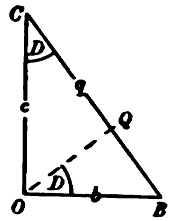

FIG. 165.

axes in this plane, we obtain similar figures which give the two remaining equations. Thus, let Fig. 165 represent the triangle formed by the axes OY, OZ and the trace BC of the face q; and let OQ be the normal to the face.

Then, $BCO = 90° - COQ = BOQ = D$.

Therefore, $\tan D = \tan BCO = b \div c$.

And, when $b = 1$, $c = \cot D$.

A similar figure in the plane XOZ gives the lengths on OX and OZ in which the trace of r (101) meets the plane; and we have

$$\tan E = c \div a.$$

30. Similar expressions for the tangents of the angles, which the faces of any other prism and dome make with the axial planes, are easily obtained. Let, in Fig. 166, LK and BC be the traces of faces of the domes $s\{0kl\}$ and $q\{011\}$ in the axial plane YOZ; and let BL' be drawn through B parallel to KL. Then, Os being the normal to $(0kl)$, the angle $\xi = BOs = 90° - L'BO = BL'O$.

Now, $OK = b \div k$, $OL = c \div l$.

Also, $OL' : OB = OL : OK$; $\therefore\ OL' = kc \div l$.

But, $\tan BOs = \tan \xi = \tan BL'O = OB \div OL'$

$$= \frac{lb}{kc} = \frac{l}{k}\tan D;\ldots\ldots\ldots\ldots(6),$$

since $b \div c = \tan D$.

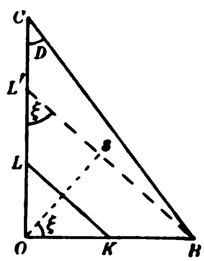

FIG. 166.

By similar figures in the axial planes ZOX and XOY, it is easy to obtain similar equations for the inclination, η, of the normal $t\,(h0l)$ to OZ; and for ζ, that of the prism-normal $N\,(hk0)$ to OX.

Thus, $$\tan\eta = \tan COt = \frac{h}{l}\tan E \;;\ldots\ldots\ldots\ldots(6^*),$$

and $$\tan\zeta = \tan AON = \frac{k}{h}\tan F\ldots\ldots\ldots\ldots(6^{**}).$$

31. The above expressions can be also obtained from equations (2) of the pole P. For, if P is made to coincide with $s\,(0kl)$; then $As = 90°$, and $\cos As = 0$, also $\cos Cs = \cos(90° - Bs) = \sin Bs$.

$$\therefore\ \frac{b\cos Bs}{k} = \frac{c\cos Cs}{l} = \frac{c\sin Bs}{l};$$

$$\therefore\ \tan Bs = \frac{l}{k}\frac{b}{c} = \frac{l}{k}\tan D\ldots\ldots\ldots\ldots(6).$$

Similarly, when P is made to coincide with $t\,(h0l)$, $Bt = 90°$, and $\cos At = \sin Ct$;

$$\therefore\ \frac{c\cos Ct}{l} = \frac{a\cos At}{h} = \frac{a\sin Ct}{h};$$

$$\therefore\ \tan Ct = \frac{h}{l}\frac{c}{a} = \frac{h}{l}\tan E\ldots\ldots\ldots\ldots(6^*).$$

And in the same way, by making P coincide with $N\,(hk0)$, we obtain $$\tan AN = \frac{k}{h}\tan F\ldots\ldots\ldots\ldots(6^{**}).$$

The three equations (6), (6*), (6**) are merely the expressions obtained from the anharmonic ratio of two axial poles and the two poles in their zone, such as A, B, (110), $(hk0)$. Thus, taking the A.R. $\{ANmB\}$, we have

$$\frac{\sin AN}{\sin Am} \div \frac{\sin BN}{\sin Bm} = \begin{vmatrix}100\\hk0\\\hline 100\\110\end{vmatrix} \div \begin{vmatrix}010\\hk0\\\hline 010\\110\end{vmatrix} = \frac{k}{h}.$$

But $\sin BN = \cos AN$, and $\sin Bm = \cos Am$,

$$\therefore\ \tan AN \div \tan Am = \frac{k}{h},$$

or $$\tan AN = \frac{k}{h}\tan F\ \ldots\ldots\ldots\ldots(6^{**}).$$

The other expressions can be obtained in a similar manner.

32. It is always best and simplest to determine the angular elements by direct measurement. If, however, the faces of the

special forms give poor reflexions, or are much striated, it may be necessary to determine them by calculation from the angles of a pyramid or sphenoid. The method employed will naturally vary with the faces on the crystal which give good reflexions, but it will almost always be possible to get angles which will enable us to obtain the elements by simple right-angled spherical triangles.

We shall illustrate the method to be followed by supposing a crystal to consist of a single form $a\,\{hkl\}$, or $\{hkl\}$, or $\mu\,\{hkl\}$, such as those given in Arts. 6, 16 and 21. With the last some other form must be present to enclose a finite space, but we shall not need to employ the measurement of the angles involving the faces of this second form.

In the case of $\{hkl\}$ and $\mu\,\{hkl\}$ we have a stereogram, Fig. 167, with the poles $P\,(hkl)$, $P'\,(\bar{h}kl)$, $P''\,(\bar{h}\bar{k}l)$ and $P'''\,(h\bar{k}l)$ symmetrically situated with respect to a dyad axis OC and the planes of symmetry BC and CA. The three different angles we can measure are: $PP' = P''P'''$, $PP''' = P'P''$, and $PP'' = P'P''' = 2CP$.

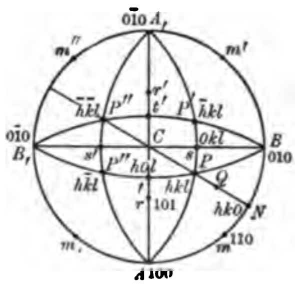

Fig. 167.

Let the zone-circle $[APP']$ intersect $[BC]$ in the pole $s\,(0kl)$, and let $[BPP''']$ intersect $[CA]$ in the pole $t\,(h0l)$, and let $[CP]$ meet $[AB]$ in the pole $N\,(hk0)$. The symbols of s, t and N are readily found by the rule of Chap. v, Art. 11.

By measurement of two of the angles given above, we obtain two sides of two of the six right-angled triangles into which the octant ABC is divided by the zone-circles through P and the axial poles. Thus, if PP' and PP'' are measured, the sides,

$$sP = PP' \div 2 = 90° - AP, \text{ and } CP = PP'' \div 2 = 90° - PN,$$

are known. Hence, two sides of each of the right-angled triangles CPs and APN are known. By Napier's rules, the third sides Cs and AN can both be found; as also the angles of the two triangles. If the elements are also known, then equations (6) and (6**) enable us to find the ratios $l : k$ and $h : k$. The indices of P are therefore determined.

If PP' and $P'P''$ or PP''' are the two measured angles, then a pair of sides of two other triangles are known. For PP' gives, as before, sP and AP; and $Pt = PP''' \div 2 = 90° - BP$. Hence, two of

the sides of each of the triangles BsP and APt are known. We can therefore find Bs, and $At = 90° - Ct$. Hence by (6) and (6*) the ratios of the indices of P are again determined.

If the form is $a\{hkl\}$, we have the two poles P, P'' above the paper associated with a pair $P_,$ and $P_{,,,}$ exactly below the points P''' and P' in the figure. The arcs between P and these poles below the paper are bisected at A and B, for $AA_,$ and $BB_,$ are dyad axes which interchange poles above with poles below the paper. Hence, measurement of any pair of the angles gives two of the angles AP, BP, CP, and therefore two sides of one of the triangles already discussed. Hence, knowing the elements, we obtain the three indices as in the previous cases.

If, in the preceding cases, we compute the arc-distance of P from each of the axial poles, we have the three arcs AP, BP, CP, and can apply the equations of the normal given in (2), and therefore find h, k, l directly. Thus, in the first case considered, AP and CP are given by measurement, and

$$\cos BP = \cos PN \cos BN = \sin CP \cos PCB.$$

33. But, if the crystal is one of a new substance, or if the observer is not anxious to refer it to parameters already selected by a previous investigator, he would naturally adopt the face P for his parametral plane (111). He could, then, by two measurements, and the computation of one of the right-angled triangles into which they enter, obtain the three arcs AP, BP, CP. Introducing these into equations (2) he would obtain the parametral ratios $a : b : c$. And in the manner already given, he can compute Bs, Ct and AN, which would, in this case, be the angular elements of the crystal.

34. If, however, the symbols of the faces are known, or if it is desired to assign to them some particular indices, the analysis indicated in Art. 32 will give the corresponding values of D, E and F, or of the parameters. For we have the same triangles, and determine the arcs AP, BP, CP from the measured angles in the same way as before. Thus, in the case first considered,

$$Bs = 90° - Cs \text{ and } BN = 90° - AN$$

are both supposed to be computed. But, from (6),

$\tan D = k \tan Bs \div l$; and from (6**), $\tan F = h \tan AN \div k$.

The third angular element E is then found from (5).

35. When several forms occur together on a crystal, the faces will, as a rule, fall into zones such that, when a few faces have been determined, the symbols of the others may be obtained by Weiss's zone-law (Chap. v, Art. 8), or by the anharmonic ratio of four tautozonal faces.

When two or more faces lie in a zone containing one of the axial planes, we have a very simple relation between their indices and the angles they make with the axial plane. Suppose the zone to be $[CP]$, Fig. 167, and let P be (hkl) and Q be (pqr). Then, from Weiss's zone-law, $h \div p = k \div q$.

Also, the A.R. $\{CPQN\}$ gives

$$\frac{\sin CP}{\sin CQ} \div \frac{\sin NP}{\sin NQ} = \frac{\begin{vmatrix} 001 \\ hkl \end{vmatrix}}{\begin{vmatrix} 001 \\ pqr \end{vmatrix}} \div \frac{\begin{vmatrix} hk0 \\ hkl \end{vmatrix}}{\begin{vmatrix} hk0 \\ pqr \end{vmatrix}} = \frac{kr}{lq}.$$

But $\sin NP = \cos CP$, and $\sin NQ = \cos CQ$;

$$\therefore \frac{\tan CP}{\tan CQ} = \frac{kr}{lq} \qquad\qquad (7).$$

If, then, the angles CP and CQ are measured and the symbol of one of the poles is known, that of the other is determined; or, conversely, knowing one of the angles and the symbols of the two poles, the remaining angle is readily computed.

36. A general expression for $\cos PQ$, where $P\,(hkl)$ and $Q\,(pqr)$ are any two poles whatever, can be obtained from equations (2) of the normal, and the known expression of analytical geometry which connects the arc PQ with the arcs AP, AQ, &c. For, when the axes are rectangular (McL. and P. *Spher. Trig.* II, p. 72),

$$\cos PQ = \cos AP \cos AQ + \cos BP \cos BQ + \cos CP \cos CQ \qquad (8)$$

But the ratios $\cos AP$, &c., are given in (2), and for Q we have the similar equations:

$$\frac{a \cos AQ}{p} = \frac{b \cos BQ}{q} = \frac{c \cos CQ}{r} = \frac{1}{\sqrt{\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} + \dfrac{r^2}{c^2}}}.$$

Substituting in (8) the values of $\cos AP$, $\cos AQ$, &c., we have

$$\cos PQ = \frac{\dfrac{hp}{a^2} + \dfrac{kq}{b^2} + \dfrac{lr}{c^2}}{\sqrt{\left(\dfrac{h^2}{a^2} + \dfrac{k^2}{b^2} + \dfrac{l^2}{c^2}\right)\left(\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} + \dfrac{r^2}{c^2}\right)}} \qquad (9).$$

This expression would be very laborious to compute, and is of little use in actual practice. When PQ is needed and the two poles do not fall into any zone already known, one of the following methods may be pursued. The symbol of the zone is determined, and those of the poles in which the zone-circle meets the axial zones $[BC]$, $[CA]$, $[AB]$. The symbols of these poles and the elements being known, their distances from the axial poles are found by equations (6), (6*), (6**); and then, by right-angled triangles, the arcs between them. We thus have a zone $[PQ]$ in which we know the symbols of, and distances between, three poles. Hence we can find by one of the methods given in Chap. VIII the distance of both P and Q from one of these poles.

Or we might find the angle which PQ subtends at one of the axial poles (C, say) viz. $PCQ = PCB - QCB$, and then find $\cos PQ$ by the well-known formula of spherical trigonometry; viz.

$$\cos PQ = \cos CP \cos CQ + \sin CP \sin CQ \cos PCQ.$$

37. The general formulæ given in the preceding Articles apply to any class of crystals in which we can take rectangular axes of reference. A complete mastery of the method of applying them to the solution of prismatic crystals will much help the student in dealing with tetragonal and cubic crystals. In these latter systems one or more of the elements, D, E, F, have special and fixed values which simplify some of the formulæ applicable to them and to prismatic crystals.

Examples.

38. *Brookite.* Taking the crystal of brookite, Fig. 169, to give on measurement the angles recorded in Art. 18, we shall first construct the stereogram, Fig. 168, and shall afterwards determine the parameters and lastly draw the crystal. From the angles it may be inferred that there are three planes of symmetry, which pass: (1) through the edges $[ee']$, $[e''e''']$ which, in Fig. 169, are nearly coincident with the paper, (2) through the edges $[mm_{,}]$, $[ee''']$, $[e_{,}e_{,,}]$, and (3) a plane perpendicular to the first pair and to the faces m and $m_{,}$.

The stereogram. The primitive being described with any convenient radius, the two diameters $aa_{,}$ and $bb_{,}$ are drawn at right angles and give the planes $\Sigma_{,}$ and Σ, respectively. From a and $a_{,}$ arcs am, $am_{,}$, &c., are measured off on the primitive each equal to 40° 5′, for $mm_{,} = m'm'' = 80° 10'$. The poles m are thus fixed.

Now $ae = (180° - ee') \div 2 = 67° 48{\cdot}5' = ae'''$. Points $\mathfrak{a}$, $\mathfrak{a}_{,}$ are then found on the primitive at arc-distance $a\mathfrak{a} = 67° 48{\cdot}5'$. A third point, which is the inter-

[illegible], and [illegible] is found on [illegible], and is at the same [illegible] from a. Through the three points a circle is drawn, [illegible] on the sphere passing through e and e'''. [illegible] ee' is described with the same radius through [illegible] the diameters through a and a_1.

[illegible] $= (180° - ee'') \div 2 = 59° 31·5'$. Hence, e and e' lie on a small [illegible] centre at b and be for arc-radius. Points β, β_1, β_{11} [illegible] are determined on the primitive and on $[bC]$ at [illegible] from b. The circle through these points meets the circles [illegible] a, a_1, &c., in the poles e and e'. An equal circle is drawn [illegible] of the diameters through β, β_1 and fixes the homologous [illegible] e'', e'''.

[illegible] $[ae'e']$, $[ae'''e'']$, $[bee''']$, $[be'e'']$, $[Cel]$, are then described.

[illegible]. We might now assume the face e to be (111), [illegible] possible pole l would be (110) and that of m would have [illegible]. This is easily done; for, from [illegible] bel, ael, we have:

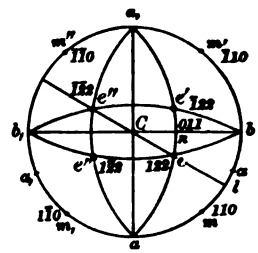

Fig. 168.

$$[illegible] \sin al \cos el,$$

$$[illegible] \cos el.$$

[illegible] the former by the latter,

$$[illegible] \therefore al = 59° 17'.$$

[illegible],

$$[illegible] \tan al = k : h;$$

[illegible], m would then be found [illegible].

[illegible] $al = 59° 17'$; and the element $D = bn$ is easily [illegible] ben. The parameters a and c can then [illegible].

[illegible] solution of this particular crystal, but it does [illegible] adopted by crystallographers, who all take [illegible]. But, knowing, as we now do, both am and al, the symbol of l is [illegible]. When am is [illegible], the equation is

$$[illegible] \tan 59° 17' \div \tan 40° 5'.$$

[illegible]

$$l \tan 59° 17' = 10·22610$$

$$l \tan 40° 5' = 9·93510$$

$$\log 2 = ·30100$$

$$[illegible], \text{ and } h = 1, \; k = 2.$$

[illegible] (120).

[illegible], in which bc and cn are both known, gives bn.

[illegible]

$$[illegible] \sin 67° 48·5'.$$

$$[illegible] = 9·80338$$

$$l \sin 67° 48·5' = 9·96657$$

$$[illegible] = 9·83671.$$

By comparison with the angular elements given in Dana's *Mineralogy* we see that n is the possible pole (011), and the angle 46° 38·25′ is the element D.

The pole e is now found to be (122); for it is the intersection of the two known zone-circles [100, 011] and [001, 120].

By computation from equations (4*) the parameters a and c are found; for when $b=1$, $a=\tan(am=40° 5')=·8416$, and $c=\cot(bn=46° 38·25')=·9444$.

Drawing the crystal. A set of unit rectangular axes, as given by Naumann and described in Chap. VI, Art. 22, are pricked through on to a sheet of paper. The length OB on the axis of OY is retained unchanged. The unit length on OX is diminished by multiplying the units of the scale which this length contains by the number $a=·8416$, or by its equivalent tan 40° 5′. The length OA thus found is measured off on OX. Similarly, the number of units of the scale which the cubic axis on OZ contains, is multiplied by cot 46° 38·25′, or $c=·9444$. The number of units thus obtained is then measured off from O on OZ, and gives OC, and $OC_,$.

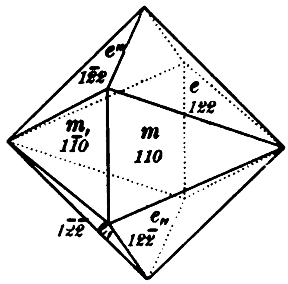

FIG. 169.

The pyramid $e\{122\}$ can now be drawn by joining A and $A_,$ to the points of bisection of OB, $OB_,$, OC and $OC_,$; or, still more easily, by taking points H and $H_,$ on OX, where $OH=OH_,=2OA$. The lines joining H and $H_,$ to B, $B_,$, C, $C_,$ give the pyramid.

We have to find the edges $[me]$, $[me_{,,}]$, &c. By the rule (Chap. V, Art. 4) the edge $[me]$ is parallel to $[110, 122]=[2\bar{2}1]$, and therefore to the diagonal through the origin of the parallelepiped having edges along the axes of $2a$, $-2b$, and c. It is simpler, however, to take the parallel diagonal through K on OY (OK being $2b$) of the similar, and similarly placed parallelepiped which has for edges, $OH=2a$, $OK=2b$ and $OC=c$. For we need only construct in the plane XOZ the parallelogram with sides parallel to the axes and passing through H, $H_,$, C, $C_,$. The lines joining the corners of the parallelogram to the points K and $K_,$ give the directions of all the edges $[me]$. These edges are now drawn through B and $B_,$, and should meet in pairs on the edges $[ee''']$, $[e_,e_{,,}]$, &c. The edges $[mm_,]$ join the coigns so obtained, and should be parallel to the axis OZ.

39. *Topaz.* Fig. 170 represents the habit and development of two crystals from the Urals in the Cambridge Museum. The crystals somewhat resemble broken cubes with irregular modifications at the coigns. The edges $[cl]$, $[cl']$, $[ml]$ of the largest crystal, which was presented by H. Waechter, B.A. of Trinity College, are about 1·25 inches long. One end is well developed and shows faces of the forms: $c\{001\}$, $s\{113\}$, $o\{112\}$, $y\{021\}$. At the other end portions of $c_,$ and $y\,(02\bar{1})$ can be seen. The faces $m\{110\}$ are narrow, and are not quite of equal size. The faces $l\{120\}$ are very broad. The second crystal is smaller and shows faces of the forms: c, s, o, $x\{123\}$, $n\{011\}$, m, l, and $r\{130\}$. The poles of the above forms, except r, are all represented in the stereogram, Fig. 171.

The crystals from the Mourne Mts., from Minas Geraes, Brazil, and from the Thomas Range, Utah, have a different habit. In them the prism-faces m and l are more nearly of the same dimensions. The base is small and often absent, whilst pyramids and domes are usually largely developed.

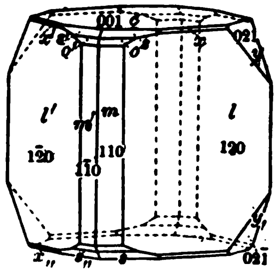

FIG. 170.

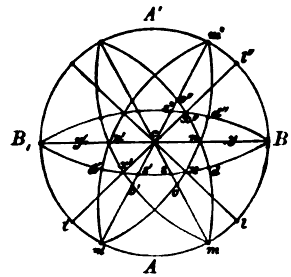

FIG. 171.

We shall adopt m as (110) and n as (011) and the corresponding parameters given by Koksharov; $a : b : c = {\cdot}52854 : 1 : {\cdot}95395$.

We shall, also, suppose the symbols, given above for the several forms, to have been determined from approximate measurements by comparison with the data of Brooke and Miller's *Mineralogy*. From the parameters we shall now calculate the angular elements, and the angles between the poles in the principal zones.

From equations (4), b being unity, we have

$$\tan F = a, \quad \therefore \ F = 27^\circ\ 51{\cdot}5';$$
$$\cot D = c, \quad \therefore \ D = 46^\circ\ 21'.$$

Hence, from (5), $\cot E = \tan D \tan F$; and $E = 61^\circ\ 0{\cdot}6'$.

Zone $[mm']$. From (6**)

$$\tan Al = 2\tan F, \quad \therefore \ Al = 46^\circ\ 35{\cdot}\dot{3}'$$
$$\tan Ar = 3\tan F, \qquad Ar = 57^\circ\ 45{\cdot}7'.$$

Zone $[m'n]$. From the right-angled triangle Bnm'', we can now determine (α) $m''n$, and (β) the angle $nm''B = \wedge\, mm'o$. Then, from the right-angled triangles mom', lxm', we can determine (γ) $m'o$, and (δ) $m'x$. The formulæ are:

(α) $\cos m''n = \cos Bm'' \cos Bn = \sin 27^\circ\ 51{\cdot}5' \cos 46^\circ\ 21'; \ \therefore \ m''n = 71^\circ\ 11'$.

(β) $\cot nm''B = \sin Bm'' \div \tan Bn = \cos 27^\circ\ 51{\cdot}5' \cot 46^\circ\ 21'; \ \therefore \ nm''B = 49^\circ\ 51{\cdot}\dot{3}'$.

(γ) $\tan m'o = \tan mm' \div \cos nm'm = \tan 55^\circ\ 43' \div \cos 49^\circ\ 51{\cdot}\dot{3}'; \ \therefore \ m'o = 66^\circ\ 16{\cdot}4'$.

(δ) $\tan m'x = \tan m'l \div \cos nm'm = \tan 74^\circ\ 26{\cdot}8' \div \cos 49^\circ\ 51{\cdot}\dot{3}'; \ \therefore \ m'x = 79^\circ\ 49{\cdot}6'$

The pole d'', Fig. 171, lies in $[bs''] = [301]$ and in $[m'n] = [11\bar{1}]$. By Chap. v, Table (23), its symbol is found to be $(\bar{1}43)$. The angle nd'' can be now determined by the A. R. of four poles.

(ϵ) From the A. R. $\{d''nom'\}$, we have

$$\frac{\sin d''n}{\sin d''o} \div \frac{\sin m'n}{\sin m'o} = \left|\begin{array}{c} \bar{1}43 \\ 011 \\ \hline \bar{1}43 \\ 112 \end{array}\right| \div \left|\begin{array}{c} 1\bar{1}0 \\ 011 \\ \hline 1\bar{1}0 \\ 112 \end{array}\right| = \frac{2}{5}.$$

$$\therefore \frac{\sin d''n}{\sin d''o} = \frac{2 \sin m'n}{5 \sin m'o} = \frac{\cdot 4 \sin 71^\circ\ 11'}{\sin 66^\circ\ 16{\cdot}4'} = \tan \theta.$$

$$\begin{aligned} L \sin 71^\circ\ 11' &= 9{\cdot}97615 \\ \log \cdot 4 &= \bar{1}{\cdot}60206 \\ &\overline{9{\cdot}57821} \\ L \sin 66^\circ\ 16{\cdot}4' &= 9{\cdot}96165 \\ L \tan (\theta = 22^\circ\ 28{\cdot}14') &= \overline{9{\cdot}61656}. \end{aligned}$$

$$\therefore \tan \tfrac{1}{2}(d''o - d''n) = \tan (45^\circ - \theta) \tan \tfrac{1}{2}(d''o + d''n).$$

But
$$d''o - d''n = on = 42^\circ\ 32{\cdot}6',$$

$$\therefore \tan \tfrac{1}{2}(d''o + d''n) = \tan 21^\circ\ 16'{\cdot}3 \div \tan 22^\circ\ 31{\cdot}86'.$$

$$\begin{aligned} L \tan 21^\circ\ 16{\cdot}3' &= 9{\cdot}59030 \\ L \tan 22^\circ\ 31{\cdot}86' &= 9{\cdot}61789 \\ L \tan 43^\circ\ 11 \quad &= \overline{9{\cdot}97241}. \end{aligned}$$

$$\therefore d''o + d''n = 86^\circ\ 22',$$
$$d''o - d''n = 42^\circ\ 32{\cdot}6'.$$

$$\therefore d''o = 64^\circ\ 27{\cdot}3', \; d''n = 21^\circ\ 54{\cdot}7', \text{ and } m''d'' = 49^\circ\ 16{\cdot}3'.$$

Zone [*com*]. From the triangle mom', we have

$$\cos om = \cos om' \div \cos mm'; \quad \therefore om = 44^\circ\ 24{\cdot}7'.$$

Hence, by equation (7),

$$\tan cs = 2 \tan co \div 3; \quad \therefore cs = 34^\circ\ 14'.$$

Zone [*Bdxs*]. From the right-angled triangle Bsm, we have

$$\cos Bs = \cos sm \cos Bm = \sin cs \sin Am: \quad \therefore Bs = 74^\circ\ 45{\cdot}5'.$$

But the point bisecting ss' is a possible pole (103) at 90° from B: call it t. Then by an equation, similar to (7), which is derived from the A.R. $\{Bzst\}$, where z is any pole (hkl) in the zone, we have $\tan Bz = \dfrac{l}{3k} \tan Bs$.

Hence, for x (123),
$$\tan Bx = \tfrac{1}{2} \tan 74^\circ\ 45{\cdot}5';$$
and for d (143),
$$\tan Bx = \tfrac{1}{4} \tan 74^\circ\ 45{\cdot}5'.$$

$$\therefore Bx = 61^\circ\ 24{\cdot}7', \quad Bd = 42^\circ\ 32{\cdot}2'.$$

The angle xl is now easily determined, for

$$\cos xl = \cos Bx \div \sin Al; \quad \therefore xl = 48^\circ\ 49'.$$

Zone [*cny*]. In this zone $\tan cy = 2 \tan (cn = 43^\circ\ 39')$. $\therefore cy = 62^\circ\ 20{\cdot}\dot{3}'$.

Many of the above angles are brought together in the following tables:

$$\left[\begin{array}{lll} Br & 32^\circ & 14{\cdot}3' \\ Bl & 43 & 24{\cdot}\dot{6} \\ Bm & 62 & 8{\cdot}5 \\ mm' & 55 & 43 \end{array}\right] \quad \left[\begin{array}{lll} cn & 43^\circ & 39' \\ cy & 62 & 20{\cdot}\dot{3} \\ nn' & 87 & 18 \end{array}\right] \quad \left[\begin{array}{lll} cs & 34^\circ & 14' \\ co & 45 & 35{\cdot}3 \\ cm & 90 & 0 \end{array}\right] \quad \left[\begin{array}{lll} Bd & 42^\circ & 32' \\ Bx & 61 & 24{\cdot}7 \\ Bs & 74 & 45{\cdot}5 \\ ss' & 30 & 29 \end{array}\right] \quad \left[\begin{array}{lll} m'o & 66^\circ & 16{\cdot}4' \\ m'x & 79 & 49{\cdot}6 \\ m'n & 108 & 49 \\ m'd'' & 130 & 43{\cdot}7 \end{array}\right]$$

CHAPTER XIV.

THE TETRAGONAL SYSTEM.

1. THE crystals possible in this system fall into seven classes, and have each a principal axis which is either a tetrad axis, or a dyad axis perpendicular to which pairs of like edges occur at right angles to one another. The classes will be discussed in the following order:

I. The *scalenohedral* (*sphenoidal-hemihedral*) class, the crystals of which have three dyad axes at right angles to one another associated with two like planes of symmetry, S and S', intersecting in one of the axes and bisecting the angles between the two others. The axis $\Delta\Delta_{,}$, Fig. 172, in which the planes S and S' meet, is the principal axis; and the two axes $O\delta$, $O\delta'$ are like dyad axes, though not interchangeable.

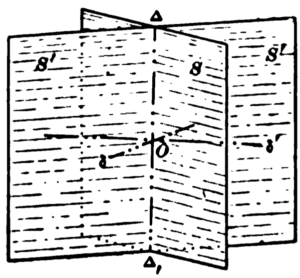

FIG. 172.

II. The *diplohedral ditetragonal* (*holohedral*, *ditetragonal-bipyramidal*) class, the crystals of which have a tetrad axis T; four dyad axes, 2δ and 2Δ, perpendicular to T; a centre of symmetry C; and five planes of symmetry, each perpendicular to one of the axes of symmetry. The planes may be denoted by Π, S or Σ according as they are perpendicular to T, Δ or δ. The above elements of symmetry may be shortly given as follows:

$$T, \ 2\delta, \ 2\Delta, \ C, \ \Pi, \ 2\Sigma, \ 2S.$$

III. The *acleistous tetragonal* (*pyramidal*, *hemimorphic-hemihedral*) class, the crystals of which have only a tetrad axis T.

IV. The *diplohedral tetragonal* (*bipyramidal, pyramidal-hemihedral*) class, in the crystals of which T is associated with a centre of symmetry C and a plane of symmetry Π perpendicular to T.

V. The *trapezohedral* class, in which four dyad axes 2δ and 2Δ are associated with the tetrad axis T.

VI. The *acleistous ditetragonal* (*ditetragonal-pyramidal*) class, the crystals of which have four planes of symmetry, $2S$ and 2Σ, intersecting in the tetrad axis T.

VII. The *sphenoidal* (*sphenoidal-tetartohedral*) class, in which every form is a sphenoid, save when the faces are parallel or perpendicular to the principal axis. One pair of opposite edges of each sphenoid are at right angles to one another and to the principal axis which is a dyad axis; and the crystals have no other element of symmetry.

I. *Scalenohedral class*; $\kappa\{hkl\}$.

2. It is clear from the definition given in Art. 1 that the planes S and S' are like planes of symmetry at 90° to one another, for they change places when the crystals are turned through 180° about either of the dyad axes $O\delta$, $O\delta'$.

Again, the axes $O\delta$, $O\delta'$ are like axes at 90° to one another, for they are reciprocal reflexions in each of the planes S and S' which bisect the angles between them; but, in this class, the axes $O\delta$, $O\delta'$ are not interchangeable. They differ also from the dyad axis $\Delta O\Delta_{,}$, for two planes of symmetry intersect in the latter; and, as we shall see later on, similar edges occur in pairs which are perpendicular to it and are at right angles to one another. This axis $\Delta O\Delta_{,}$ is a principal axis, and in translucent crystals coincides with the direction of the optic axis.

3. It is obviously convenient to select the three dyad axes as axes of reference, for they remain at right angles to one another at all temperatures. The principal axis is always taken as OZ. Again, since the axes of X and Y are reciprocal reflexions in each of the planes of symmetry S and S', equal lengths on them must always correspond to one another. If the representation of forms possible on a crystal is to accord with the symmetry, equal lengths on OX and OY must, therefore, be taken for parameters. It follows that

only one element varies with the substance in crystals of this class. This element may be taken to be the ratio of the parameter c measured on the principal axis to that measured along either OX or OY, which we shall call a. We shall take $O\delta$ to be OX, and $O\delta'$ to be OY; and when we desire to denote that lengths equal to a, or to $a \div h$, are measured on OY we shall denote them by $a_{,}$, or $a_{,} \div h$.

We shall see in the course of the Chapter that, as a consequence of the principal axis, the crystals of each class of the system can be referred to three rectangular axes of which OZ is the principal axis; and that equal parameters can be taken on the axes of X and Y.

4. *The pinakoid*, {001}. The simplest form on a crystal of this class is a pinakoid, the faces of which are perpendicular to the principal axis and to the planes S and S'. Such faces are possible (Chap. IX, Prop. 3): they are parallel to the axes of X and Y, and are interchangeable by a semi-revolution about either of them. The form is {001}, and comprises the faces:

$$001,\ 00\bar{1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

The possible crystals shown in Figs. 174, 176, are each terminated by the pinakoid.

5. *The tetragonal prism*, {110}. By Chap. IX, Prop. 1, a plane of symmetry is always parallel to a possible face. Let us suppose, in Figs. 173 and 174, a face $m_{,}$ to be drawn through A on OX parallel to S; and let it meet $OY_{,}$ at $A_{,}$. Then since S bisects the angle XOY, and $AA_{,}$ is parallel to the trace of S on the plane XOY, the angles $OAA_{,}$, $OA_{,}A$ are equal, each of them being 45°. Hence $OA_{,} = OA$. These lengths, being taken as the parameters on OX and OY, may be denoted by a. But the face $m_{,}$ through $AA_{,}$ parallel to S is also parallel to OZ; and its symbol is therefore $(1\bar{1}0)$. Similarly, there must be a parallel face m' on the opposite side of S which has the symbol $(\bar{1}10)$. These two faces also interchange places by rotation through 180° about OZ.

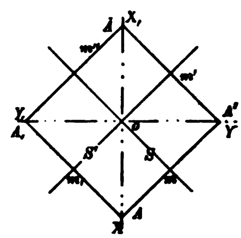

FIG. 173.

Again, rotation about OX through 180° interchanges equal positive and negative lengths on OY, and also the planes S and S'. The face $m_{,}$ is brought into the position given by m, and m' into that given by m''. The two new faces have the symbols (110), $(\bar{1}\bar{1}0)$.

Again, since m is parallel to S' and $m_,$ to S, it follows that the angles $mm_,$, mm', &c., are all equal to 90°. The form {110}, Fig. 174, is, therefore, a rectangular prism having the faces:

$$110,\ \bar{1}10,\ \bar{1}\bar{1}0,\ 1\bar{1}0 \ldots\ldots\ldots\ldots\ldots\ldots\ \ldots\text{(b)}.$$

This prism differs from the corresponding prism of the prismatic system inasmuch as the angles are permanently 90°. In a prismatic crystal the axes retain their direction whilst the temperature varies, but the coefficients of expansion along the axes are different; hence the angles of the prism and domes vary slightly with the temperature. In crystals of the class now under consideration the relations connecting the planes and axes of symmetry do not vary with the substance, or with the temperature as long as it is not raised to a point at which the crystalline structure is destroyed. The parameters on the axes of X and Y therefore remain equal when the temperature is changed; but the ratio of $a:c$ changes with the temperature. The same holds for crystals of all classes of this system.

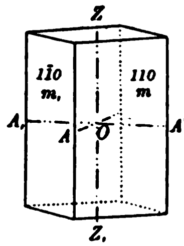

FIG. 174.

6. *The tetragonal prism*, {100}. A face parallel to the principal axis and to one of the other dyad axes, OY (say), is clearly possible, and is necessarily perpendicular to the remaining axis OX. The face may be drawn through A on OX, when its symbol is (100); and its trace on the plane XOY is given by $MAM_,$ in Fig. 175. The planes of symmetry meet the plane XOY in the traces marked S and S', and the face (100) in vertical lines through M and $M_,$. But through these vertical lines homologous faces pass which are inclined to S and S' at the same angles as (100). Faces of this form therefore pass through the traces $MA'M'$, $M_,A_,M''$, where $\wedge A'MO = \wedge OMA$, and $\wedge A_,M_,O = \wedge OM_,A$. But

$$\wedge OMA = \wedge OM_,A = 45°.$$

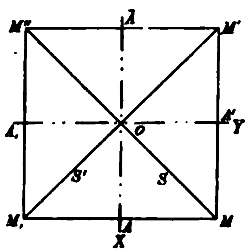

FIG. 175.

Hence $\wedge AMA' = \wedge AM_,A_, = 90°$; and the new faces are parallel to one another and to OX. Their symbols are therefore (010), (0$\bar{1}$0). Similarly, each of these faces is repeated over the planes S' and S

in the same face ($\bar{1}00$) which passes through the trace $M'\bar{A}M''$, where the angles between the faces meeting at M' and M'' are 90°.

We have therefore a second rectangular prism {100}, Fig. 176, which has the four faces:

100, 010, $\bar{1}00$, $0\bar{1}0$ .... (c).

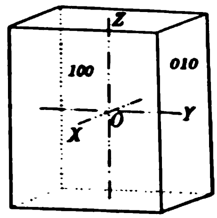

Fig. 176.

The faces of this prism truncate the edges of the prism {110}; and vice versâ, the faces of {110} truncate the edges of {100}.

7. *The ditetragonal prism*, {$hk0$}. If a face ($hk0$) parallel to OZ occurs on a crystal, it meets the plane XOY in a line HK, Fig. 177, where $OH = a \div h$, $OK = a_{,} \div k$. Let HK meet the trace of S in M. Through M a new face can be drawn parallel to OZ and inclined to S at the same angle as ($hk0$) makes with S. Let the new face meet the plane XOY in the trace $H'K'$. Now in the two triangles HOM, $K'OM$, we have $\wedge HOM = \wedge MOK'$, each being 45°; and $\wedge HMO = \wedge OMK'$, the equal angles on opposite sides of a plane of symmetry: also the side OM is common to both triangles. Hence the remaining angles and sides are equal. Therefore $\wedge OHM = \wedge OK'M$; and

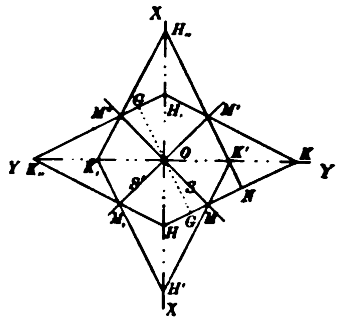

Fig. 177.

$$OK' = OH = a_{,} \div h.$$

Again, in the triangles HOK, $H'OK'$, we have $\wedge OHK = \wedge OK'H'$, and $\wedge HOK = 90°$ common; also the side OH = the side OK'. Therefore the remaining sides and angles are equal; and

$$OH' = OK = a \div k.$$

The face through the trace $H'K'$ has therefore the symbol ($kh0$).

A semi-revolution about OX interchanges positive and negative lengths on OY, and brings the traces HK, $H'K'$ into the positions given by $HK_{,,}$ and $H'K_{,}$; where

$$OK_{,,} = -OK = a_{,} \div \bar{k}, \text{ and } OK_{,} = -OK' = a_{,} \div \bar{h}.$$

The vertical faces through the traces $HK_{,,}$ and $H'K_{,}$ are ($h\bar{k}0$), ($k\bar{h}0$), respectively.

Again, by a semi-revolution about OY, the above four faces are brought into the positions of faces which meet OY at the same points as before, but in which the signs of the intercepts on OX are changed. The four faces taken in succession from $K'M'H_{\prime\prime}$ are:

$$\bar{k}h0,\ \bar{h}k0,\ \bar{h}\bar{k}0,\ \bar{k}\bar{h}0.$$

The form $\{hk0\}$, Fig. 178, is therefore a ditetragonal prism of eight faces which have the symbols:

$$hk0,\ kh0,\ \bar{k}h0,\ \bar{h}k0,\ \bar{h}\bar{k}0,\ \bar{k}\bar{h}0,\ k\bar{h}0,\ h\bar{k}0 \ \ldots\ldots\ldots \text{(d)}.$$

The alternate angles F over the edges passing through the points M, M', &c., are all equal, and so are the angles over the edges through H, K', $H_{\prime}$, $K_{\prime}$. But the angles F are never equal to those of the other set. Further, alternate faces are at right angles to one another. This important relation is easily proved from Fig. 177.

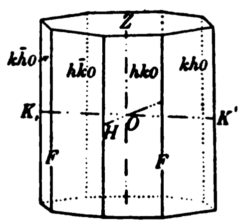

Fig. 178.

Let $H_{\prime\prime}K'$ be produced to meet HK in N. Then the external angle

$$K'NH = \wedge NKK' + \wedge NK'K.$$

But $\qquad \wedge NK'K = \wedge OK'H_{\prime\prime} = \wedge OK'H' = \wedge OHK.$

$$\therefore\ \wedge K'NH = \wedge OKH + \wedge OHK = 180^\circ - \wedge HOK = 90^\circ.$$

The reader should observe that two faces symmetrical with respect to S or S' have the indices h and k in reverse order. This is obvious in the case of the adjacent faces which meet in lines through the points M. But it is also true of faces like those through HK and $K_{\prime}H_{\prime\prime}$ which, if produced, will meet in the plane S'. These two faces have the symbols $(hk0)$, $(\bar{k}\bar{h}0)$, and are reciprocal reflexions with respect to S'.

Again, if OG is the normal to $(hk0)$, it is parallel to $H_{\prime\prime}K'N$, since $K'NK = 90^\circ$.

$$\therefore\ \wedge XOG = \wedge OH_{\prime\prime}K' = \wedge OKH.$$

$$\therefore\ \tan XOG = \tan OKH = OH \div OK = \frac{a}{h} \div \frac{a}{k} = \frac{k}{h} \ldots\ldots\ldots (1).$$

Hence, the inclination of a face $(hk0)$ to one of the vertical axial planes, and therefore also to the planes of symmetry, is determined as soon as the indices are known, and is independent of the parameter c on the vertical axis, which is the only element varying with the substance. The angles of ditetragonal prisms can therefore be calculated once for all, and will be the same for all

substances crystallizing in the tetragonal system. It is easy to construct a table of angles for such prisms, similar to the following, which gives the angles for a few cases of common occurrence.

$\{hk0\}$	$100 \wedge hk0$	$hk0 \wedge 110$	$F = hk0 \wedge kh0$.
$\{310\}$	18° 26′	26° 34′	53° 8′
$\{210\}$	26 34	18 26	36 52
$\{320\}$	33 41	11 19	22 38.

8. When the faces of the forms are inclined to the vertical axis and to the horizontal plane, the forms are closed figures, which differ from the preceding inasmuch as each single form completely encloses a finite portion of space.

The tetragonal bipyramid, $\{h0l\}$. Let one of the faces be parallel to OY and meet OZ at L, where $OL = c \div l$. Such a face can be supposed to be drawn through $MAM_,$ of Fig. 175, where OA is now replaced by $OH = a \div h$. A semi-revolution about OY brings the face into a parallel position, where it passes through $M'M''$ and $L_,$, and meets the axes of X and Z at distances $a \div \bar{h}$, $c \div \bar{l}$. The two faces have therefore the symbols $(h0l)$, $(\bar{h}0\bar{l})$. If, again, the crystal is turned through 180° about OX, the trace $MAM_,$ remains the same, but the points M and $M_,$, and the points L and $L_,$ are interchanged, we therefore have the face $MM_,L_,$ of Fig. 179, the symbol of which is $(h0\bar{l})$: the parallel face is $(\bar{h}0l)$. The above four faces are now repeated over the edges LM, $ML_,$, &c., in which they meet the planes S and S' in four new faces which are parallel to OX and meet OY at H', $H_,$, where $OH' = a_, \div h$. The symbols of the new faces are: $0hl$, $0h\bar{l}$, $0\bar{h}l$, $0\bar{h}\bar{l}$. The form $\{h0l\}$, Fig. 179 is a tetragonal bipyramid, the horizontal edges of which are at 90° to one another; it consists of the faces:

$$\left.\begin{matrix} h0l, & 0hl, & \bar{h}0l, & 0\bar{h}l \\ h0\bar{l}, & 0h\bar{l}, & \bar{h}0\bar{l}, & 0\bar{h}\bar{l} \end{matrix}\right\} \ldots\ldots\ldots (e).$$

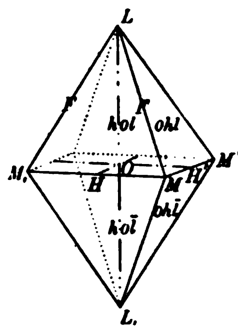

FIG. 179.

Each face is an isosceles triangle, and the angles F over polar edges (p. 112), such as LM, are all equal: so are also the angles over the horizontal edges $MM_,$, MM', &c.; but the angles over these latter edges are in no case equal to the angles F.

The face e (101) of the pyramid $\{101\}$ occupies a position similar

to that of $r\,(101)$ of a prismatic crystal. The equation connecting the parameters on OZ and OX must be the same as that given for $c : a$ and the face r in (4) of Chap. XIII; for a section of the pyramid by the plane XOZ gives a right-angled triangle LOH, in which the angle LHO is equal to the angle between OZ and the normal to e.

Hence, $\tan ZOe = \tan LHO = OL \div OH.$

But for $e\,(101)$, $OL = c$, and $OH = OA = a$. Hence, denoting the pole of (001) by C, and the arc Ce by E, which we shall call the *angular element* of the crystal (see Chap. XIII, Art. 26), we have

$$\tan E = \tan Ce = \tan ZOe = c \div a \qquad (2).$$

Again, the face $e'\,(011)$ of the same pyramid occupies a similar position to $q\,(011)$ in a prismatic crystal. But the faces $e\,(101)$ and $e'\,(011)$ of the pyramid are inclined to the plane XOY at the same angles, for the two faces are reciprocal reflexions in the vertical plane S. Hence, the angle

$$D = 010 \wedge 011 = 90^\circ - 001 \wedge 011 = 90^\circ - 001 \wedge 101 = 90^\circ - E.$$

It follows that a crystal of this class has only one element which varies with the substance. The element may be given either by the angle E, or by the ratio of the parameters c and a. When the numerical values of the parameters are introduced, it is usual to make a the unit of length. The relation between the angular element E and the *linear* element c is then given by

$$c = \tan E \qquad (2^*).$$

The parameter c may be greater or less than 1; and it can only be equal to unity under exceptional circumstances: even if at any temperature $c = a$, a change of temperature will alter the ratio of $c : a$; for the coefficient of expansion along the principal axis differs from that along any line at right angles to it.

9. We have now exhausted all the possible types of the special forms in the class, which have parallel faces. It will be noticed that the parallelism of the faces in them is due to the faces being in each case parallel to one or other of the dyad axes.

The sphenoid, $\kappa\,\{hhl\}$. Suppose a face to be drawn through the line AA' of Figs. 173 and 180 to meet the principal axis at the point L; then, since AA' is perpendicular to S, every face drawn through it, or parallel to it, is also perpendicular to S. Hence, if OA is $a \div h$ and OL is $c \div l$, the symbol of the face is (hhl).

Let this face be turned through 180° about OZ, the point L

remains unchanged whilst the line AA' is transposed to the parallel line $\bar{A}A_{,}$, meeting OX and OY at equal distances in the negative direction. The new position of the face is given by the symbol $(\bar{h}\bar{h}l)$.

Similarly, a semi-revolution about OX leaves A unchanged but interchanges equal positive and negative lengths on OY and OZ. The face $q_1q_2q_3$ has, therefore, the symbol $(h\bar{h}\bar{l})$. By a similar rotation about OY the face is brought into the position of qq_2q_3, of which the symbol is $(\bar{h}h\bar{l})$. Repetition of the rotations gives no new faces. The form, therefore, consists of the faces:

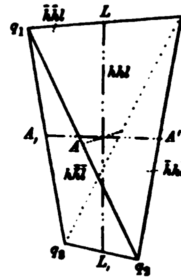

Fig. 180.

$$hhl,\ h\bar{h}\bar{l},\ \bar{h}\bar{h}l,\ \bar{h}h\bar{l}\ \ldots\ldots\ldots\ldots \text{(f)}.$$

Each of the faces is perpendicular to S or S'. Further, the edge qq_1 is parallel to AA' and $\bar{A}A_{,}$, and lies in the plane S'. This plane bisects the angle between the pair of faces (hhl), $(\bar{h}\bar{h}l)$ which meet in qq_1. Similarly, the plane S contains the opposite edge q_2q_3, and bisects the angle between the two faces $(h\bar{h}\bar{l})$, $(\bar{h}h\bar{l})$ which meet in the edge. The edges qq_1, q_2q_3 are therefore like and interchangeable edges, which are at right angles to one another and to the principal axis. It is also clear that the faces are all equal and similar isosceles triangles, for each face is bisected by a plane of symmetry; for instance, the face qq_1q_2 is bisected by S in the line Lq_2. The angles over the slanting edges passing through A, A', &c., are all equal; but they are not equal to the angles over the horizontal edges. Again, a semi-revolution about a dyad axis, OX (say), changes only the ends q_1 and q_2 of the edge meeting it, but not the direction of the edge. The edge q_1q_2 is therefore perpendicular to OX, and is truncated by the face (100), when the sphenoid and $\{100\}$ occur together. The other slanting edges qq_3, &c., are all similarly truncated by the other faces of the tetragonal prism $\{100\}$.

A complementary sphenoid is possible, the faces of which are parallel to those of the preceding sphenoid. Its symbol is $\kappa\{h\bar{h}\bar{l}\}$, and it consists of the faces: $h\bar{h}l$, $\bar{h}hl$, $\bar{h}\bar{h}\bar{l}$, $hh\bar{l}$.

Since the faces of the two forms are parallel, the faces will be equal and similar, and the angles over the corresponding edges will be equal. The two forms can be placed in similar positions by rotating one through 90° about the principal axis. Complementary forms connected by the fact that they can be brought to occupy the same space are said to be tautomorphous (p. 210).

10. *The disphenoid, or scalenohedron*, $\kappa\{hkl\}$. If a plane is drawn through a line in the plane XOY, such as HK of Fig. 177, to meet the vertical axis in L at the distance $c \div l$, we obtain a face which has the symbol (hkl). Such a face meets the plane of symmetry S in a line joining L to M in the plane XOY, through which and $H'K'$ a second face must pass. The new face $H'K'L$ is symmetrical to the first with respect to S; and has the intercepts $a \div k : a_{,} \div h : c \div l$, and the symbol (khl).

A semi-revolution about OZ brings the above two faces into positions in which they meet the plane XOY in the lines $H_{,}K_{,,}$ and $H_{,,}K_{,}$ of Fig. 177. The new faces have, therefore, the symbols $(\bar{h}\bar{k}l)$, $(\bar{k}\bar{h}l)$. Again, as was proved in Art. 7, the pair of lines HK and $H_{,,}K_{,}$ are symmetrical with respect to S', and the faces through them and L must be so too. These faces must therefore intersect in an edge, Ln (say) of Fig. 181, lying in S'. For a similar reason, the line of intersection Ln' of (khl) and $(\bar{h}\bar{k}l)$ lies in S'.

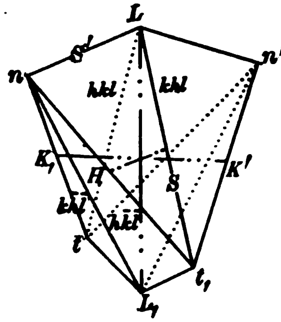

Fig. 181.

If now the above faces are turned through 180° about OX, they are brought into positions in which they meet OZ at $L_{,}$, where $OL_{,} = c \div \bar{l}$, and meet the plane XOY in the lines $HK_{,,}$, $K_{,}H'$, &c., of Fig. 177. The new faces have, therefore, the symbols: $h\bar{k}\bar{l}$, $k\bar{h}\bar{l}$, $\bar{k}h\bar{l}$, $\bar{h}k\bar{l}$.

These faces are necessarily symmetrical with respect to S and S', for the original set of four faces with which they change places are symmetrical with respect to the same planes.

A semi-revolution about OY gives no new faces, for successive rotations of 180° about OZ and OX are together equivalent to a single rotation of 180° about OY. Hence, the disphenoid $\kappa\{hkl\}$, Fig. 181, consists of:

$$\left.\begin{array}{llll} hkl, & khl, & \bar{k}\bar{h}l, & \bar{h}\bar{k}l \\ h\bar{k}\bar{l}, & k\bar{h}\bar{l}, & \bar{k}h\bar{l}, & \bar{h}k\bar{l} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (g).$$

The faces are equal and similar scalene triangles, pairs of which intersecting in the sloping edges $nt_{,}$, $n't_{,}$, &c., are interchangeable by a semi-revolution about the axis to which these edges are perpendicular. The pairs of faces which meet in the edges $Lt_{,}$, Ln, &c. are reciprocal reflexions in the planes of symmetry through these

edges, and are therefore antistrophic. The angles over the dissimilar edges of different faces are unequal, whilst those over similar edges bounding different faces are all equal. Hence each disphenoid has three different angles: (i) that over an edge, such as $Lt_{,}$, lying in S or S'; (ii) that over a dissimilar edge, such as Ln, also lying in S or S'; and (iii) that over an edge, such as $nt_{,}$ which is perpendicular to, and bisected by, one of the like dyad axes.

11. The planes through $L_{,}$ and the lines HK, $H'K'$ of Fig. 177 are clearly possible faces; for they have rational indices and are parallel to a pair of faces of the disphenoid $\kappa\{hkl\}$, although they do not belong to this form. Hence we obtain a complementary disphenoid $\kappa\{hk\bar{l}\}$, Fig. 182, the faces of which are parallel to those of the form discussed in Art. 10. The form has the following faces:

$$\left.\begin{matrix} hk\bar{l}, & kh\bar{l}, & \bar{k}\bar{h}\bar{l}, & \bar{h}\bar{k}\bar{l} \\ h\bar{k}l, & k\bar{h}l, & \bar{k}hl, & \bar{h}kl \end{matrix}\right\} \ldots\ldots(\text{h}).$$

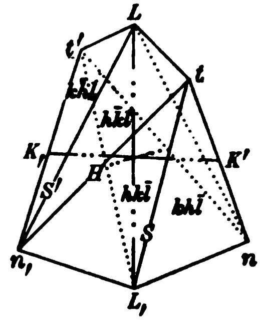

FIG. 182.

The angles of the complementary disphenoids are necessarily equal, for pairs of parallel faces make equal angles with one another. Since also the lines of Fig. 177, in which the faces of both forms meet the plane XOY can be interchanged by a rotation of 90° about OZ, it follows that the same rotation must bring the faces of $\kappa\{hk\bar{l}\}$ into the position of those of $\kappa\{hkl\}$, and *vice versâ*. Hence the complementary forms are tautomorphous.

Crystals showing sphenoids and disphenoids were said to be hemihedral with inclined faces, and this term is still used in descriptive works.

12. If a bipyramid $\{h0l\}$ occurs on a crystal, measurement of one angle will enable us to determine the symbol, when the angular element E, or the parametral ratio $c : a$, is known; or conversely, to determine the element when the indices h and l are given. Thus, if the angle over one of the edges $MM_{,}$ of Fig. 179 is measured, we know the angle $LHO = \wedge LHL_{,} \div 2$. But, if C is (001) and n is $(h0l)$, then $\wedge Cn = \wedge LHO$.

$$\text{Hence, } \tan Cn = \tan LHO = OL \div OH = \frac{c}{l} \div \frac{a}{h} = \frac{h}{l}\frac{c}{a}.$$

$$= (\text{from } (2)) \frac{h}{l} \tan E \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The same expression can be easily obtained from the A.R. $\{CenA\}$, where e is (101) and A is (100).

If, however, the measured angle is that marked F in Fig. 179, *i.e.* $h0l \wedge 0hl$, the angle $Cn = 001 \wedge h0l$ must be found from the right-angled spherical triangle having a right-angle at C, and arcs $= Cn$ for the sides meeting at C. Hence, $\cos F = \cos^2 Cn$.

But, $\sec^2 Cn = 1 + \tan^2 Cn = (\text{from } (3))\ \dfrac{l^2 + h^2 \tan^2 E}{l^2}$.

$$\therefore \cos F = \frac{1}{\sec^2 Cn} = \frac{l^2}{l^2 + h^2 \tan^2 E} \quad \text{.............. (4).}$$

13. Should the only forms meeting the principal axis be sphenoids, one of them (the faces of which we shall denote by o) is selected as $\kappa\{111\}$, and the others have then the symbols $\kappa\{hhl\}$. Now the face (hhl) meets the axes at $a \div h$, $a_, \div h$, $c \div l$. If displaced parallel to itself so as to pass through A and A'—where $OA = OA' = a$—the intercepts are a, $a_,$, $hc \div l$. Let Fig. 183 represent part of a section in the plane S, where Op is the normal (hhl), $OC = hc \div l$, and Om is the length intercepted by AA' on OM of Fig. 173. Hence,

$$Om = OA \cos 45° = a \div \surd 2.$$

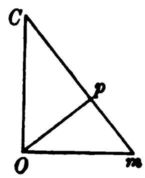

FIG. 183.

But, since the angles at p and COm are right-angles, $COp = 90° - pOm = CmO$.

$$\therefore \tan COp = \tan CmO = \frac{OC}{Om} = \frac{hc}{la \cos 45°}$$

$$= (\text{from } (2))\ \frac{h}{l}\surd 2 \tan E \quad \text{........................ (5).}$$

When $h = l$, the sphenoid becomes $o = \kappa\{111\}$, and we have

$$\tan Co = \frac{c}{a} \sec 45° = \tan E \sec 45° = \surd 2 \tan E \quad \text{........................ (6).}$$

The parameter c, and the angular element E, can, therefore, be calculated.

14. In the discussion of the very simple problems, which have been so far solved, stereograms were not required; but more general problems are best solved by their aid. In such diagrams the principal axis is always taken to be the diameter through the eye: the pole C(001) therefore occupies the centre. On the primitive, arcs $Am = mA' = \&c. = 45°$ are marked off by a protractor, the pole A(100) being placed at the bottom, and A'(010) at the

right extremity of the horizontal diameter. At the same time the positions of poles such as f, Fig. 184, are determined from a knowledge of the angle Af. If the element E is known, the poles $e\,(101)$, $e'\,(011)$, &c., are placed on the diametral zones $[AC]$, $[A'C]$, &c., by the method of Chap. VII, Prob. 1. The zone-circles $[Ae']$, $[A'e]$, &c., can then be drawn: they determine by their intersections the pole $o\,(111)$ and its homologues. Or, if sphenoids are the known forms, Cp can be marked off on $[Cm]$ by the method given for e. Two poles p lie on $[Cm]$ above the paper, and two poles of the form are given by circlets to be placed round points at the same distance from C on $[Cm_{,}]$. A disphenoid, such as $t = \kappa\,(hkl)$ in Fig. 184, is most quickly and correctly placed by the intersection of zones such as $[Cf]$ and $[m_{,}p]$, which can generally be easily determined and drawn. Four poles above the paper are given by dots occupying opposite quadrants, and four poles below the paper are given by circlets occupying the other pair of opposite quadrants.

15. Should a disphenoid be the only form present, or be the form with brightest and smoothest faces, we may have one of the two following problems to solve. i. Assigning definite indices to the faces, it may be required to find the element E or the parameter c. Measurement of one angle of the disphenoid suffices to give the element. ii. Knowing the element E, the symbol of the disphenoid may be needed: two angles must then be measured.

i. Let Fig. 184 represent a projection of the poles t of the disphenoid $\kappa\,\{hkl\}$, and let p be a pole of the possible sphenoid which truncates the longer edges, such as $Lt_{,}$ of Fig. 181. Now the zone-circle $[tt'] = [hkl,\ khl]$ must pass through $m_{,}$, the pole of the plane S with respect to which the two faces are symmetrical. The plane S meets the sphere in the great circle projected in the diametral zone $[Cm] = [1\bar{1}0]$, on which the pole p lies. Hence p may be denoted by (eeg). But since it lies in the zone

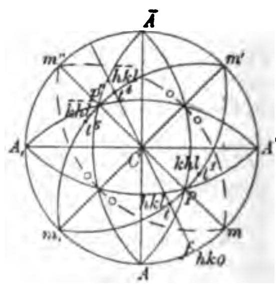

Fig. 184.

$$[m_{,}t] = [\bar{l},\ \bar{l},\ h + k],$$

we have by Weiss's zone-law

$$-\,2le + g\,(h + k) = 0.$$

$\therefore$ we may take $e = h + k$, and $g = 2l$.

But the indices of p are connected with the angle Cp by equation (5) of Art. 13, in which e takes the place of h and g of l.

$$\therefore \tan Cp = \frac{e}{g}\sqrt{2}\tan E = \frac{h+k}{l\sqrt{2}}\tan E \ldots\ldots\ldots\ldots(7).$$

Or if the angle Co is known, where o is (111), we can by combining equations (5) and (6) obtain the following relation:

$$\tan Cp \div \tan Co = \frac{e}{g} = \frac{h+k}{2l} \ldots\ldots\ldots\ldots\ldots\ldots(8).$$

This equation is a particular case of the general one holding for zones in which two of the poles are at 90° from one another, and corresponds to the equation given in Chap. XIII, Art. 35.

Again, if f is the pole in which the zone-circle $[Ct]$ meets $[AA'] = [001]$, then f has the symbol $(hk0)$. From equation (1), we know

$$\tan Af = k \div h.$$

The indices h, k, l, being known, the arc $Af = 45° - p\,Ct$ can be found, or may be taken from a table like that given in Art. 7.

We shall now consider the cases which may arise.

α. Suppose $\wedge tt_4$ to be the measured angle. Then $Ct = tt_4 \div 2$, and $\hat{\wedge}\, pCt$ are both known; and from the right-angled triangle Cpt, Cp can be found. For $\cos pCt = \tan Cp \cot Ct$.

The element E is then found from (7).

β. If tt^1 is the measured angle, then $pt = tt^1 \div 2$. Hence, in the right-angled triangle pCt the side pt and the angle pCt are known.

$$\therefore \sin Cp = \tan pt \cot pCt.$$

The element E can then be found from (7).

γ. If tt^5 is the measured angle, then $mt = (180° - tt^5) \div 2$; for the zone through the poles t, t^5 includes m (110). But $mf = 45° - Af$. We therefore know two sides of the right-angled triangle mft, from which $ft = 90° - Ct$ can be found. When Ct has been computed, we know Ct and $\wedge pCt$, and the element is determined in the way described under (*α*).

16. ii. To find the symbol of the form, we must find both Af and Cp. When these arcs are determined, the indices are obtained from equations (1) and (7). Measurement of tt^1 gives pt, of tt^5 gives $mt = (180° - tt^5) \div 2$, and of tt^4 gives Ct.

α. If pt and mt are given from the measured angles, the arc $mp = 90° - Cp$ can be found from the right-angled triangle mpt by the formula

$$\sin Cp = \cos mp = \cos mt \div \cos pt.$$

Then $mf = \wedge pCt$ can be found from the triangle pCt of which Cp and pt are known. Hence, $Af = 45° - mf$, Cp, and E being known, we can find the indices from equations (1) and (7).

β. If pt and Ct are given from the measurements, we find Cp and $\wedge pCt$ from the right-angled triangle pCt. Hence Cp and $Af = 45° - \wedge pCt$ are both determined, and the indices can be found from (1) and (7).

γ. If Ct and mt are the angles determined by measurement, then from the triangle Ctm we have

$$\cos mt = \sin Ct \cos mCt = \sin Ct \cos mf.$$

The arc Cp is then found from the triangle pCt of which Ct and $\wedge pCt$ are now known; the rest of the solution is the same as before.

17. We can now prove that the arc Ct, where t is the pole (hkl), is connected with the element E and the indices of t by the following equation:

$$\tan^2 Ct = \frac{h^2 + k^2}{l^2} \tan^2 E \text{........} (9).$$

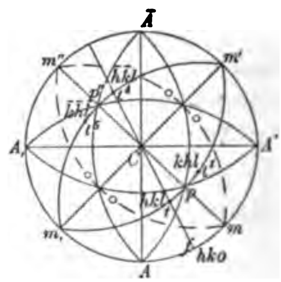

FIG. 184.

From the right-angled spherical triangle Cpt, Fig. 184, we have

$$\cos pCt = \tan Cp \cot Ct;$$

$$\therefore \tan^2 Ct = \tan^2 Cp \sec^2 pCt = \tan^2 Cp \sec^2 mf,$$

since $\wedge pCt = \text{arc } mf$.

But $mf = 45° - Af$, and $\tan Af = k \div h$.

$$\therefore \sec^2 mf = 1 + \tan^2(45° - Af) = 1 + \left(\frac{1 - \tan Af}{1 + \tan Af}\right)^2$$

$$= \frac{2(1 + \tan^2 Af)}{(1 + \tan Af)^2} = \frac{2(h^2 + k^2)}{(h + k)^2}.$$

Hence, replacing $\tan Cp$ by the expression given in (7), we have

$$\tan^2 Ct = \frac{(h + k)^2}{2 l^2} \tan^2 E \times \frac{2(h^2 + k^2)}{(h + k)^2} = \frac{h^2 + k^2}{l^2} \tan^2 E.$$

The employment of this equation will simplify the solution of the problem discussed in Art. 15.

18. The formulæ established in preceding Articles can easily be obtained from the general equations of the normal $P(hkl)$. Taking the parameters on the axes of X and Y to be unity, and that on the principal axis to be c; and remembering that, since the axes are rectangular, the axial points X, Y, Z coincide with the axial poles A, A', C; the equations become

$$\frac{\cos AP}{h} = \frac{\cos A'P}{k} = \frac{c\cos CP}{l} = \frac{1}{\sqrt{h^2+k^2+l^2 \div c^2}} \quad \ldots\ldots (10).$$

The last term is obtained from the relation

$$\cos^2 AP + \cos^2 A'P + \cos^2 CP = 1,$$

which holds whenever the axes are rectangular. For, if each of the three first terms is made equal to r, we have

$$rh = \cos AP, \quad \therefore\ r^2h^2 = \cos^2 AP,$$

$$rk = \cos A'P, \quad \therefore\ r^2k^2 = \cos^2 A'P,$$

$$rl \div c = \cos CP, \quad \therefore\ r^2l^2 \div c^2 = \cos^2 CP;$$

$$\therefore\ r^2(h^2+k^2+l^2 \div c^2) = \cos^2 AP + \cos^2 A'P + \cos^2 CP = 1.$$

$$\therefore\ r = \frac{1}{\sqrt{h^2+k^2+l^2 \div c^2}}.$$

If now P is taken to be coincident with $f(hk0)$, we have

$$\frac{\cos Af}{h} = \frac{\cos A'f}{k} = \frac{\sin Af}{k},$$

since $A'f = 90° - Af$.

$$\therefore\ \tan Af = \frac{k}{h} \quad \ldots\ldots (1).$$

If P is made to coincide with $e(101)$, then

$$Ae = 90° - Ce = 90° - E, \text{ and } A'e = 90°.$$

Introducing these values in (10), we have

$$\cos(Ae = 90° - Ce) = \sin Ce = c\cos Ce;$$

$$\therefore\ c = \tan Ce = \tan E \quad \ldots\ldots (2^*).$$

Again, if P is brought to coincide with $o(111)$, then from the right-angled spherical triangles Amo, $A'mo$,

$$\cos Ao = \cos A'o = \cos Am \cos mo = \cos 45° \sin Co.$$

Introducing in (10), we have

$$\cos 45° \sin Co = c\cos Co;$$

$$\therefore\ \tan Co = c\sec 45° = c\sqrt{2} = \sqrt{2}\tan E \quad \ldots\ldots (6).$$

Similarly, if P coincides with $p\,(hhl)$,

$$\cos Ap = \cos Am \cos mp = \cos 45^\circ \sin Cp.$$

$\therefore$ from (10),

$$\frac{\cos 45^\circ \sin Cp}{h} = \frac{c \cos Cp}{l};$$

$$\therefore \tan Cp = \frac{h}{l} c \sqrt{2} = \frac{h}{l} \sqrt{2} \tan E \ldots\ldots\ldots\ldots\ldots(5).$$

From the right-angled triangle APf (which may be taken to be the same as the triangle Atf of Fig. 184), we have

$$\cos AP = \cos Af \cos fP = \cos Af \sin CP.$$

Introducing in (10), we have

$$\frac{\cos Af \sin CP}{h} = \frac{c \cos CP}{l} = (\text{from } 2^*) \frac{\cos CP \tan E}{l}.$$

$$\therefore \tan^2 CP = \frac{h^2}{l^2} \tan^2 E \sec^2 Af$$

$$= (\text{from } (1)) \frac{h^2}{l^2} \tan^2 E \times \frac{h^2 + k^2}{h^2} = \frac{h^2 + k^2}{l^2} \tan^2 E \ldots\ldots(9).$$

The formula for the cosine of the angle between any two poles $P\,(hkl)$ and $Q\,(pqr)$ can be obtained from equations (10) and the general expression (Chap. XIII, Art. 36) which holds when the axes are rectangular. For

$$\cos PQ = \cos AP \cos AQ + \cos A'P \cos A'Q + \cos CP \cos CQ$$

$$= (\text{from } (10)) \frac{hp + kq + lr \div c^2}{\sqrt{(h^2 + k^2 + l^2 \div c^2)(p^2 + q^2 + r^2 \div c^2)}} \ldots\ldots(11).$$

Examples.

19. *Copper Pyrites* (Chalcopyrite), $CuFeS_2$, affords good instances of crystals of this class. Fig. 185 represents a sphenoid $r = \kappa\{332\}$; Fig. 186

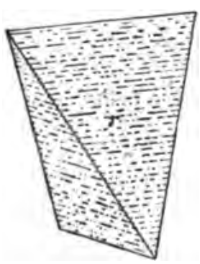

FIG. 185.

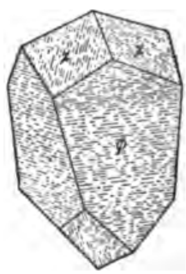

FIG. 186.

represents a crystal showing the sphenoid $\phi = \kappa\{772\}$ with the disphenoid $\chi = \kappa\{122\}$: these crystals from French Creek, Pennsylvania, were described by Professor Penfield (*Am. Jour. of Sci.* [iii] XL, p. 207, 1890), who states that the symbols of ϕ and χ are doubtful. In Fig. 187 the sphenoid $o = \kappa\{111\}$, which is often found alone, is associated with the complementary sphenoid $\omega = \kappa\{1\bar{1}1\}$ and the tetragonal pyramid $z\{201\}$. The faces of o are usually large, dull and striated; those of ω are usually small and bright. Other forms occasionally observed on the crystals are: $c\{001\}$, $e\{101\}$ and $s = \kappa\{513\}$. The poles of the above forms, with the exception of those of r, ϕ and χ, are shown in Fig. 188.

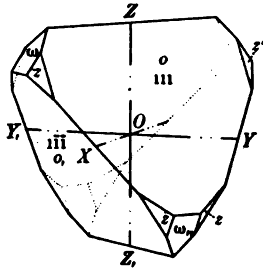

FIG. 187.

The best angle to measure would be $\omega\omega'$ over the horizontal edge of the sphenoid $\kappa\{1\bar{1}1\}$: this angle was determined by Haidinger to be 108° 40′. Hence, C being (001), $C\omega = Co = 54° \ 20'$.

∴ from equations (2*) and (6)

$$c = \tan E = \tan Co \div \sqrt{2} = \tan 54° \ 20' \div \sqrt{2}.$$

$$L \tan 54° \ 20' = 10{\cdot}14406$$

$$\log \sqrt{2} = \underline{\ \ {\cdot}15051}$$

$$9{\cdot}99355.$$

$$\therefore \ E = 44° \ 34{\cdot}5'; \text{ and } c = {\cdot}98525.$$

The parameter c differs from unity by an almost insignificant amount. The angle E is nearly 45°, and $oo'' = \omega\omega'$ is not far removed from 109° 28′ (the angle of the regular tetrahedron). The crystals therefore have somewhat the appearance of those of one of the classes of the cubic system; and it was not until 1822 that the crystals were proved by Haidinger (*Mem. Wern. Nat. Hist. Soc. Edin.* IV, p. 1, 1822) to belong to the tetragonal system.

We can now determine the angle $o\omega = 2oe$. For, in the right-angled triangle Coe, Fig. 188, we know

$$Co = 54° \ 20' \text{ and } Ce = E = 44° \ 34{\cdot}5'.$$

Hence, $$\cos oe = \cos Co \div \cos Ce,$$

$$\therefore \ oe = 35° \ 3{\cdot}\dot{6}' \text{ and } o\omega = 70° \ 7{\cdot}\dot{3}'.$$

From Fig. 187, it is seen that o meets z and z' in parallel edges; and that the face ω also meets z and $z_,$ in parallel edges. Hence, Fig. 188, z is the pole in which the zone-circles through o and ω perpendicular to the planes of symmetry intersect. These circles pass through $m_,(1\bar{1}0)$ and $m(110)$, respectively. The symbol of the pole z is, therefore, obtained by (23) of Chapter v The zone $[m_,o]$ is $[11\bar{2}]$ and $[m\omega]$ is $[\bar{1}12]$; therefore z is (201), and the pole lies

in the zone [CA], and the face is that of a tetragonal pyramid. The angle zo can now be found from the right-angled spherical triangle Coz by the expression—

$$\sin Co = \tan oz \cot 45° = \tan oz.$$

$$\therefore\ oz = 39°\ 5{\cdot}5', \text{ and } m_{,}z = mz = 90° - oz = 50°\ 54{\cdot}5'.$$

Poles. A single measurement in the zone [zo] suffices to give the symbol of any pole lying in it, for we know the symbols of three faces and the angles between them. Thus, if os is 28° 26·5′ by measurement, and if s is (hkl), the A.R. {$ozsm_{,}$} gives

$$\tan oz \div \tan os = \left|\begin{matrix}111\\201\end{matrix}\middle/\begin{matrix}111\\hkl\end{matrix}\right| \div \left|\begin{matrix}1\bar{1}0\\201\end{matrix}\middle/\begin{matrix}1\bar{1}0\\hkl\end{matrix}\right| = \frac{h+k}{h-k}.$$

$L \tan (oz = 39°\ 5{\cdot}5') = 9{\cdot}90978$
$L \tan (os = 28°\ 26{\cdot}5') = 9{\cdot}73371$
$\log 1{\cdot}5 = {\cdot}17607.$

$$\therefore\ \frac{h+k}{h-k} = \frac{3}{2};\ \text{and } h=5,\ k=1.$$

But s lies in [$om_{,}$]=[$11\bar{2}$].

$$\therefore\ h+k-2l=0;\ \text{and } l=3.$$

The pole s is therefore (513), and belongs to the disphenoid κ {513}.

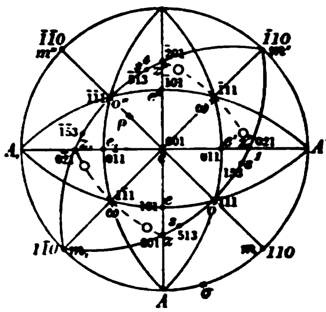

Fig. 188.

Pole r. Knowing the symbol of the sphenoid r to be κ {332}, we can find the angles. For, supposing the pole r to be inserted in Fig. 188, we have the A.R. {$Corm$} in which $\wedge\ Cm = 90°$.

$$\therefore\ \tan Cr \div \tan Co = 3 \div 2.$$

But $Co = 54°\ 20'$; therefore, by computation, $Cr = 64°\ 25{\cdot}5'$, and $rr'' = 128°\ 51'$. is the angle over the horizontal edge.

The angle over one of the slanting edges, such as q_1q_2 of Fig. 180, can be now obtained from the right-angled spherical triangle Amr; for the angles over such edges are bisected at the poles A, A', &c. Thus,

$$\cos Ar = \cos Am \sin Cr = \cos 45° \sin 64°\ 25{\cdot}5'.$$

$\therefore\ Ar = 50°\ 22'$, and the angle $rr_{,}$ over the edge is 100° 44′.

The stereogram. Fig. 188 is constructed as follows. On the primitive, arcs $Am = mA' =$ &c. $= 45°$ are marked off, and diameters through A, m, A', &c., are then drawn. On [CA] an arc $Ce = 44°\ 34{\cdot}5'$ is determined (Chap. VII, Prob. 1). The homologous poles e', e'', $e_{,}$, are then introduced by a pair of compasses, for the four poles lie on a small circle with C as centre. The circles [Ae'], [$A'e$], &c., are now described and fix the points o and ω. The zone-circles [$m_{,}o$], [$m_{,}o''$], and the similar zones through m below the paper (only partially shown by interrupted strokes), are then drawn. They determine by their intersections with [CA] and [CA'] the poles z of {201}.

To find the position of s, the pole ρ of [$m_{,}o$] is determined by the construction of Chap. VII, Prob. 4; its position is marked by a cross. The arc

$m\sigma = o s = 28° 26·5'$ is then measured off on the primitive. The straight line $\rho\sigma$ intersects $[om_{,}]$ in s. The homologous poles s^1, &c., are introduced by a pair of compasses; the four poles of the form below the paper being indicated by circlets. In order to avoid confusing the figure, the poles r and the circlets giving the positions of the inferior poles of the other forms are omitted.

Drawing the crystal, Fig. 187. The sphenoid o is first completely drawn, the edges through Z, $Z_{,}$ being parallel respectively to XY and $XY_{,}$; those through X, Y, &c., being parallel to ZY, XZ, &c. Through points taken at equal distances from Z and $Z_{,}$ (marked off by proportional compasses) the edges $[o\omega]$, &c., separating the complementary sphenoids are then drawn parallel to the opposite sides of each face o. The parallel edges $[oz]$, $[oz']$, are then drawn parallel to the diagonal of the face o of the sphenoid; and similarly $[\omega z]$ is parallel to the diagonal of the face ω. These edges and their homologues are drawn through points on $[o\omega]$, &c., at equal distances from the vertices of the faces ω, these distances being obtained by proportional compasses. The remaining lines $[zo_{,}]$, &c., are then drawn, and complete the figure.

II. *Diplohedral ditetragonal class*; $\{hkl\}$.

20. If a centre of symmetry is added to the elements of symmetry which distinguish the crystals of the last class, we obtain a class which we shall denote as the *diplohedral ditetragonal* class of the tetragonal system. In a centro-symmetrical crystal there must be a plane of symmetry perpendicular to each axis of even degree and an axis of symmetry of even degree perpendicular to each plane of symmetry (Chap. IX, Prop. 4). Hence, there is a plane of symmetry Π perpendicular to the principal axis, and two like planes of symmetry Σ, Σ', perpendicular to each of the dyad axes δ, δ', which were, in the last section, taken as the axes OX and OY. The plane Σ contains OZ and OY, the plane Σ' contains OZ and OX.

Again, the lines of intersection of Π with S and S' are dyad axes (Chap. IX, Prop. 8). They will be denoted by Δ and Δ' respectively. They are like axes, for they are interchangeable by a rotation of 180° about OX or OY.

The principal axis is now the intersection of four planes of symmetry, and perpendicular to four dyad axes. It is therefore, by Chap. IX, Props. 8 and 11, a tetrad axis. The crystals are therefore characterised by the following elements of symmetry:

$$T,\ 2\delta,\ 2\Delta,\ C,\ \Pi,\ 2\Sigma,\ 2S.$$

The planes and axes of symmetry are shown in Fig. 189. The like planes S are indicated by a series of horizontal lines, the Σ planes by a series of vertical lines, and Π by two sets of lines crossing at right angles. The axes of symmetry are indicated by lines of strokes alternating with dots the number of the latter giving the degree.

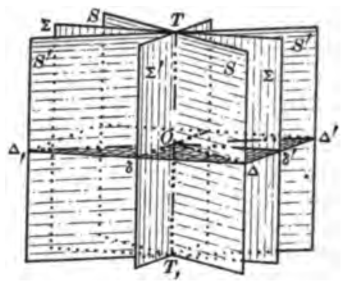

FIG. 189.

21. It is clear that the axes of reference adopted in the last section can advantageously be retained in this class; for, geometrically, crystals of this class differ from those of the scalenohedral class only in having pairs of parallel faces always associated together. The parameters on OX and OY can, therefore, be taken to be equal, for the lines are interchangeable. The only unknown element is the parameter c on the tetrad axis OZ, or the corresponding angular element $E = 001 \wedge 101$ from which c is computed. All the analytical formulæ established in previous Articles of this Chapter are therefore immediately applicable to crystals of this class.

22. It is obvious that the special forms with parallel faces, described in Arts. 4—8, are not modified by the introduction of a centre of symmetry; and the student can easily see that those forms are geometrically symmetrical with respect to the new elements of symmetry which result from the presence of a centre of symmetry. Hence, the following forms are common to both classes:

The pinakoid $\{001\}$ consisting of: 001, $00\bar{1}$ (Art. 4).

The tetragonal prism $\{110\}$ consisting of:

$$110,\ \bar{1}10,\ \bar{1}\bar{1}0,\ 1\bar{1}0 \text{ (Art. 5).}$$

The tetragonal prism $\{100\}$ consisting of:

$$100,\ 010,\ \bar{1}00,\ 0\bar{1}0 \text{ (Art. 6).}$$

The ditetragonal prism $\{hk0\}$ consisting of:

$$hk0,\ kh0,\ \bar{k}h0,\ \bar{h}k0,\ \bar{h}\bar{k}0,\ \bar{k}\bar{h}0,\ k\bar{h}0,\ h\bar{k}0 \text{ (Art. 7).}$$

The tetragonal pyramid $\{h0l\}$ consisting of:

$$h0l,\ 0hl,\ \bar{h}0l,\ 0\bar{h}l,\ h0\bar{l},\ 0h\bar{l},\ \bar{h}0\bar{l},\ 0\bar{h}\bar{l} \text{ (Art. 8).}$$

23. *The tetragonal bipyramid*, $\{hhl\}$. The sphenoid $\kappa\{hhl\}$ is changed by the introduction of parallel faces to a tetragonal bipyramid $\{hhl\}$, Fig. 190, which consists of the faces:

$$hhl,\ \bar{h}hl,\ \bar{h}\bar{h}l,\ h\bar{h}l,\ hh\bar{l},\ \bar{h}h\bar{l},\ \bar{h}\bar{h}\bar{l},\ h\bar{h}\bar{l} \ldots (1).$$

The student will, on looking back to Art. 9, perceive the relations of the pyramid and the two complementary sphenoids. Geometrically the pyramid $\{hhl\}$ only differs from $\{h0l\}$ in having different angles. It is easy to find the condition that the angles of a pyramid of one kind should be equal to the corresponding angles of a pyramid of the other kind; and thus to show that the assumption contravenes the law of rational indices.

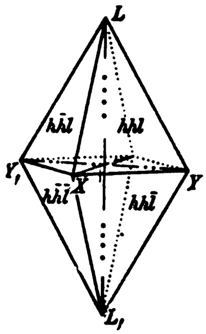

FIG. 190.

Proof. Let Fig. 191 show the poles of the two forms $P\{hhl\}$ and $Q\{p0r\}$, and let the zone-circles $[AP]$, $[A'P]$, &c., meet the zones $[CA']$, &c., in n', n, &c. Then the pole n is $(h0l)$.

Since the corresponding angles of the two pyramids P and Q are to be equal, the arc $CP =$ arc CQ. Now by equation (3),

$$\tan CQ \div \tan E = p \div r;$$

and by the same equation,

$$\tan Cn \div \tan E = h \div l.$$

$$\therefore \tan CQ \div \tan Cn = lp \div hr.$$

But, from the right-angled triangle CPn,

$$\cos 45° = \tan Cn \cot CP = \tan Cn \cot CQ,$$

since $CP = CQ$.

$$\therefore lp \div hr = \sqrt{2}, \text{ or } \frac{l}{h} = \frac{r}{p}\sqrt{2}.$$

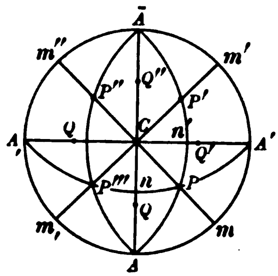

FIG. 191.

If (hhl) is a possible face, then $p \div r$ is irrational, and the pyramid Q is not possible on the crystal. If, however, the pyramid Q is possible, then $l \div h$ is irrational and the pyramid P is impossible.

Structurally the pyramid $\{hhl\}$ differs from a pyramid $\{p0r\}$, for in the former the coigns lie in the dyad axes δ, $\delta_{,}$; and in the latter the coigns lie in Δ and $\Delta_{,}$: and though these latter are also dyad axes, they are essentially different from the former pair. Were the one pair δ interchangeable with the other pair Δ of dyad axes, the principal axis would become an axis of octad symmetry, and this is impossible (Chap. IX, Prop. 6).

There is nothing, however, in a crystal to indicate which pair of

dyad axes is to be taken as the axes of reference; and when a crystal is first described, it is immaterial which pair is put in the positions OX and OY. But, when once the axes have been selected, it is necessary to retain the same pair, if it is desired to compare the symbols of the faces with those given by other observers.

24. *The ditetragonal bipyramid*, $\{hkl\}$. The addition of parallel faces to the disphenoid of Art. 10 converts it into a diplohedral ditetragonal pyramid, Fig. 192. For, clearly, two pairs of parallel faces can be drawn through the parallel lines HK, $H_{,}K_{,,}$ of Fig. 177 to meet the vertical axis OZ at opposite points L and $L_{,}$, where

$$OL = OL_{,} = c \div l.$$

The two faces meeting in HMK have the symbols (hkl), $(hk\bar{l})$; the pair through $H_{,}K_{,,}$ have the symbols $(\bar{h}\bar{k}\bar{l})$, $(\bar{h}\bar{k}l)$. The form $\{hkl\}$ therefore consists of the faces:

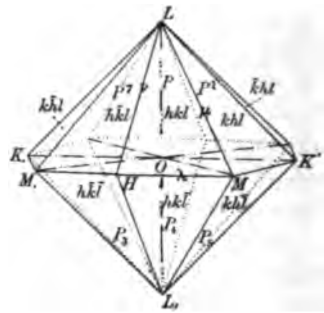

Fig. 192.

$$\left.\begin{array}{l} hkl,\ khl,\ \bar{k}hl,\ \bar{h}kl,\ \bar{h}\bar{k}l,\ \bar{k}\bar{h}l,\ k\bar{h}l,\ h\bar{k}l \\ hk\bar{l},\ kh\bar{l},\ \bar{k}h\bar{l},\ \bar{h}k\bar{l},\ \bar{h}\bar{k}\bar{l},\ \bar{k}\bar{h}\bar{l},\ k\bar{h}\bar{l},\ h\bar{k}\bar{l} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (j).$$

All the faces are equal scalene triangles having their edges in three dissimilar planes of symmetry. The angles between pairs of adjacent faces which meet in the same, or in like, planes of symmetry are equal, those over edges lying in dissimilar planes of symmetry are always unequal. Hence, the three unequal angles λ, μ, ν over the edges of a face are the only angles possible between adjacent faces which meet in edges in an equably developed form. We shall generally express such a relation between the angles of a form by the statement that the form has only a particular number of different angles: in this case the number is three.

25. That the angles λ, μ, ν are unequal may be proved as follows. The ditetragon, $HMK'\ldots$, of Fig. 177, Art. 7, can never become a regular octagon; for, were it a regular octagon, the angle $\wedge OHM = \wedge OMH$, and OG the normal to the vertical face through HM would make $22° \, 30'$ with OX. Suppose the prism-face through HM to be $(hk0)$ and to be inclined to the vertical axial plane XOZ at $90° - 22° \, 30'$. Then $XOG = 22° \, 30'$; and from equation (1) $\tan(XOG = 22° \, 30') = k \div h$.

By a well-known formula, $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$ (Todhunter's *Trig.* p. 56).

But, when $\theta = 22° 30'$, $2\theta = 45°$, and $\tan 2\theta = 1$.

$$\therefore\ 2\tan\theta \div (1 - \tan^2\theta) = 1;$$

$$\therefore\ \tan^2\theta + 2\tan\theta - 1 = 0.$$

Solving the equation, $\tan\theta = \sqrt{2} - 1 = k \div h$.

The indices are irrational, and the plane is not a possible face. It follows that the figure HMK'... cannot be a regular octagon. But the angles μ and ν over the polar edges (p. 112) can only be equal, when the coigns H and M are equally distant from O and the figure HMK'... is a regular octagon. The angles μ and ν are, therefore, always unequal.

It is clear also that, if the edges HMK and HL meet the axes of Y and Z at equal distances from O, the angles λ over HM and ν over HL are equal. But $OK = a, \div k$, and $OL = c \div l$. Therefore,

$$c \div l = a \div k; \text{ or } c \div a = \tan E = l \div k.$$

Hence, $c \div a$ is a rational number under all circumstances. But the tetrad axis OZ and the dyad axis OY are different physically; and the coefficient of thermal expansion along OZ is different from that along OY. Hence, $c \div a$ varies by insensibly small increments as the temperature changes. But the integers k and l cannot so change. Therefore the angles λ and ν cannot be equal, except at some special temperature.

In a similar manner it may be shown that λ and μ are unequal.

26. We can easily obtain from (11) expressions for $\cos\lambda$, $\cos\mu$, $\cos\nu$, which enable us to establish the proposition generally. These expressions may also under exceptional circumstances be convenient for determining the angles, when the indices and parameter are known, but they are not adapted to logarithmic computation, and the method is laborious.

Let P be (hkl), $P^1(khl)$, $P^7(h\bar{k}l)$ and $P_4(hk\bar{l})$; then, from equations (10) of Art. 18, we have:

for P, $$\frac{\cos AP}{h} = \frac{\cos A'P}{k} = \frac{c\cos CP}{l} = \frac{1}{\sqrt{h^2+k^2+l^2\div c^2}} = \frac{1}{N}\ \text{(say)};$$

for P^1, $$\frac{\cos AP^1}{k} = \frac{\cos A'P^1}{h} = \frac{c\cos CP^1}{l} = \frac{1}{N};$$

for P^7, $$\frac{\cos AP^7}{h} = \frac{\cos A'P^7}{-k} = \frac{c\cos CP^7}{l} = \frac{1}{N};$$

for P_4, $$\frac{\cos AP_4}{h} = \frac{\cos A'P_4}{k} = \frac{c\cos CP_4}{-l} = \frac{1}{N}.$$

$$\therefore\ \cos\mu = \cos PP^1 = \cos AP\cos AP^1 + \cos A'P\cos A'P^1 + \cos CP\cos CP^1$$
$$= (2hk + l^2 \div c^2) \div N^2 \qquad (12).$$

$$\cos\nu = \cos PP^7 = (h^2 - k^2 + l^2 \div c^2) \div N^2 \qquad (13).$$

$$\cos\lambda = \cos PP_4 = (h^2 + k^2 - l^2 \div c^2) \div N^2 \qquad (14).$$

Hence, if $\mu = \nu$, $h^2 - k^2 = 2hk$; and $k \div h$ has the irrational value $\sqrt{2} - 1$.

If $\lambda = \nu$, $\therefore\ h^2 - k^2 + l^2 \div c^2 = h^2 + k^2 - l^2 \div c^2$; $\therefore\ c^2 = l^2 \div k^2$.

But c changes by insensible increments as the temperature varies, which the integers l and k cannot do. Hence, λ cannot be equal to ν.

Similarly, if $\lambda = \mu$, $c^2 = 2l^2 \div (h-k)^2$, which cannot remain true as the temperature changes.

Examples.

27. The crystals of the following substances belong to this class:—cassiterite, SnO_2; rutile, TiO_2; anatase, TiO_2; zircon, $ZrO_2.SiO_2$; thorite, $ThO_2.SiO_2$; phosgenite, $(PbCl)_2CO_3$; torbernite, $CuO(UO_3)_2P_2O_5.8H_2O$; apophyllite, $H_7KCa_4(SiO_3)_8.4\frac{1}{2}H_2O$; vesuvianite, $(Ca, Fe)_6Al(F, OH)Al_2(SiO_4)_5$ (?).

Cassiterite. The crystals are usually short tetragonal prisms $m\{110\}$, the edges of which are truncated by $A\{100\}$ and often also bevelled by narrow faces $h\{210\}$. The crystals are most commonly terminated by faces $s\{111\}$, Fig. 195; and occasionally by faces of $e\{101\}$ and $z\{321\}$. These latter forms most frequently occur on a somewhat rare variety from Cornwall represented by Fig. 193. Crystals of the habit shown in Fig. 195 are generally much twinned, and are often hard to decipher. Optically, they are uniaxal and positive, *i.e.* the ordinary refractive index, ω, is less than the principal extraordinary index, ϵ.

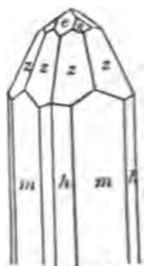

FIG. 193.

In crystals of the habit given in Fig. 195, the best measurements are obtained in the zones $[s,As]$, $[sms_{,,}]$. The faces A, m, &c., in the zone $[Am]$ are usually deeply striated parallel to the vertical axis, and the images given when this zone is adjusted on the reflecting goniometer consist of bands, such that the angles read are of little value. By measuring the angles in $[s,As]$ and $[A'ss''']$, the student can prove the equality of the angles ss' and ss''', and that

$$\wedge mm' = \wedge sCs' = 90°;$$

from them $\wedge Cs$ can be calculated and will be found equal to that given by measurement of $[sms_{,,}]$. Such measurements enable him to place on the stereogram, Fig. 194, the poles A, m, s and e and their homologues.

i. The face m is taken to be (110), then A is (100). A knowledge of $mh = 18° 26'$, or of $Ah = 26° 34'$, proves that h is {210} (see table in Art. 7).

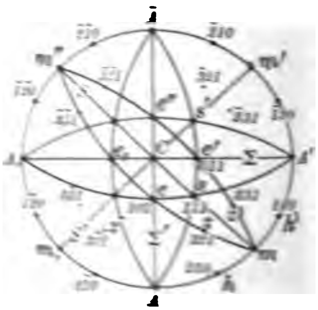

FIG. 194.

ii. Taking s to be (111), we fix the parameter c, and can then find it and the angular element F from equation (6). Thus, if sm = [illegible] (111) is found by measurement to be [illegible], then $Cs = 45° 34'$, and

$$\tan F = c = \tan (Cs = 45° 34') \div \sqrt{2}$$

$F =$ [illegible], and $c =$ [illegible]

iii. [illegible] The Cornish crystals in the [illegible] collection, which show the form z, are [illegible] combinations of s, m and z; but the [illegible] s and e are occasionally slightly [illegible]. To determine the symbol of the

form, two angles in different zones must be measured, unless the faces e are present and the zone $[mze]$ can be measured.

α. In this latter case we can determine either the angle mz or ze, and we know that the faces are tautozonal. Let z be (hkl), then, since it lies in $[me]=[1\bar{1}\bar{1}]$, $h-k-l=0$.

Again, let T be the possible pole midway between e and $e_,$ in which the zone $[me]$ meets $[Cm_,]$; then T has the symbol $(1\bar{1}2)$, and $mT=90°$. Taking the A.R. $[mzeT]$, we have

$$\tan mz \div \tan me = \begin{vmatrix} 110 \\ hkl \\ \overline{110} \\ 101 \end{vmatrix} \div \begin{vmatrix} 1\bar{1}2 \\ hkl \\ \overline{1\bar{1}2} \\ 101 \end{vmatrix} = \frac{l}{l+2k}.$$

If now the angles me and mz given by measurement are both fairly good, they may be introduced into this equation when the ratio of $l \div k$ is determined. Then, by the equation of the zone, the three indices can be found. If, however, the measured angles are not good, it will be best to use the theoretical value of me, which can be readily obtained when $E=Ce$ is known. For, from the right-angled spherical triangle Ame,

$$\cos me = \cos Am \cos Ae = \cos 45° \sin Ce.$$

If then either mz, or ze, given by measurement, is fairly good, the equation representing the A.R. gives the symbol of z.

β. But if e is not present, we are not at liberty to assume that z is in the zone $[me]$, and the method given in Art. 16 must be employed. Suppose the angles $mz=25°\ 0'$ and $zz^1=20°\ 54'$ to be obtained by measurement. Taking, as in that Article, p on $[Cm]$ midway between z and z^1 to be (eeg), where $e=h+k$ and $g=2l$, we have from the right-angled spherical triangle mzp,

$$\cos mz = \cos zp \cos mp = \cos zp \sin Cp.$$

$$\therefore \sin Cp = \cos 25°\ 0' \div \cos 10°\ 27';$$

and, by computation, $Cp=67°\ 9{\cdot}6'$.

But, by equations (6) and (7),

$$\frac{e}{g}=\frac{h+k}{2l}=\tan Cp \div \tan Cs = \tan 67°\ 9{\cdot}6' \div \tan 43°\ 34\ ;$$

$$\therefore \text{ by computation } \frac{h+k}{2l}=\frac{5}{2}.$$

$$\therefore\ h+k-5l=0 \quad\text{.................................}(15).$$

Knowing now Cp and zp, the angle pCz can be found; and this is equal to mf of Art. 16 $=45°-Af$.

The expression is $\cot pCz = \cot zp \sin Cp$,

$$\therefore \cot mf = \cot 10°\ 27' \sin 67°\ 9{\cdot}6'.$$

$$\therefore\ mf=11°\ 19', \text{ and } Af=33°\ 41'.$$

Therefore, by the table given in Art. 7, f is (320). Introducing the value 3 for h, and 2 for k in equation (*15*), we have $l=1$; and the pole z is (321).

The angle zs can, also, be now computed; for

$$\cos zs = \cos zp \cos sp = \cos 10°\ 27' \cos (Cp-Cs) = \cos 10°\ 27' \cos 23°\ 35{\cdot}6'.$$

$$\therefore\ zs = 25°\ 41'.$$

γ. To determine the symbol of z from measurement of $mz = 25° 0'$ and $sz = 25° 41'$.

We now know the three sides of the triangle smz, for $ms = 46° 26'$. Hence, by the formula which gives the angles of a spherical triangle when the sides are known (McL. and P. *Spher. Trig.* I. p. 47),

$$\tan \tfrac{1}{2} msz = \sqrt{\frac{\sin 2° 7{\cdot}5' \sin 22° 52{\cdot}5'}{\sin 48° 33{\cdot}5' \sin 23° 33{\cdot}5'}},$$

and $\wedge\, msz = 24° 44{\cdot}5'$.

From the right-angled triangle szp, the arcs ps and pz can be now found; and the symbol of z is then found in the way given under case β.

For, $\tan ps = \tan (sz = 25° 41') \cos (msz = 24° 44{\cdot}5')$,

and $\sin pz = \sin msz \sin sz$.

Hence, by computation, $ps = 23° 35{\cdot}6'$, and $pz = 10° 27'$.

$$\therefore\ zz^1 = 20° 54', \text{ and } Cp = 67° 9{\cdot}6'.$$

iii. A stereogram, Fig. 194, of the poles is made as follows. The poles A, m, &c., of the tetragonal zone $[AmA']$ having been inserted in the primitive by a protractor, diametral zones $[CA]$, $[Cm]$, &c., are drawn. These zones coincide with the circles in which the planes of symmetry meet the sphere. On $[AC]$ an arc $Ce = 33° 55{\cdot}5'$, or on $[Cm]$ an arc $Cs = 43° 34'$, is marked off by the construction of Chap. VII, Prob. 1. The zone-circles $[Ae']$, $[A'e]$, $[me]$, &c., are then easily described. The poles z are the points of intersection of zone-circles, such as $[me]$ and $[hs]$, and can therefore be easily placed.

iv. The crystal is drawn as follows. The cubic axes being projected by the method of Mohs or Naumann (Chap. VI), lengths OC and $OC_{,}$ $(= {\cdot}6726\ OA'')$ are measured off on the vertical axis. The lines joining C and $C_{,}$ to the axial points A, A', &c., on the axes of X and Y give the polar edges of the pyramid s. The upper and lower pyramid are then separated by a prism $\{110\}$, the edges having the length which corresponds to the particular crystal. To introduce the faces $\{100\}$, cut off from the polar edges by proportional compasses equal lengths, such as will give, approximately, the relative dimensions of the two prisms $\{110\}$ and $\{100\}$. The edge $[As]$ is parallel to the polar edge $[ss']$, and similarly for all the homologous edges; the vertical edges $[Am]$ can then be drawn, and the figure completed.

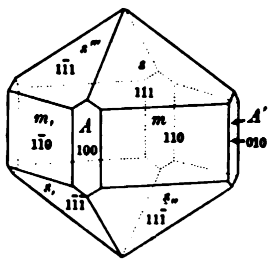

FIG. 195.

28. *Zircon* is found in crystals resembling those of cassiterite. The parameter and symbols of the faces would be determined in the way described in Art. 27. The crystals from some localities have the habits represented in Figs. 196 and 197. Taking p to be (111), it is easy to verify the symbols of the faces represented in Fig. 197 from the following angles: $Ax = 31° 43'$, $xp = 29° 57'$. The element

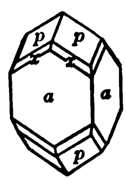

FIG. 196.

$E = 32° 38'$ can also be found from the same data; and then $c = \cdot6404$ can be computed. The crystals are uniaxal and positive.

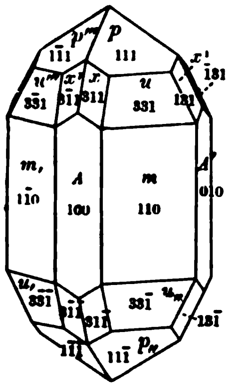

FIG. 197.

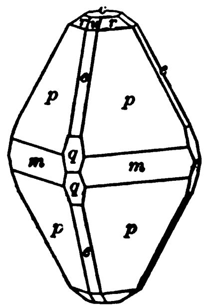

FIG. 198.

Anatase usually occurs in small crystals which are very acute pyramids. This pyramid is taken to be {111} and is generally indicated by the letter p as in Fig. 198. The angle of the pyramid over the face m is 43° 24′. Therefore $mp = 21° 42'$ and $Cp = 68° 18'$. Hence, by equation (6), $E = 60° 38'$; and $c = 1 \cdot 777$. The forms shown in Fig. 198 are: m {110}, p {111}, r {115}, c {001}, u {105}, e {101} and q {201}. The crystals have a good cleavage parallel to c (001); and they are uniaxal and negative.

Apophyllite occurs in simple prisms a {100} with the pyramid p {111}, Fig. 199. The angle ap being 52° 0′, the arc mp can be computed; for $\cos ap = \cos am \cos mp$. Hence, by equation (6), E is found to be 51° 22·5′, and $c = 1 \cdot 2515$.

In Figs. 200 and 201 crystals of different habits are shown; the additional forms being c {001} and y {310}. There is a good cleavage with a remarkable pearly lustre parallel to (001). The crystals have very weak double refraction; and are sometimes positive, sometimes negative.

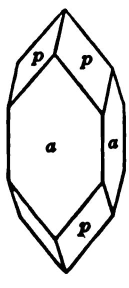

FIG. 199.

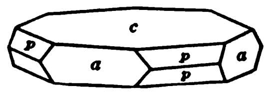

FIG. 200.

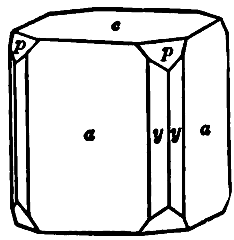

FIG. 201.

Between crossed Nicols plates of positive crystals show a dark cross and dull purple rings on a white ground: some plates of negative crystals from Utö absorb most of the light and show a grey cross on a violet ground, whilst others show both phenomena in adjoining portions of the same plate, which otherwise seems to be homogeneous (Des Cloizeaux).

29. *Vesuvianite* (idocrase) is generally found in stout prismatic crystals of the habit shown in Fig. 205. The faces *m* are large, and their edges are truncated by *a* {100}; the edges [*ma*] being often modified by faces of the forms {210} or {310}: all these faces are, as a rule, striated parallel to the principal axis. The faces of *c* {001} and of the pyramid *u* {111} are generally well developed, and are frequently the only faces associated with the prisms; but numerous small faces are often found to modify the edges and coigns. The stereogram, Fig. 202, from Brooke and Miller's *Mineralogy* gives the forms recognised in 1852.

The double refraction is weak; and the crystals are generally negative, although they are sometimes positive: the plates occasionally show segments which are irregularly biaxal.

The poles shown in Fig. 202 are: *a* {010}, *m* {110}, *h* {130}, *f* {120}, *c* {001}, *n* {113}, *y* {112}, *u* {111}, *w* {221}, *t* {331}, *r* {441}, *e* {011}, *g* {021}, *v* {151}, *x* {141}, *s* {131}, *z* {121}, *o* {241}, *i* {132}.

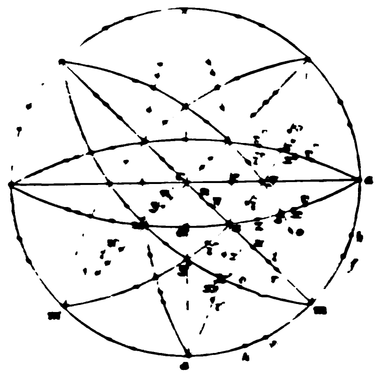

Fig. 202.

We shall now show how to find the angles in the principal zones, from a knowledge of the angle *cu* and the indices given.

i. In the prism-zone, all the angles are fixed; and *ah*, *af* are given in Art. 7.

ii. The angle *cu* is given by Miller as 37° 7′, and by Dana as 37° 13·5′. The angles of crystals from different localities vary, and no single value of *cu* fits all of them. We shall in the following computations adopt Miller's angle.

Hence, from equation (6) Art. 18,

$$c = \tan E = \tan (cu = 37° \; 7') \div \sqrt{2};$$

$\therefore E = 28° \; 9'$, and the parameter $c = \cdot 5351$.

iii. Let *p* be any pole (*hhl*) in the zone [*cum*]; then, from the A. R. {*cpum*}, we have

$$\tan cp \div \tan cu = h \div l,$$

or

$$\tan cp = h \tan cu \div l.$$

Assigning, in turn, to p the indices of the poles, n, y, &c. we have:—

for n (113), $\tan cn = \tan 37^\circ 7' \div 3$, $\therefore cn = 14^\circ 9{\cdot}5'$;
,, y (112), $\tan cy = \tan 37^\circ 7' \div 2$, $\therefore cy = 20^\circ 48{\cdot}5'$;
,, w (221), $\tan cw = 2 \tan 37^\circ 7'$, $\therefore cw = 56^\circ 32{\cdot}75'$;
,, t (331), $\tan ct = 3 \tan 37^\circ 7'$, $\therefore ct = 66^\circ 13{\cdot}6'$;
,, r (441), $\tan cr = 4 \tan 37^\circ 7'$, $\therefore cr = 71^\circ 48'$.

iv. We have already determined $\wedge\, ce = E$ to be $28^\circ 9'$, $\therefore$ for g (021),

$$\tan cg = 2 \tan (ce = 28^\circ 9'), \ \therefore cg = 46^\circ 56{\cdot}5'.$$

v. *Zone* [*avxsu*]. From the right-angled triangle aum, we can find au; and then, taking Q to be any pole in the zone [au], we can, from the A.R. $\{aQue'\}$, obtain a formula which enables us to find each of the angles. From $\triangle\, amu$,

$$\cot au \tan am = \cos (mau = a'e') = \sin ce'.$$

But, $\tan (am = 45^\circ) = 1$, $\therefore \cot au = \sin (ce' = 28^\circ 9')$.

From A.R. $\{aQue'\}$, $\tan aQ \div \tan au = h \div k$.

$$\therefore \cot aQ = k \cot au \div h = k \sin ce' \div h.$$

Hence, for v (151), $\cot av = 5 \sin 28^\circ 9'$, $\therefore av = 22^\circ 58{\cdot}3'$;
,, x (141), $\cot ax = 4 \sin 28^\circ 9'$, $\therefore ax = 27^\circ 55'$;
,, s (131), $\cot as = 3 \sin 28^\circ 9'$, $\therefore as = 35^\circ 14{\cdot}\dot{6}'$;
,, z (121), $\cot az = 2 \sin 28^\circ 9'$, $\therefore az = 46^\circ 39{\cdot}\dot{6}'$;
,, u (111), $\cot au = \sin 28^\circ 9'$, $\therefore au = 64^\circ 44{\cdot}5'$.

vi. *Zone* [*mosgi''*]. The angles in this zone can be obtained in an exactly similar manner. For,

$$\cos mg = \cos am \cos ag = \cos 45^\circ \sin (cg = 46^\circ 56{\cdot}5'). \quad \therefore mg = 58^\circ 53{\cdot}5'.$$

For any pole $P\,(hkl)$ in this zone, the A.R. $\{mPgu''\}$ gives, since $mu'' = 90^\circ$,

$$\tan mP \div \tan mg = \left|\begin{matrix} 110 \\ hkl \\ \overline{1}\overline{1}0 \\ 021 \end{matrix}\right| \div \left|\begin{matrix} \overline{1}11 \\ hkl \\ \overline{1}11 \\ 021 \end{matrix}\right| = \frac{l}{k-l} = \frac{l}{h+l};$$

the two latter equations being obtained by taking different pairs of columns.

If the angles have been measured, the two equations give the symbols of o, s, i''. But taking the symbols given, we have:—

for o (241), $\tan mo = \tan 58^\circ 53{\cdot}5' \div 3$, $\therefore mo = 28^\circ 55'$;
,, s (131), $\tan ms = \tan 58^\circ 53{\cdot}5' \div 2$, $\therefore ms = 39^\circ 39'$;
,, i'' ($\overline{1}32$), $\tan mi'' = 2 \tan 58^\circ 53{\cdot}5'$, $\therefore mi'' = 73^\circ 12{\cdot}\dot{6}'$.

vii. *Drawing.* To illustrate the method of drawing Fig. 205 we give Figs. 203 and 204. The unit cubic axes, projected in the way described in Chap. VI, Arts. 11 or 23, are pricked through from a permanent card. The vertical axis OA'' is alone altered, by multiplying it by $c = {\cdot}5851$. We thus get OC and $OC_{\prime}$, as shown in Fig. 203. The pyramid u is obtained by joining C and $C_{\prime}$ to the points A, A', &c. From the edges CA, CA', &c., equal lengths $C\delta$,

$C\delta'$, &c., are cut off by proportional compasses, as shown in Fig. 204. Pairs of the points $\delta\delta'$, &c., are joined and give the face c (001). A second set of equal

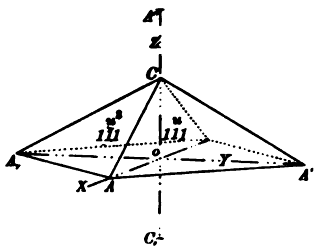

FIG. 203.

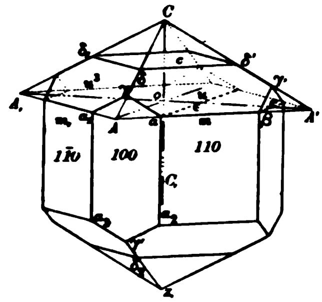

FIG. 204.

lengths $A\gamma$, $A'\gamma'$, &c., are cut off in a similar manner from the edges CA, CA', &c. Through γ the lines γa, γa_1 are drawn parallel to CA' and CA, respectively to meet AA' and $AA_,$ in a and a_1. Similarly through γ, and the corresponding point on $CA_,$, lines $\gamma'\beta$, &c., are drawn parallel to CA and $C\bar{A}$. Through the points a, a_1, β, &c., vertical lines are drawn of any desired length so as to correspond with the development of the particular crystal.

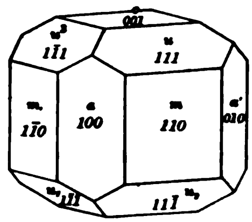

FIG. 205.

Through a_2, a_3 lines are drawn parallel to AA', $AA_,$ and to γa_1, γa. Through γ' the lower edge $\gamma' Z_,$ of the pyramid is drawn parallel to $C_,A$ of Fig. 203, and $Z_,\delta_2$ is cut off by proportional compasses to represent the same length as $C\delta$. The rest of the construction is obvious. The coigns can now be pricked through to a fresh paper and Fig. 205 produced.

Should it be desired to introduce new faces, the directions of the edges must be determined, and lines parallel to them be drawn through points marked off by proportional compasses on the homologous edges already introduced.

III. *Acleistous (polar) tetragonal class;* $\tau\ \{hkl\}$.

30. Crystals of this class have only a single tetrad axis T, and no other element of symmetry. The faces must, therefore, occur in sets of four, which change places with one another after each rotation of 90°, or any multiple of 90°, about the tetrad axis. There is one exception to this rule in the case of the possible face (Chap. IX, Prop. 3, Cor. 1) which is perpendicular to the axis. This face cannot be repeated, and constitutes a pedion which is frequently

called the base. The general form is an acleistous tetragonal pyramid which cannot by itself enclose a finite portion of space. The class may therefore be called the *acleistous tetragonal* class. No crystal of the class can consist of a single form, but must be a combination of two or more forms. The possible crystal shown in Fig. 206 consists of the tetragonal pyramid $\tau\{h0l\}$ with the pedion $\tau\{00\bar{1}\}$.

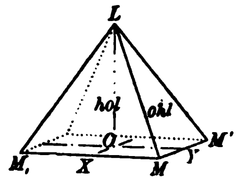

FIG. 206.

The axis of symmetry is uniterminal and the crystals are hemimorphic; they should show pyro-electric phenomena.

31. The tetrad axis is taken as the axis of Z, and any pair of edges perpendicular to it which are at right angles to one another give the directions of the axes of X and Y. These latter axes are parallel to the edges in which the pedion is met by the pyramid to which the symbol $\tau\{101\}$, or $\tau\{h0l\}$, is assigned.

Fig. 206, representing the pyramid $\tau\{h0l\}$ and the pedion $\tau\{00\bar{1}\}$, gives the directions of the axes of X, Y and Z. But we have already seen that the plane $M_,LM'$ through opposite polar edges of the pyramid is parallel to a possible face; and $MM_, = MM'$, therefore the line $M_,M'$ makes 45° with the axes of X and Y. The possible vertical face through X parallel to $M_,M'$ must therefore meet OY at a point such that $OY = OX$. Hence, we can take equal parameters, on these axes; and such parameters are clearly the most convenient. The parameters may be given by $a : a_, : c$, or by $1 : 1 : c$.

If there are a number of tetragonal pyramids in different azimuths, *i.e.* such that they meet the pedion in lines inclined at various angles to one another, one of the pyramids is taken to give the directions of the axes, and the others are then assigned symbols which correspond with their positions. There is nothing, however, to limit the choice of the pyramid selected to give the axes, and the crystallographer is influenced chiefly by a desire to give the simplest indices to the faces. But in every case the possible prism-face equally inclined to the axial planes XOZ and YOZ is taken to be (110), so that equal parameters are measured on OX and OY.

32. *The tetragonal pyramid*, $\tau\{h0l\}$. If the face parallel to OY in Fig. 206 is taken to be $(h0l)$, then the homologous face, obtained

by a rotation of 90°, passes through MM' and L, and has the symbol $(0hl)$. The remaining two faces are $(\bar{h}0l)$ and $(0\bar{h}l)$. Hence, the form $\tau\{h0l\}$ consists of the faces:

$$h0l,\ 0hl,\ \bar{h}0l,\ 0\bar{h}l \text{ ....(k).}$$

Again, if a section of this crystal is taken in the plane LOX, we have a right-angled triangle LXO, in which $OX = a \div h$, and $OL = c \div l$.

Hence, $$\tan LXO = OL \div OX = \frac{h}{l}\frac{c}{a}.$$

If n is the pole of this face, and C is the pole (001), then $\wedge\, Cn = \wedge\, LXO$.

$$\therefore\ \tan Cn = \frac{h}{l}\frac{c}{a}.$$

If we take the particular case of the pyramid in which h and l are both unity, and if e is the pole (101), then

$$\tan Ce = c \div a = \tan E.$$

We, therefore, have the same formulæ for determining the element $c \div a$, or E, as those given in (2) and (3) of Arts. 8 and 12.

33. The special forms are: (*1*) pedions; (*2*) tetragonal prisms, $\{110\}$, $\{100\}$, $\tau\{hk0\}$.

1. The pedions are the faces perpendicular to the tetrad axis; and if both faces are present, they form complementary pedions and have different characters. Since they are parallel to the axes of X and Y, the one has the symbol $\tau\{001\}$, and the other the symbol $\tau\{00\bar{1}\}$.

2. $\{110\}$. We have already had in Art. 31 a prism-face (110) parallel to $M_{,}LM'$ of Fig. 206. This face must be associated with three others, the positions of which are obtained by rotations of 90°, 180° and 270° about OZ. The prism is geometrically the same as $\{110\}$ in the two preceding classes, and the Greek prefix may be omitted.

The prism $\{110\}$ has the faces: $110,\ \bar{1}10,\ 1\bar{1}0,\ \bar{1}\bar{1}0$.

$\{100\}$. Again, faces through the similar and interchangeable edges $M_{,}M$, MM', &c. of Fig. 206 can be drawn parallel to OZ. Since the faces are parallel to two of the axes, the symbols are 100, 010, $\bar{1}00$, $0\bar{1}0$. They constitute a prism similar to that which has its faces parallel to the vertical axial planes in the two preceding classes. The symbol may therefore be written $\{100\}$.

The tetragonal prism, $\tau\{hk0\}$. If a prism occur in any general azimuth, one of its faces may be supposed to be drawn through the line HMK of Fig. 177. By rotations of 90°, 180° and 270° this line is brought successively into the positions $K'H_{,,}$, $H_,K_{,,}$, $K_,H'$. For we saw in Art. 7 that alternate sides of the ditetragon of this figure are at 90° to one another. Hence the prism $\tau\{hk0\}$ consists of the faces:

$$hk0,\ \bar{k}h0,\ \bar{h}\bar{k}0,\ k\bar{h}0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{l}).$$

A similar prism $\tau\{kh0\}$ can be formed by drawing the faces through the remaining edges of Fig. 177; but it has no necessary connection with $\tau\{hk0\}$, though the faces can be placed in a similar geometrical position by rotation through 180° about the line OM or the line $OM_,$. But such a rotation interchanges different ends of the tetrad axis; and the facial development and physical characters at the two ends are dissimilar. The two prisms cannot therefore be brought into similar positions without disturbing the arrangement of parts having the same physical characters.

34. *The tetragonal pyramid*, $\tau\{hhl\}$. Through the lines AA', $AA_,$, &c., of Fig. 173 in which the faces of the prism $\{110\}$ meet the plane XOY, faces can be drawn to meet the tetrad axis at a finite distance on either side of the origin. One of the faces which meet at an apex L has the intercepts $OA : OA' : OL$, or $a : a_, : hc \div l$, if $OL = hc \div l$. The face has therefore the symbol (hhl). The three other faces all pass through L, and the signs of the first two indices are alone changed. Hence $\tau\{hhl\}$ consists of:

$$hhl,\ \bar{h}hl,\ \bar{h}\bar{h}l,\ h\bar{h}l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{m}).$$

Again, if o is used to represent the faces of the particular case in which $h = l$, then the pyramid o is $\tau\{111\}$; it comprises the faces: $111,\ \bar{1}11,\ \bar{1}\bar{1}1,\ 1\bar{1}1$.

The inclinations of these two pyramids to the pedion, or to the plane XOY, are given by equations (5) and (6) of Art. 13.

35. *The tetragonal pyramid*, $\tau\{hkl\}$. Similarly, through the alternate edges HK, $K'H_{,,}$, $H_,K_{,,}$, $K_,H'$ of Fig. 177 faces can be drawn to meet the tetrad axis at a point L, where $OL = c \div l$. These four faces are interchangeable by rotation about the tetrad axis and constitute a pyramid $\tau\{hkl\}$, and have the symbols:

$$hkl,\ \bar{k}hl,\ \bar{h}\bar{k}l,\ k\bar{h}l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{n}).$$

In all the preceding pyramids the apex L may be on either side

of the origin; and if we remember that l may represent either a positive or a negative number, the above expressions include all cases which can occur.

36. We have therefore pyramids $\tau\{h0l\}$, $\tau\{hhl\}$, $\tau\{hkl\}$ which may be said to belong to three series. In the first the edges of the base are parallel to the axes, in the second they are at 45° to the axes, and in the third they may occupy any azimuth which is limited by the equation $\tan XOG = k \div h$. But we have already stated that *any* pyramid may be taken to give the directions of X and Y. There can therefore be no essential distinction between pyramids of the different series. The only difference which will be caused by changing the series, to which the pyramid giving the element belongs, is to alter the value of the element (whether it be E or $c \div a$); and this new value can be determined when we know the pyramids which have been changed and the element corresponding to one of them. Thus, let $e\,(101)$ and $o\,(111)$ be the faces of two pyramids in the first representation, and let E or $c \div a$ be the corresponding element; and in the second representation let o be made (101), so that Co is E' and the corresponding parametral ratio is $c' : a'$.

Now, by equation (2),

$$\tan E = \tan Ce = c \div a.$$

And by equation (6),

$$\tan Co = c\sqrt{2} \div a = \sqrt{2} \tan E.$$

But $Co = E'$, and $\tan E' = c' \div a'$.

If a and a' are both taken to be unity, then

$$c' = \tan E' = \tan Co = \sqrt{2} \tan E = c\sqrt{2}.$$

The new parameter c' is therefore the original c multiplied by $\sqrt{2}$.

From equation (9) we can find E'' and c'' the values of the elements when a face t of the pyramid $\tau\{hkl\}$ is taken to be (101). For $Ct = E''$, and $c'' = \tan E''$ when $a'' = 1$.

$$\therefore\ c'' = \tan E'' = \sqrt{\frac{h^2+k^2}{l^2}} \tan E = \frac{c}{l}\sqrt{h^2+k^2}.$$

Thus, if (hkl) is (211), $c'' = c\sqrt{5}$.

37. If, however, the crystals are regarded as merohedral, the pyramids $\tau\{h0l\}$, $\tau\{hhl\}$, have each one-half the faces in the bi-pyramids $\{h0l\}$ and $\{hhl\}$ of Class II, which has the greatest symmetry possible in the system, and the forms of which are regarded

as holohedral. The pyramids $\tau\{h0l\}$ and $\tau\{hhl\}$ are therefore hemihedral. But the pyramid $\tau\{hkl\}$ has only one-fourth the faces of the ditetragonal bipyramid $\{hkl\}$ of Class II, and is therefore tetartohedral. As, however, it is a matter of choice on the part of the crystallographer which of the series of pyramids is selected to give the axes, the different series cannot be some of them hemihedral and the others tetartohedral. For by taking (hkl) to give the element c'' we change a tetartohedral form into a hemihedral one; or rather, a form which was in the first representation regarded as tetartohedral has in the second case to be regarded as hemihedral.

Similarly, the prisms $\{100\}$, $\{110\}$ are regarded as holohedral, whilst $\tau\{hk0\}$ is taken to be hemihedral; and by a variation in the axes of reference OX and OY, we can vary the manner in which these prisms are regarded.

We see, therefore, that the views underlying the idea of merohedrism lead to inconsistencies, and to representations of the crystals which are not in accordance with the facts. These difficulties are avoided when each class is treated as consisting of a group of crystals having definite elements of symmetry, of which the facial development and the physical characters are the consequence.

38. It is clear that analytically the method of determining the symbols of the faces and the parametral ratio from the measured angles, or of determining the angles from the symbols and element, must in this class be exactly the same as that given in preceding sections.

39. Crystals of *wulfenite* ($PbMoO_4$) have been described which show forms of this kind. The crystal shown in Fig. 207 (after Naumann) includes the forms: $n=\tau\{111\}$, $\mathrm{n}=\tau\{11\bar{1}\}$, $e=\tau\{10\bar{1}\}$, $\delta=\tau\{432\}$, $x=\tau\{31\bar{1}\}$. The element $E=57°\ 37{\cdot}3'$, and $c=1{\cdot}5771$.

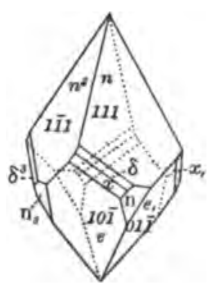

Fig. 207.

The principal axis being one of uniterminal symmetry should be a pyro-electric axis: this has not been established in wulfenite; but in barium-antimonyl dextrotartrate, $Ba(SbO)_2(C_4H_4O_6)_2 . H_2O$, the crystals of which are held to belong also to this class, the tetrad axis has been found to be a pyro-electric axis.

Crystals of these two substances do not rotate the plane of polarization.

IV. *Diplohedral tetragonal class*; $\pi\{hkl\}$.

40. Crystals of this class possess a tetrad axis, a centre of symmetry, and therefore, by Chap. IX, Prop. 4, a plane of symmetry Π, perpendicular to the tetrad axis. The forms can, therefore, be deduced from those of the preceding class by the addition of faces parallel to those given in Arts. 32—35. Or the new faces can be obtained by regarding the plane Π as a mirror in which the faces are reciprocal reflexions.

41. We can clearly take axes having to one another the same relation as the axes of crystals of class III; *i.e.* OZ is the tetrad axis, OX and OY are parallel to the horizontal edges of any tetragonal bipyramid, and equal parameters are taken on them. The element c, or E, is obtained from the possible or actual pyramid to which the symbol $\{101\}$ is given.

The analytical relations of the element, angles and face-indices, and the methods of solution of the problems which may occur, are identical with those of crystals belonging to the preceding classes.

42. The prisms given in class III remain unaltered, for the faces, being parallel to a tetrad axis, occur in pairs which are parallel to one another. Hence:

$$\{100\} \text{ consists of } 100,\ 010,\ \bar{1}00,\ 0\bar{1}0.$$
$$\{110\} \quad ,, \quad ,, \quad 110,\ \bar{1}10,\ \bar{1}\bar{1}0,\ 1\bar{1}0.$$
$$\pi\{hk0\} = \tau\{hk0\} \quad ,, \quad ,, \quad hk0,\ \bar{k}h0,\ \bar{h}\bar{k}0,\ k\bar{h}0.$$

The faces perpendicular to the tetrad axis both occur together and form the pinakoid $\{001\}$.

43. The tetragonal pyramids become diplohedral, and each of them encloses a finite portion of space. They can, therefore, occur alone on crystals of this class.

The bipyramid $\{h0l\}$, Fig. 179, having its faces parallel to the axes of X and Y, is geometrically identical with that described in Arts. 8 and 22 as a possible form in each of classes I and II. The bipyramid includes the eight faces given in table **e**.

The bipyramid $\{hhl\}$, which has its horizontal edges parallel to the diagonals of the square formed by bipyramids of the first series, is geometrically identical with $\{hhl\}$ of Art. 23 and may be represented

by Fig. 190: it comprises the faces given in table i. The tetragonal bipyramids of these two series are therefore apparently holohedral.

The bipyramid $\pi\{hkl\}$ consists of the eight faces of table j, Art. 24, which pass through the sides of the square formed by the alternate sides of the ditetragon of Fig. 177, and have the symbols:

$$hkl,\ \bar{k}hl,\ \bar{h}\bar{k}l,\ k\bar{h}l,\ hk\bar{l},\ \bar{k}h\bar{l},\ \bar{h}\bar{k}\bar{l},\ k\bar{h}\bar{l} \ldots\ldots\ldots\ldots(\text{o}).$$

The general appearance of the form is the same as that of each of the bipyramids $\{h0l\}$ and $\{hhl\}$; but, as it has only one-half the faces of the bipyramid of Art. 24, it was regarded as being hemihedral with parallel faces.

44. Since a bipyramid of any series may be selected arbitrarily to give the axes of X and Y, and the element, there is no essential distinction between the tetragonal bipyramids of the different series. Also, when a change is made in the axes and element, by selecting a bipyramid of a different series to give them, the values of the element in the two representations are connected by the equations given in Art. 36.

It is clearly inconsistent with the symmetry of the crystals, of which these several forms are the immediate consequence, to regard two of the series as holohedral and the others as hemihedral. Different crystallographers might, by a mere change in the series to which the bipyramid selected as $\{101\}$ belongs, make a definite bipyramid holohedral in the one case and hemihedral in the other.

The same remarks apply to prisms $\{100\}$, $\{110\}$ which are geometrically identical with those of the three previous classes, whilst $\pi\{hk0\}$ has only half the faces present in $\{hk0\}$ of classes I and II.

45. Crystals of scapolite (wernerite), $m\mathrm{Ca_4Al_2Si_6O_{25}} + n\mathrm{Na_4Al_3Si_9O_{24}Cl}$; of scheelite, $\mathrm{CaWO_4}$; of stolzite, $\mathrm{PbWO_4}$; of erythroglucine, $\mathrm{C_4H_{10}O_4}$; belong to this class.

Fig. 208 represents a crystal of *meionite*, in which variety of scapolite $m:n$ ranges from 1 : 0 to 3 : 1. The forms are: $a\{100\}$, $m\{110\}$, $r\{111\}$, and $z=\pi\{311\}$. The angle $mr=58°\ 9'$, and from equation (6) $c=\cdot43925$. The double refraction is weak and negative, and it diminishes as the amount of sodium increases.

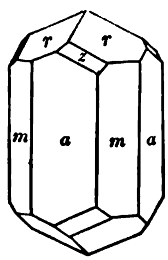

FIG. 208.

Crystals of *scheelite* are optically positive. Fig. 209 shows a crystal having the forms: $e\{101\}$, $o\{111\}$, $h=\pi\{313\}$, $s=\pi\{131\}$. The faces o are usually present and are sometimes those which predominate: as a rule,

they are bright and smooth, and give good reflexions. The faces $e\{101\}$ are striated parallel to the edges $[eh]$, and give poor reflexions. Hence the element is computed from measurement of the angle $oo_{,,} = 49° \ 27'$.

$$\therefore \ Co = 90° - oo_{,,} \div 2 = 65° \ 16{\cdot}5'.$$

By equation (6),

$$c = \tan E = \tan (Co = 65° \ 16{\cdot}5') \div \sqrt{2}.$$

$$\therefore \ E = 56° \ 55{\cdot}8', \ c = 1{\cdot}5356.$$

Zone $[ehos]$. Measurement of the angles in the zone $[eh]$ enables us to determine the symbols of the faces; or, vice versâ, knowing the symbols and the element, we can determine the angles which h, o and s make with e. For, as shown in Fig. 210, the poles lie in the zone $[A'eA_{,}]$. Hence, if T is taken to be a pole (hkh) lying in this zone, we have from the A.R. $\{A'Toe\}$

$$\tan A'T \div \tan A'o = \left|\frac{\begin{matrix}010\\ hkh\end{matrix}}{\begin{matrix}010\\ 111\end{matrix}}\right| \div \left|\frac{\begin{matrix}101\\ hkh\end{matrix}}{\begin{matrix}101\\ 111\end{matrix}}\right| = \frac{h}{k}.$$

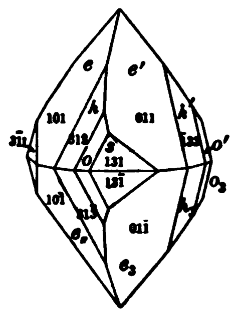

Fig. 209.

But, from the right-angled triangle $A'om$, we have

$$\cos A'o = \cos (A'm = 45°) \cos (mo = 24° \ 43{\cdot}5'),$$

$$\therefore \ A'o = 50° \ 2{\cdot}25'.$$

Hence, if T is taken to be s (131),

$$\therefore \ \tan A's = \tan 50° \ 2{\cdot}25' \div 3; \text{ and } A's = 21° \ 41{\cdot}5';$$

and if to be h (313),

$$\therefore \ \tan A'h = 3 \tan 50° \ 2{\cdot}25', \ \therefore \ A'h = 74° \ 23{\cdot}6'.$$

The construction of the stereogram, Fig. 210, presents no difficulties. The poles A, m, &c., are marked off on the primitive at 45° to one another. The diametral zones $[CA]$, $[Cm]$, &c., are then drawn; and by the construction of Chap. VII, Prop. 1, the pole e, or o, is determined on $[CA]$ or $[Cm]$. The zone-circles $[A'e]$, $[Ae']$, &c., are then described.

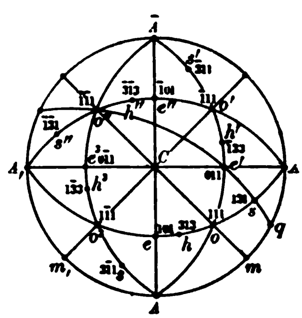

Fig. 210.

The simplest way of determining the points h and s is to find f and f' in $[Am]$, where f is (310) and f' (130), and $Af = A'f' = 18° \ 26'$. The diameters through the points f and f' give, by their intersection with the zone-circle $[eo]$, the poles h and s. The poles below the paper would be given by circlets surrounding each dot.

In Fig. 209 the axes have been turned 45° more to the left than is usual, so that $o\,(111)$ is in the position usually occupied by $(h0l)$. The development is better shown in this position than in one conforming to the common arrangement.

V. *Trapezohedral class; a* $\{hkl\}$.

46. In crystals of this class the tetrad axis is associated with four dyad axes all perpendicular to it, which necessarily consist of two pairs of like axes. The one pair may as in class II be denoted by Δ, $\Delta_{,}$; the other pair by δ, $\delta_{,}$. Again, the like axes Δ, $\Delta_{,}$ must be at 90° to one another, and the angles between them be bisected by the other pair δ, $\delta_{,}$; for a rotation of 90° about the tetrad axis interchanges like elements of the crystal. This class differs from II inasmuch as it has no centre and no planes of symmetry. The general form may, therefore, be denoted by the symbol $a\{hkl\}$.

47. The tetrad axis being OZ, a pair of like dyad axes which we may denote as the pair δ, $\delta_{,}$, are taken as OX and OY. The possible face perpendicular to Δ is then the most convenient one to select to give the parameters on OX and OY, for it meets them at equal distances from the origin, and $a = b$. Any plane through the trace of this face in the plane XOY, which meets OZ at a finite distance from the origin, may be taken as parametral plane (111), and will give the parameter c by equation (6). Hence we have rectangular axes with equal parameters on OX and OY, but a different parameter on OZ. The arrangement of axes and parametral plane is the same as that in classes I and II, save that in crystals of this class there are no planes of symmetry. The presence, or absence, of planes of symmetry does not, however, affect the formulæ connecting angles, parameters and face-indices, and consequently all those established for class I hold for this class also.

48. Such of the forms of class II as have faces parallel to the axes of symmetry will be common to this class also. For rotation through 180°, or 2 × 90°, about a dyad, or tetrad, axis necessarily interchanges a pair of parallel faces which are both parallel to the axis of rotation. Hence:

The pinakoid $\{001\}$ has the faces (001), $(00\bar{1})$.

The two tetragonal prisms, $m\{110\}$ and $A\{100\}$, will each consist of four faces, which have the symbols given in Arts. 5 and 6; and the ditetragonal prism $\{hk0\}$ will consist of the eight faces given in Art. 7.

The tetragonal bipyramid $\{h0l\}$ will be the same as that given in Art. 8.

The tetragonal bipyramid $\{hhl\}$ consists of the eight faces given in the similar form in Art. 23.

In all the above forms the Greek prefix is usually omitted; for, geometrically, the forms do not differ from those of class II. And, as has been seen, several of them are common to the other classes already discussed.

49. The general form $\alpha\{hkl\}$, Fig. 211, in which h and k are finite and unequal, is the only one geometrically characteristic of the class. It consists of eight four-sided similar faces which are all metastrophic. Each face has two like polar edges, such as $L\alpha$ and $L\gamma$, which change places when the crystal is turned through 90° about the tetrad axis. The median edges $\alpha\beta$ and $\beta\gamma$ of each face are dissimilar and are inclined to the axial plane XOY at different angles; and each of them is perpendicular to, and bisected by, a dissimilar dyad axis. The form is called a *trapezohedron*, and gives its name to the class. Geometrically, it may be regarded as made up of that set of the faces of the bipyramid $\{hkl\}$ of Art. 24, which are interchangeable by rotation about the axes of symmetry. Hence, $\alpha\{hkl\}$ has the following faces:

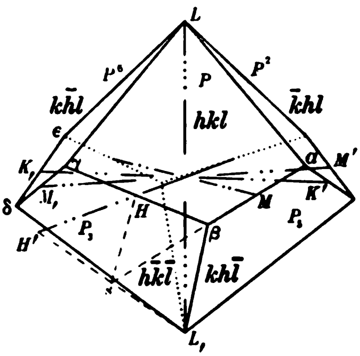

Fig. 211.

$$hkl,\ kh\bar{l},\ \bar{k}hl,\ \bar{h}k\bar{l},\ \bar{h}\bar{k}l,\ \bar{k}\bar{h}\bar{l},\ k\bar{h}l,\ h\bar{k}\bar{l}\ \ldots\ldots\ldots\ldots(\text{D}).$$

It will be noticed that the first two indices of the pairs of faces, which meet in like median edges, $\alpha\beta$, $\gamma\delta$, &c., perpendicular to Δ or $\Delta_{,}$, interchange places, since a rotation of 180° about either of these axes interchanges the axes of X and Y. The same rotation interchanges opposite ends of the axis of Z, and the last index changes sign. A pair of faces, meeting in edges perpendicular to δ or $\delta_{,}$, have h and k in the same order, but the signs of the intercepts on the two axes of reference, which the edge does not meet, are different; for a rotation of 180° about OX interchanges equal and opposite lengths on OY and OZ, and an equal rotation about OY interchanges equal and opposite lengths on OX and OZ. By paying

attention to the effect of rotation about the axes of symmetry, the symbols of all the faces can be directly obtained without reference to Art. 24.

50. It is clear that the trapezohedron has three different angles between faces which meet in edges; viz. those over polar edges $L\alpha$, and angles over the dissimilar median edges $\alpha\beta$ and $\beta\gamma$; and no two of these angles can be permanently equal. We need not discuss the possibility of making the angle over $L\alpha$ equal to either of the other two; for, if a case were found, an alteration of temperature, causing a change in the ratio of $c : a$, would necessarily disturb the equality and would make the angles unequal.

The simplest and most direct proof, that the angle PP_5 over the edge $\alpha\beta$ is not equal to $\wedge PP_3$ over $\beta\gamma$, is obtained in the same way as that, given in Art. 26, for the corresponding relation of the angles of $\{hkl\}$. For using the equations and notation of that Article, we have:

for $P_5\,(kh\bar{l})$, $$\frac{\cos AP_5}{k}=\frac{\cos A'P_5}{h}=\frac{c\cos CP_5}{-l}=\frac{1}{N};$$

for $P_3\,(h\bar{k}\bar{l})$, $$\frac{\cos AP_3}{h}=\frac{\cos A'P_3}{-k}=\frac{c\cos CP_3}{-l}=\frac{1}{N};$$

$$\cos PP_5=\left(2hk-\frac{l^2}{c^2}\right)\div N^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16);$$

and $$\cos PP_3=\left(h^2-k^2-\frac{l^2}{c^2}\right)\div N^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Hence, if $PP_5=PP_3$,

$$2hk-\frac{l^2}{c^2}=h^2-k^2-\frac{l^2}{c^2};$$

$$\therefore\; k^2+2hk-h^2=0.$$

And $$k\div h=\sqrt{2}-1.$$

We can also prove the proposition as follows. Assume $PP_3=PP_5$. Then $\wedge HP=\wedge MP$: for $\wedge HP=PP_3\div 2$, and $\wedge MP=PP_5\div 2$. Hence the plane through the tetrad axis and the normal OP must meet the sphere with centre at O in a circle which intersects the zone-circle $[AmA']$ in a pole f lying exactly midway between A and m; *i.e.* $Af=22°\ 30'$. But it has been shown in Art. 25 that no prism-face can be inclined to a vertical axial plane at $22°\ 30'$. Hence, PP_3 can never be equal to PP_5.

51. The form $\alpha\{hkl\}$ may be drawn as follows. The axes having been projected (Chap. VI, Art. 13), points H, H', $H_{,}$, $H_{,,}$, K, K', $K_{,}$, $K_{,,}$ are determined on the axes of X and Y, where $OH=a\div h$, $OH'=OK=a\div k$, &c., and the ditetragon of Fig. 212 is then drawn in the projected plane

XOY; the points M, M', &c., and N and its homologues being also obtained. The upper polar edges $L\alpha$, &c., are the lines joining L on OZ (where $OL = OL_{,} = c \div l$) to N and the three homologous points, in which alternate

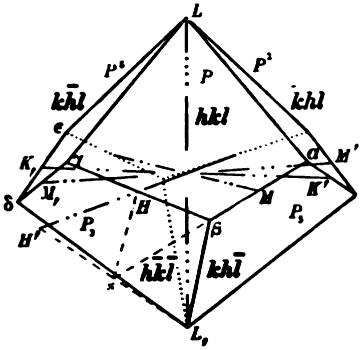

Fig. 211.

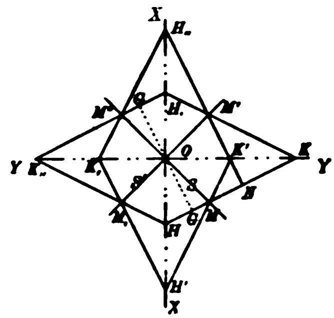

Fig. 212.

sides of the ditetragon intersect. For the face (hkl) passes through HK, and $(\bar{k}hl)$ through the alternate trace $H_{,,}K'$. We have, now, to determine the points α, γ, &c., on these polar edges. The median edge through H is parallel to the zone axis $[hkl, h\bar{k}\bar{l}]$, which is the same as $[hkl, \bar{h}kl]$: but these latter faces pass through the line LK, which therefore gives the direction of the edge $\beta H\gamma$. The point γ is determined by the intersection of this edge with $L\gamma$; and $H\beta = H\gamma$ is then measured off by a pair of compasses. Similarly, $K'\alpha$ is parallel to the line $L_{,}H'$, and $K_{,}\delta$ to LH'. The points α and ϵ are therefore found; and the points equidistant from K' and $K_{,}$ are then determined by a pair of compasses. The remaining median edges $\alpha\beta$, $\gamma\delta$, &c., are then easily drawn and will be found to pass through M, $M_{,}$, &c. Finally the edges $L_{,}\beta$, $L_{,}\delta$, &c., are drawn and complete the trapezohedron.

Or we may determine the median edges $\alpha M\beta$, &c., as follows. The line LH is produced to meet $L_{,}H'$ in s. Hs and $L_{,}H'$ are shown in Fig. 211 by lines of interrupted strokes. The line Ms gives the direction of the edge $\alpha M\beta$. If the direction $L\alpha$ is known, α is determined; and $M\beta = M\alpha$. If $\gamma H\beta$ is known, β is determined; and α obtained by a pair of compasses. The homologous edges $M_{,}\delta$, &c., are obtained by a similar construction.

The student will be guided in his selection of the method he pursues by considerations of the accuracy attainable, in any particular case, in the determination of the points N and s.

52. The complementary form $\alpha\{khl\}$ has all its faces, and, therefore, all its edges, parallel respectively to those of $\alpha\{hkl\}$. Hence,

Fig. 213 showing α {$k\bar{h}l$} is immediately made by drawing through K' a line parallel to $\delta\epsilon$ of Fig. 211 and through $K_{,}$ a line parallel to aK'. Similarly, the edges through H and $H_{,}$ are reciprocally interchanged. Again the polar edges through L in Fig. 213 are parallel to those through $L_{,}$ in Fig. 211, and vice versâ. The remaining median edges through M are then easily drawn.

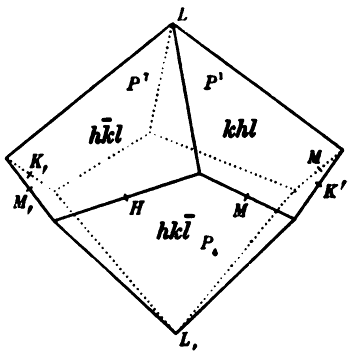

FIG. 213.

53. The two forms are clearly reciprocal reflexions with respect to any one of the axial planes, and are enantiomorphous. With this character a difference in the optical properties may be expected. Arguing from the analogy of quartz (Chap. XVI, class IV), crystals of this class should rotate the plane of polarization of a plane-polarised beam traversing the crystal along the principal axis. The direction of rotation may be with, or against, the hands of a watch. If crystals, on which α {hkl} is developed, rotate the plane of polarization clockwise, those in which the complementary form is developed may be expected to give a rotation counter-clockwise. Unfortunately, no crystals, save those of guanidine carbonate, have yet been observed showing either of the two general forms; but crystals on which special forms are alone developed have been found to cause rotation of the plane of polarization. The substances are therefore placed in this class; and we may hope, by variation of the conditions under which their crystals are grown, to establish the correlation by the development of general forms.

54. Tetragonal crystals of the following substances rotate the plane of polarization of a beam transmitted along the principal axis and are therefore placed in this class.

Ethylene-diamine sulphate, $C_2H_4(NH_2)_2 . H_2SO_4$. $c = 1{\cdot}4943$. The crystals are combinations of the pinakoid c {001} with one or several bipyramids {111}, {221}, {101}, {201}. The crystals are optically positive; and by a plate 1 mm. thick the rotation for sodium light of the plane of polarization is 15° 30′.

Guanidine carbonate, $(CH_5N_3)_2H_2CO_3$. $c = {\cdot}9910$. The crystals are bipyramids {111} with the forms {001}, {100} slightly developed: small dull faces of a trapezohedron have been observed, but the symbols could

XOY; the points M, M', &c., and N and its homologues being also obtained. The upper polar edges La, &c., are the lines joining L on OZ (where $OL = OL_, = c \div l$) to N and the three homologous points, in which alternate

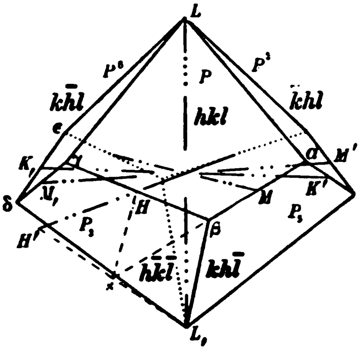

FIG. 211.

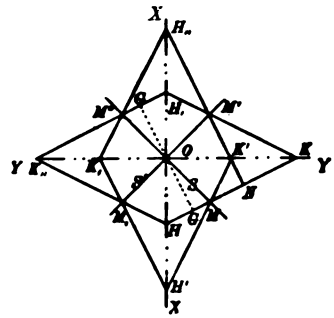

FIG. 212.

sides of the ditetragon intersect. For the face $(hk\bar{l})$ passes through HK, and $(\bar{k}hl)$ through the alternate trace $H_{,,}K'$. We have, now, to determine the points a, γ, &c., on these polar edges. The median edge through H is parallel to the zone axis $[hkl, h\bar{k}\bar{l}]$, which is the same as $[hkl, \bar{h}kl]$: but these latter faces pass through the line LK, which therefore gives the direction of the edge $\beta H\gamma$. The point γ is determined by the intersection of this edge with $L\gamma$; and $H\beta = H\gamma$ is then measured off by a pair of compasses. Similarly, $K'a$ is parallel to the line $L_,H'$, and $K_,\delta$ to LH'. The points a and ϵ are therefore found; and the points equidistant from K' and $K_,$ are then determined by a pair of compasses. The remaining median edges $a\beta$, $\gamma\delta$, &c., are then easily drawn and will be found to pass through M, $M_,$, &c. Finally the edges $L_,\beta$, $L_,\delta$, &c., are drawn and complete the trapezohedron.

Or we may determine the median edges $aM\beta$, &c., as follows. The line LH is produced to meet $L_,H'$ in s. Hs and $L_,H'$ are shown in Fig. 211 by lines of interrupted strokes. The line Ms gives the direction of the edge $aM\beta$. If the direction La is known, a is determined; and $M\beta = Ma$. If $\gamma H\beta$ is known, β is determined; and a obtained by a pair of compasses. The homologous edges $M_,\delta$, &c., are obtained by a similar construction.

The student will be guided in his selection of the method he pursues by considerations of the accuracy attainable, in any particular case, in the determination of the points N and s.

52. The complementary form $a\{khl\}$ has all its faces, and, therefore, all its edges, parallel respectively to those of $a\{hkl\}$. Hence,

Fig. 213 showing $a\{khl\}$ is immediately made by drawing through K' a line parallel to $\delta\epsilon$ of Fig. 211 and through $K_{,}$ a line parallel to aK'. Similarly, the edges through H and $H_{,}$ are reciprocally interchanged. Again the polar edges through L in Fig. 213 are parallel to those through $L_{,}$ in Fig. 211, and vice versâ. The remaining median edges through M are then easily drawn.

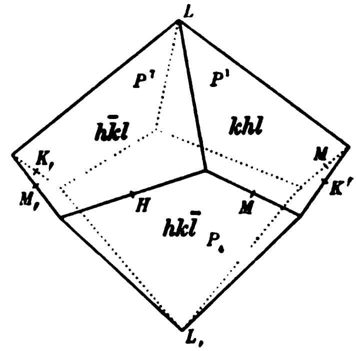

FIG. 213.

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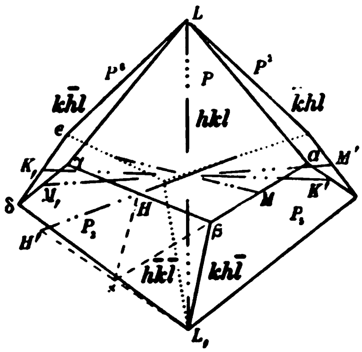
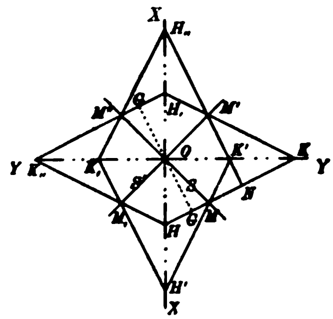


FIG. 211.



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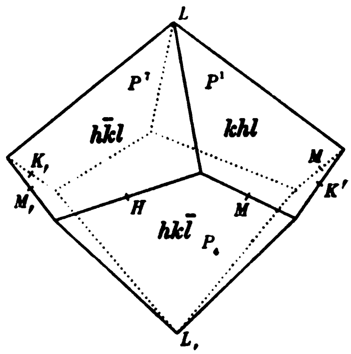

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Guanidine carbonate, $(CH_5N_3)_2H_2CO_3$. $c = {\cdot}9910$. The crystals are bipyramids {111} with the forms {001}, {100} slightly developed: small dull faces of a trapezohedron have been observed, but the symbols could

not be determined. The double refraction is weak, and negative. The rotation for a plate 1 mm. thick is 12° 35′ (Li), 14° 34′ (Na), 17° 4′ (Tl) (Bodewig, *Pogg. Ann.* CLVII, p. 122, 1876.).

Diacetyl-phenolphtaleïn, $C_{20}H_{12}O_4(C_2H_3O)_2$. $c = 1{\cdot}3593$. The crystals are optically negative, and rotate the plane of polarization sometimes to the right, sometimes to the left: the rotation for 1 mm. is 17·1° (Li), 19·7° (Na), 23·8° (Tl).

Strychnine sulphate, $(C_{21}H_{22}N_2O_2)_2 . H_2SO_4 . 6H_2O$. $c = 3{\cdot}312$. The crystals are combinations of {111} with {001}. There is a good cleavage parallel to (001); and cleavage plates, etched by dilute hydrochloric acid, show fine rectangular striæ, inclined to the edges of the plate on opposite sides of it, in directions which conform to symmetry with respect to a dyad axis perpendicular to the edges of the plate, but not to a centre of symmetry. The crystals are negative and lævogyral; and the rotation for 1 mm. is about 9° for red light.

VI. *Acleistous ditetragonal class*; $\mu\{hkl\}$.

55. Crystals of this class have a tetrad axis and four planes of symmetry intersecting in it, but no other element of symmetry. The planes of symmetry consist of two pairs S and Σ of like planes; for each pair of like planes are at 90° to one another and change places when the crystal is turned through 90° about the tetrad axis. The planes of one pair, also, bisect the angles between the other pair. Geometrically, the forms of this class consist of those planes which in the corresponding forms of class II meet at only one apex on the principal axis. From this point of view the planes meeting the axis at the opposite apex form a tautomorphous complementary form. Forms with faces parallel to the principal axis, *i.e.* prisms, will be identical with those of class II. The tetrad axis is uniterminal and is an axis of pyro- and piezo-electric polarity; and the crystals are hemimorphic.

56. The possible zone-axes perpendicular to Σ and Σ' can be taken as the axes OX and OY; and, as was shown in class I, the parameters on them are equal, for the angles between them are bisected by the planes S and S'. The only parameter, which varies with the substance, is c measured on the principal axis. It is determined in the manner given in Art. 8.

57. The prisms {110}, {100}, $\{hk0\}$ have clearly the same faces as the similarly placed prisms of classes I and II, and need no Greek prefix: they, respectively, comprise the faces given in tables **b**, **c** and **d**.

The other forms possible are the following:

1. Pedions; of which $\mu\{001\}$ is one; the complementary pedion being $\mu\{00\bar{1}\}$.

2. Acleistous tetragonal pyramids:

$$\mu\{h0l\} \text{ consisting of } h0l,\ 0hl,\ \bar{h}0l,\ 0\bar{h}l;$$
$$\mu\{hhl\} \quad ,, \quad ,, \quad hhl,\ \bar{h}hl,\ \bar{h}\bar{h}l,\ h\bar{h}l.$$

These pyramids have the same faces as those denoted by $\tau\{h0l\}$ and $\tau\{hhl\}$ in class III. The simple possible crystal represented in Fig. 206 may belong both to III and to this class.

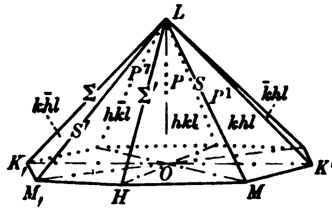


Fig. 214.

3. *Acleistous ditetragonal pyramids* like Fig. 214 of which the general symbol is $\mu\{hkl\}$. The form comprises the faces:

$$hkl,\ khl,\ \bar{k}hl,\ \bar{h}kl,\ \bar{h}\bar{k}l,\ \bar{k}\bar{h}l,\ k\bar{h}l,\ h\bar{k}l \ldots\ldots\ldots\ldots\ldots (\mathfrak{a}).$$

None of the forms are closed, and the crystals must in all cases be combinations of two or more forms. The class will be called the *acleistous ditetragonal* class.

58. Crystals of silver fluoride, $AgF \cdot H_2O$; of succin-iodimide, $C_4H_4O_2NI$; and of penta-erythrite, $C_5H_{12}O_4$, belong to this class.

A crystal of *succin-iodimide* having the forms $m\{110\}$, $o=\mu\{111\}$, $v=\mu\{221\}$ and $v'=\mu\{22\bar{1}\}$ is shown in Fig. 215. The element c given by Professor Groth is $\cdot 8733$. Hence, by equation (6), $Co=51°\,0'$, $mo=39°\,0'$; and $mv=22°\,2{\cdot}5'$. The more obtusely terminated end is the antilogous pole, the more acutely terminated end the analogous pole. This is indicated by the + and − signs which give the electrifications with falling temperature.

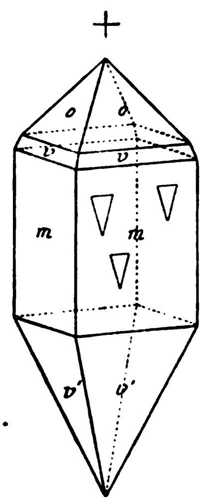


Fig. 215.

Crystals are sometimes obtained which, as the temperature falls, manifest negative electrifications at the opposite ends of the principal axis, so that both ends are analogous poles. This peculiarity is explained by supposing the crystal to be a composite or twinned crystal (Chap. XVIII) in which two similar halves are joined along a pedion. Corrosion-experiments have confirmed this view; for, whilst normal untwinned crystals give on the faces $m\{110\}$ a single series of pits having the shape of isosceles

triangles with their apices all directed towards the analogous pole, as shown by the three triangles drawn on m in Fig. 215, the crystals having analogous poles at opposite ends of a face m, give two series of triangular pits with apices directed respectively towards the opposite ends.

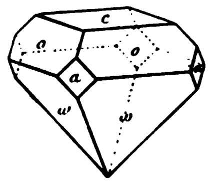


Fig. 216.

Fig. 216 shows a crystal of *penta-erythrite* having the forms: $a\,\{100\}$, $o=\mu\,\{111\}$, $\omega=\mu\,\{11\bar{1}\}$, $c=\mu\,\{001\}$. The parameter $c = 1{\cdot}0236$. The crystals are optically negative, and often show anomalous optic phenomena.

VII. *Sphenoidal class;* $\tau_\sigma\,\{hkl\}$.

59. Attention has been called to the possibility of a class of crystals in which every form is a tetragonal sphenoid, save when the faces are parallel or perpendicular to the principal axis (though no crystal of the kind has ever been found). Such a form, placed

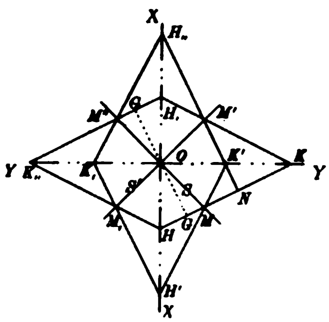


Fig. 217.

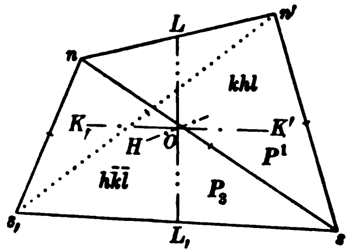


Fig. 218.[1]

in a general azimuth, may be represented by Fig. 218, and may be obtained by drawing through the opposite sides $H'K'$, $H_{,,}K_{,}$ of Fig. 217 a pair of faces to meet the principal axis at L, where $OL = c \div l$. The two faces have the symbols (khl), $(\bar{k}\bar{h}l)$; and are interchangeable by a semi-revolution about OZ. They are associated with a second pair of faces which pass through the parallel sides $HK_{,,}$ and $H_{,}K$ of the ditetragon of Fig. 217, and

[1] The sphenoid $\tau_\sigma\,\{khl\}$ has been drawn in preference to $\tau_\sigma\,\{hkl\}$ because the figure gives the geometrical relations of the sphenoid more clearly than a drawing of the latter. The more distant points H', $K_{,,}$, &c., of Fig. 217 are not shown in Fig 218: the positions of M, M', &c., are indicated by short strokes bisecting ns, $n's$, &c.

meet OZ at $L_{,}$, where $OL_{,} = c \div \bar{l}$: the new faces have therefore the symbols $(h\bar{k}\bar{l})$, $(\bar{h}k\bar{l})$. But the lines $H'K'$, $H_{,}K$, $H_{,,}K_{,}$, $HK_{,,}$ are alternate sides of the ditetragon, and are, when taken in succession, at 90° to one another (see Art. 7). The figure formed by these sides in the plane XOY is therefore a square. But the edge nn' is parallel to $H'K'$, and $ss_{,}$ to $HK_{,,}$. These two edges are therefore perpendicular to the principal axis and are at right angles to one another. Hence the form $\tau_\sigma\{khl\}$ is a sphenoid similar to that described in Art. 9, but it is now placed in a general azimuth with respect to the axial planes; it consists of the four faces:

$$khl,\ \bar{h}k\bar{l},\ \bar{k}\bar{h}l,\ h\bar{k}\bar{l}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(r)}.$$

Again, the plane $Lss_{,}$ is at right angles to the edge nn', and the line Ls in the face (khl) must therefore be at 90° to nn'; but by a semi-revolution about OZ, n and n' are interchanged, and $Ln = Ln'$. Hence the triangles sLn, sLn' are equal; for two sides nL, Ls of the one are equal to two sides $n'L$, Ls of the other, and the included angles are equal. Therefore $sn = sn'$; and the face snn' is an isosceles triangle. Similarly, $ns = ns_{,}$; and $n's = n's'$. The faces are therefore equal isosceles triangles.

It follows that any particular sphenoid is divided symmetrically by the two perpendicular planes which pass through the principal axis and the horizontal edges, such as nn' and $ss_{,}$, in the same way that the sphenoid of Art. 9 is divided by the planes of symmetry S and S'. But if a crystal has several sphenoids in different azimuths, as, for instance, the three sphenoids $\tau_\sigma\{101\}$, $\tau_\sigma\{111\}$, $\tau_\sigma\{hkl\}$ of Fig. 219, then the pair of vertical planes which bisect the faces of one of them will not divide the others symmetrically; and the crystal has no planes of symmetry.

60. Of the sphenoids possible on a crystal of this class one is selected to give by its horizontal edges the axes of X and Y; and as the faces meet the horizontal plane through the middle point of the principal axis in a square it follows that equal parameters can be taken on the axes of X and Y. This can also be proved from the relations of the tetragonal prisms which we shall show can occur on such crystals.

The sphenoids of the series taken to give the axes of X and Y will have the general symbol $\tau_\sigma\{h0l\}$, the faces of which are:

$$h0l,\ 0h\bar{l},\ \bar{h}0l,\ 0\bar{h}\bar{l}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(s)}.$$

One of the sphenoids of this series is selected to give the parametral ratio $c : a$, and its symbol is $\tau_\sigma\{101\}$. Measurement of the angle over one of the horizontal edges gives

$$101 \wedge \bar{1}01 = 2Ce = 2E,$$

if, as in Fig. 219, e is the pole (101).

Hence, as in (2) of Art. 8,

$$c \div a = \tan E.$$

If n is the face $(h0l)$ of a sphenoid $\tau_\sigma\{h0l\}$, then

$$\tan Cn = \frac{h}{l} \tan E.$$

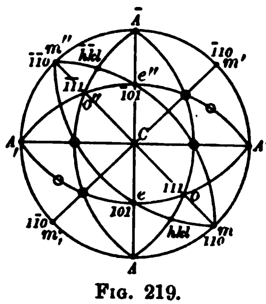


FIG. 219.

This latter sphenoid can be easily derived from the pyramid $\{h0l\}$ shown in Fig. 220 by drawing through L an edge parallel to $MM_{,}$ and through $L_{,}$ an edge parallel to MM'. Again, through M an edge is drawn parallel to $LM_{,}$, for the edge $[h0l,\ 0h\bar{l}]$ is parallel to $[h0l,\ 0\bar{h}l]$. Similarly, through $M_{,}$ an edge is drawn parallel to LM. Let q be the point at which these new edges meet the horizontal line through $L_{,}$; then it is clear that the triangle $MqM_{,}$ is isosceles, for the plane figure $LMqM_{,}$ is a rhombus. Hence the face $(h0l)$ of the sphenoid is an isosceles triangle having its sides double those of $MqM_{,}$. The same is true of the other faces of the sphenoid $\tau_\sigma\{h0l\}$.

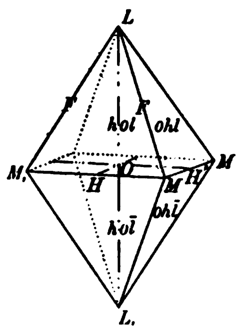


FIG. 220.

61. Again, since the edge Mq of the sphenoid $\tau_\sigma 0l\}\ h\}$ is parallel to $LM_{,}$ of Fig. 220, the face through Mq parallel to the principal axis OZ is a possible face and is parallel to the plane through $LOM_{,}$. But this latter plane passes through $M_{,}M'$, and the triangle $M_{,}MM'$ is an isosceles triangle having the angles at $M_{,}$ and M' each 45°. The trace $M_{,}M'$ is therefore inclined at 45° to the axes OX and OY. Hence the vertical face through Mq meets these axes at equal distances from the origin; and equal parameters can, as was stated in Art. 60, be taken on the axes of X and Y: the face is then (110). But through each of the other slanting edges of the sphenoid $\tau_\sigma\{h0l\}$, a similar vertical face can be drawn, which truncates the edge and meets the axes at equal distances from the origin. We

thus have a rectangular prism {110} similar to that obtained in each of the preceding classes, and comprising the same faces: the poles are shown in Fig. 219.

The vertical faces through the sides of the square, in which the sphenoid τ_σ {101} or τ_σ {$h0l$} meets the axial plane XOY, compose a second tetragonal prism {100}, the poles of which are at A, A', &c., of Fig. 219. This prism is also similar to that obtained in all the other classes, and comprises the same faces.

The face parallel to the pair of horizontal edges of any sphenoid is a possible face; and since no distinction has, in the definition of the forms of the class, been made between opposite ends of the principal axis, we must have the two faces of the pinakoid {001}.

62. Again, a crystal may have sphenoids τ_σ {hhl} geometrically identical with κ {hhl} of class I; and only differing from the latter in that the two vertical planes through the principal axis which bisect the faces are no longer planes of symmetry. The vertical planes truncating the slanting edges of the sphenoids of this series compose the tetragonal prism {100}; the vertical faces parallel to the planes bisecting the faces of the sphenoid compose the prism {110}.

63. It is clear that from every sphenoid two tetragonal prisms can be derived: the one is that formed by the vertical faces through the sides of the square in which the sphenoid meets the plane XOY; these faces are parallel to the vertical planes drawn through each of the horizontal edges bisecting pairs of the faces of the sphenoid. The other is that formed by the vertical faces which truncate the slanting edges of the sphenoid; the faces of this latter prism are parallel to the diagonals of the square formed by the former prism. If the symbol of the sphenoid is given, those of the two prisms can be easily determined. Thus, taking τ_σ {khl}, the former prism is τ_σ {$kh0$} and comprises the faces: $kh0$, $\bar{h}k0$, $\bar{k}\bar{h}0$, $h\bar{k}0$. The symbol of the second prism can be obtained (i) by Weiss's zone-law; or (ii) from the fact that its faces are inclined to those of τ_σ {$kh0$} at angles of 45°. Let the second prism have the symbol τ_σ {$qp0$}.

i. If the face ($qp0$) truncates the edge ns of Fig. 218, it lies in the zone [khl, $h\bar{k}\bar{l}$]. Hence by Weiss's zone-law,

$$q(h-k)-p(h+k)=0—$$

an equation which gives the ratio of $q:p$.

ii. Let OF be the normal to $(qp0)$, and OG' be the normal to $(kh0)$. Then $XOG' = 45° + XOF$. But, by equation (1),

$$\tan XOF = p \div q;$$

and
$$\tan XOG' = h \div k.$$

$$\therefore \frac{p}{q} = \tan(XOG' - 45°) = \frac{\tan XOG' - \tan 45°}{1 + \tan 45° \tan XOG'}.$$

If $\tan XOG'$ is replaced by $h \div k$ and $\tan 45°$ by 1, we have

$$\frac{p}{q} = \frac{h \div k - 1}{1 + h \div k} = \frac{h - k}{h + k};$$

which is the same result as before.

A zone, in which a face inclined at 45° to any face of the zone is possible, will be called *tetragonal* (Chap. IX, Art. 12).

64. Some crystallographers give the relations of the faces of the sphenoid $\tau_\sigma\{khl\}$ in the following manner. A face (khl) of the form is taken and rotated about the principal axis OZ through 90°. The transposed face now occupies a position which is the reflexion in a mirror, situated in XOY, of a second face of the form. This face must therefore slope in a direction opposite to that of the first face so as to meet OZ at $L_{,}$. The second face meets the plane XOY in $HK_{,,}$ of Fig. 217; and its symbol is $(h\bar{k}\bar{l})$. On a second rotation of 90° about OZ in the same direction the same operation is repeated: the face $(h\bar{k}\bar{l})$ being, after rotation, reflected in the plane XOY, the homologous face $(\bar{k}\bar{h}l)$ must meet the principal axis in the same point L as the first face. The joint result of the two operations is equivalent to a simple rotation of 180° about OZ. When the operation is, again, repeated, a face $(\bar{h}k\bar{l})$ meeting OZ at $L_{,}$ is obtained; and a third repetition of the operation brings the face to its original position. The lines $H'K'$, $HK_{,,}$, &c., in which the transposed face meets the plane XOY after each rotation, necessarily form a square.

The special forms derived from this principle are: (*1*) a pinakoid $\{001\}$, (*2*) tetragonal prisms $\tau_\sigma\{kh0\}$, &c., such as have already been described.

CHAPTER XV.

THE CUBIC SYSTEM.

1. IN Chapter x, the cubic system was stated to include all crystals having four triad axes, the directions of which are given by the diagonals of a cube: they are therefore similarly situated with respect to three rectangular axes, parallel to the edges of the cube, which are either tetrad or dyad axes. The crystals fall into the five following classes:

I. The *plagihedral* (*pentagonal-icositetrahedral*) class, in which the rectangular axes are tetrad axes, and the opposite ends of the triad axes are similar and interchangeable. These seven axes involve the presence of six like dyad axes which are parallel to the face-diagonals of the cube of which the triad axes are the diagonals.

II. The *hexakis-octahedral* (*diplohedral ditrigonal*, *holohedral*) class, the crystals of which have a centre of symmetry associated with the thirteen axes of symmetry described under class I. It follows that the crystals have also nine planes of symmetry—three *cubic planes*, each perpendicular to a tetrad axis; and six *dodecahedral planes*, each perpendicular to one of the dyad axes.

III. The *tetrahedral* (*polar trigonal*, *tetrahedral-pentagonal-dodecahedral*, *tetartohedral*) class, in which the rectangular axes, parallel to the edges of the cube, are dyad axes, and the triad axes are uniterminal and hemimorphic. The crystals have no other element of symmetry.

IV. The *dyakis-dodecahedral* (*diplohedral trigonal*, *parallel-faced hemihedral*) class, in which the rectangular axes are dyad axes and are associated with a centre of symmetry and with three planes of symmetry, each perpendicular to one of the dyad axes: the distribution of faces about opposite ends of each triad axis is similar, but the ends are not interchangeable.

V. The ***hexakis-tetrahedral*** (*polar ditrigonal*, *hemihedral with inclined faces*) class, in which the three dyad and four triad axes of symmetry characteristic of class III, are associated with six dodecahedral planes of symmetry intersecting in sets of three in each of the triad axes and in pairs in each of the dyad axes.

2. We shall begin with the two classes of crystals having three tetrad axes at right angles to one another. These axes are necessarily like and interchangeable; for rotation through 90° about any one of them interchanges the two others, and therefore any faces similarly situated with respect to each of them.

In Chap. IX, Prop. 12, ii (*c*), it was shown that the above is the only possible arrangement of several tetrad axes; but it is easy to establish the proposition independently. For assume two tetrad axes to be inclined to one another at an angle $a < 90°$, and let their directions be given by the radii of a sphere emerging at T and T_1 of Fig. 221: further, let a be the least angle possible between any pair of such axes. If the sphere and crystal—supposed to be rigidly connected together—are turned about the diameter T through 90°, then T_1 is brought to T_2 on the great circle CT, where $CTT_1 = 90°$, and $\wedge\, TT_2 = \wedge\, TT_1$. Then the diameter through T_2 must be the direction of a tetrad axis similar to that emerging at T_1. Similarly, by a rotation of 90° about the diameter through T_1, T is brought to T_3 on the great circle CT_1; and the diameter through T_3 must be the direction of a tetrad axis similar to that emerging at T: also $\wedge\, T_3T_1 = \wedge\, TT_1$. But the point C, in which the great circles CT, CT_1 intersect, is the pole of the great circle TT_1, for the angles at T and T_1 are right angles. It follows, therefore, that T_2T_3 is less than TT_1. This contravenes the limitation imposed on a, viz. that it is the least angle between tetrad axes. It is also clear that, if T_2 and T_3 are taken as the initial pair of axes, we can in a similar manner find new tetrad axes inclined to one another at a still smaller angle; and that the process can be continued indefinitely. But in a crystalline substance, bounded by planes inclined to one another at finite angles, there cannot be an infinite number of tetrad axes making indefinitely small angles with one another. Hence a cannot be less than 90°.

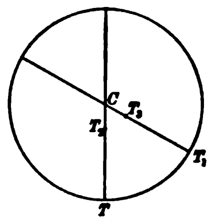


FIG. 221.

If, however, T and T_1 are 90° from one another, the points T_2

and T_2 of Fig. 221 obtained by a rotation of 90° about T and T_1 respectively coincide in the point C, the pole of the great circle TT_1. There can therefore be three tetrad axes mutually at right angles to one another; and, as a rotation of 90° about any one of them interchanges the remaining two, they are all like and interchangeable tetrad axes; *i.e.* the distribution and arrangement of faces about each of them are similar, and the faces at each end of the axes are interchangeable.

3. It was shown in Chap. IX, Prop. 12, ii (*c*), that the three tetrad axes are associated with four like triad axes, which are the diagonals of the cube having its edges parallel to the three tetrad axes. This proposition can also be established independently.

a. Let, in Fig. 222, the three tetrad axes meet the sphere at the points T, T', T'', and construct according to Euler's theorem (Chap. IX, Prop. 7), the triangle $T\rho T'$, where

$$\wedge\ \rho TT' = \wedge \rho T'T = 45°.$$

Then ρ is the extremity of a diameter, rotation about which is equivalent to successive rotations of 90° about the diameters through T and T'. The spherical triangle $T\rho T'$ is isosceles; and the great circle $T''\rho\delta''$ bisects the side TT', and meets it at 90° in the point δ''.

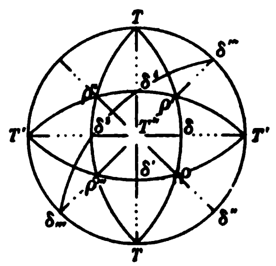


FIG. 222.

Hence in the right-angled spherical triangle $T\rho\delta''$, we have

$$\cos T\rho\delta'' = \sin \rho TT' \cos T\delta'' = \sin 45° \cos 45° = 1 \div 2$$

$$\therefore\ \wedge\ T\rho\delta'' = 60°, \text{ and } \wedge\ T\rho T' = 2T\rho\delta'' = 120°.$$

The external angle $\delta\rho T'$ of the triangle at ρ is 60°, and the least angle of rotation about the diameter through ρ is 120°: the diameter is therefore a triad axis.

The same point ρ and the same angle of rotation are obtained, if successive rotations of 90° clockwise about the diameters through T'' and T' are taken. Hence all the angles at ρ are 60°; and $\wedge\ T\rho = \wedge\ T'\rho = \wedge\ T''\rho$. The diameter through ρ is therefore a diagonal of the cube having its edges parallel to the tetrad axes.

Similarly, the points ρ', ρ'', ρ''' in each of the remaining octants above the paper are the extremities of diameters which are triad axes similar to, and interchangeable with, $O\rho$: they also are diagonals of the cube.

β. We can also prove the proposition in the following way. In Fig. 223 (*a*), the sphere, with the crystal rigidly attached to it, is in the initial position; and the tetrad axes emerge at T, T' and T'''. The sphere is then turned through 90° about the diameter through T''' in the direction of the arrow. The point T is transposed to T', and T' to $\overline{T}$, when the sphere is given by Fig. 223 (*b*).

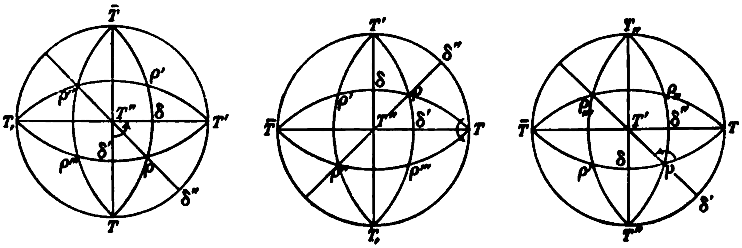


Fig. 223 (*a*). Fig. 223 (*b*) Fig. 223 (*c*).

The sphere is now turned through 90° about the right-and-left diameter, which has its extremity at T of Fig. 223 (*b*); the direction of rotation being indicated by the arrow at T. The positions of the several axes after this second rotation are shown in Fig. 223 (*c*); and are exactly the same as if the crystal had been turned once through 120° in the direction of the arrow about the diameter through ρ, which is equally inclined to the diameters through T, T' and T'''.

In the same way it may be shown that the diameters through ρ', ρ'', ρ''' of Fig. 223 (*a*) are also triad axes equally inclined to the adjacent tetrad axes. It is also clear that successive rotations of 90° about the diameter through T''' brings the axis ρ into the position of each of the other similar triad axes ρ', ρ'', ρ'''.

4. Successive rotations about a tetrad axis T and a triad axis ρ, Fig. 223 (*a*), are, by Euler's theorem, equivalent to a single rotation of 180° about an axis emerging at δ'', the apex of the triangle $\rho T\delta''$. For the angle $T\rho\delta'' = 60°$, and $\wedge\ \rho T\delta'' = 45°$, each being one-half the angle of rotation about the respective axis: the angle at δ'' is also a right angle and the equivalent angle of rotation about $O\delta''$ is 180°. Hence δ'' is the extremity of a dyad axis.

Similarly, the other points δ, δ', &c., bisecting each of the quadrants $T'T'''$, $T'''T$, &c., are also the extremities of similar dyad axes interchangeable with δ'' and each other.

5. We have therefore thirteen axes of symmetry—three tetrad, four triad, and six dyad axes—which meet the sphere in the points T, ρ and δ of Fig. 222, respectively; and which are also shown in Fig. 224. In this latter diagram the tetrad axes are parallel to the

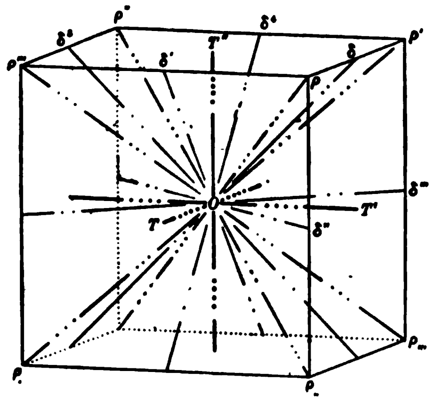


FIG. 224.

edges of the cube, and are indicated by lines of strokes and four dots; the triad axes are the diagonals of the cube, and are indicated by lines of strokes and three dots; and the dyad axes are parallel to the face-diagonals, and are given by lines of strokes and two dots.

It is now necessary to prove that successive rotations about any pair of these axes give rise to no new axes. The combinations which remain to be examined are those in which pairs of dyad axes, pairs of triad axes, or a dyad and triad axis together or each with a tetrad axis, are those of successive rotation.

6. If two dyad axes are taken as those of successive rotations, we have two cases to consider: (i) that in which the plane of the dyad axes contains a pair of tetrad axes; (ii) that in which their plane does not contain a tetrad axis.

i. Suppose the pair to be δ'', δ''', those in the plane of the primitive of Fig. 225. Then Euler's construction gives the triangle $\delta''T''\delta'''$; and OT'' is the equivalent axis of rotation. But, since the angle $\delta''T''\delta''' = 90°$, the external angle at T'' is also $90°$; and the angle of rotation about OT'' is $180°$. Hence successive rotations of $180°$ about the pair of dyad axes are equivalent to a single rotation of $180°$ about the tetrad axis OT''. No new axis, and no new rotation has been introduced; for $180° = 2 \times 90°$ is made up of two rotations of $90°$ about the tetrad axis.

ii. Suppose δ^4 and δ^5 of Fig. 225 to be the pair of dyad axes. Now

δ^4 and δ' are in the same plane with T''' and at 45° from it: they are therefore at 90° from one another. The axis δ^4 is also at 90° from T': it is therefore at 90° from every point in the great circle $T'\rho\delta'$, and the arc $\rho\delta^4=90°$. Similarly, δ^5 is at 90° from both δ and T, and from every point in the great circle $T\rho\delta$: hence arc $\rho\delta^5=90°$. In the same way it can be proved that $\rho\delta'''=\rho\delta_{'''}=90°$. It follows that ρ is the pole of the great circle $\delta^4\delta^5$ which passes through $\delta'''\delta_{'''}$. Further, every great circle through ρ must meet the great circle $\delta'''\delta^4\delta^5$ at right angles. But to find the axis equivalent to successive rotations about the axes δ^4 and δ^5, we have, by Euler's theorem, to draw great circles through δ^4 and δ^5 at right angles to the great circle $\delta^4\delta^5$; these circles meet at ρ, which is, therefore, the extremity of the equivalent axis.

Again, from the right-angled triangle $\delta^4T'''\delta^5$,

$$\cos \delta^4\delta^5 = \cos T'''\delta^4 \cos T'''\delta^5 = \cos^2 45° = 1 \div 2.$$

$\therefore \wedge \delta^4\delta^5 = 60°$. Similarly,

$$\wedge \delta'''\delta^4 = \wedge \delta^5\delta_{'''} = 60°.$$

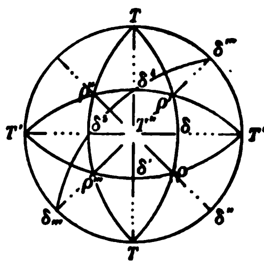


FIG. 225.

Hence, we have three dyad axes in the plane of the great circle $\delta'''\delta^4\delta^5$ inclined to one another at angles of 60°. By Chap. IX, Prop. 11, the diameter $O\rho$ perpendicular to their plane is a triad axis. Or we can establish this independently as follows. The arc $\delta^4\delta^5 = \wedge \delta^4\rho\delta^5$ subtended at the pole of the great circle $\delta^4\delta^5$. Hence, the external angle of the triangle $\delta^4\rho\delta^5$ at ρ is 120°. The angle of rotation about the equivalent axis $O\rho$ is therefore

$$2 \times 120° = 360° - 120°.$$

Hence, after rotations of 180° about $O\delta^4$ and $O\delta^5$, the crystal is in exactly the same position as if it had been turned through 120° about $O\rho$ in a direction opposite to that required by Euler's theorem. The least angle of rotation being 120°, the axis is a triad axis.

In the same way it may be proved that the directions of any other pair of dyad axes (not in a plane with a tetrad axis) are at 60° to one another and perpendicular to one or other of the triad axes. Hence, successive rotations about any pair are equivalent to a rotation of 120° about the triad axis perpendicular to both the dyad axes.

7. Successive rotations about a pair of triad axes give no new axis. For if ρ and ρ' are taken, the triangle formed according to Euler's theorem is $\rho T'''\rho'$, or $\rho T'\rho'$ according to the order and direction of the rotations. The angles $\rho T'''\rho'$ and $\rho T'\rho'$ are each 90°. Hence the two rotations are equivalent to a rotation of 180° about T''' or T'. These are the same as two rotations of 90° about each axis; and hence we have no new axis and no new rotation.

If the two axes ρ and ρ'' of Fig. 225 are taken, the triangle formed by Euler's construction is $\rho\rho'\rho''$ or $\rho\rho'''\rho''$ according to the order and direction of the first rotations. But the angles $\rho\rho'\rho''$, $\rho\rho'''\rho''$ are each 120°. The angle of rotation about the equivalent axis ρ' or ρ''' is therefore 120°, *i.e.* that about a triad axis.

8. Successive rotations about a tetrad axis T'' and an adjacent dyad axis δ are equivalent to a rotation of 240° about the adjacent triad axis ρ or ρ'. But this is the same as a rotation of 120° in the opposite direction about the same axis. Similarly, a combination of an adjacent ρ and δ gives one of the adjacent tetrad axes.

9. The only remaining combinations are those in which a tetrad or triad axis is combined with a dyad axis inclined to it at 90°.

Rotations about a tetrad axis T'' and a dyad axis δ'' are equivalent to one of 180° about T or T'. Hence we have no new axis or rotation.

Similarly, successive rotations about ρ and δ^4 are equivalent to a single rotation of 180° about δ''' or δ^6. For according to Euler's theorem we have to construct a spherical triangle $\rho\delta^4\delta'''$ or $\rho\delta^4\delta^6$, in which the angles at ρ are 60° and those at δ^4 are 90°. Hence, the apex is at δ''' or δ^6, and the angle of rotation is $2 \times \rho\delta'''\delta^4 = 2\rho\delta^6\delta^4 = 180°$.

10. The angles between the several axes of symmetry are clearly fixed and constant, for the triad and dyad axes are a necessary consequence of the coexistence of three tetrad axes, and it was proved that these latter must be at 90°. The angles can be all determined without difficulty. The triad axes $O\rho$, &c., are all equally inclined to each of the tetrad axes; hence, in Fig. 225, $T\rho = T'\rho = T''\rho$. Further, the planes containing each a tetrad and a triad axis, such as $T''\rho$, bisects the angle between the two other tetrad axes. Hence, $T\delta'' = \delta''T' = \&c. = 45°$. But the arc $T\delta''$ is equal to the angle $TT''\delta''$, for T'' is the pole of the great circle TT'. The angles $\rho T''T = \rho T''T' = \rho TT' = \&c. = 45°$.

Hence, from the right-angled triangle $T''\rho\delta$, we have

$$\sin T''\delta = \tan \rho\delta \cot \delta T''\rho.$$

But $\wedge T''\delta = \wedge \delta T''\rho = 45°$, and $\cot 45° = 1$.

$$\therefore \tan \rho\delta = \sin 45° \text{ ; and } \wedge \rho\delta = 35° \, 16'.$$

Hence $$\wedge \rho\rho' = 2\rho\delta = 70° \, 32' = \wedge \rho'\rho'' = \&c.$$

also $$\text{arc } T\rho = 90° - \rho\delta = 54° \, 44' = \rho T' = \rho T''.$$

$$\therefore \wedge \rho\rho'' = 2\rho T'' = 109° \, 28' = \wedge \rho'\rho''' = \&c.$$

It has already been shown (Art. 6, ii.) that

$$\wedge \delta^4\delta^6 = \wedge \delta'''\delta^4 = \&c. = 60°.$$

11. It is clear that each octant in Fig. 225 included between tetrad axes is made up of six equal triangles having a common apex at a point ρ, where the triad axis emerges. Rotation through 120° about $O\rho$ will necessarily interchange three of these triangles. Thus, taking the first octant $TT'T''$, we have the three equal triangles $T\rho\delta''$, $T'\rho\delta$, $T''\rho\delta'$, interchangeable by rotation about ρ. We have also the three similar equal triangles $T'\rho\delta''$, $T''\rho\delta$, $T\rho\delta'$ which are interchangeable with one another but not with any one of the first set.

If now the sphere is turned through 90° about a tetrad axis, such as T'', each of the above sets of three triangles is interchanged with a similar set in either of the adjacent octants. Further, if the sphere is turned through 180° about a dyad axis, $\delta_{,}$ say, lying in the plane dividing the octants, we must clearly interchange similar sets of three triangles. It is easily seen that the sets interchanged by the latter rotation are identical with the sets interchanged by a rotation of 90° about T'', although the individual triangles of the two sets which change places with one another are different.

Hence the sphere is divided into forty-eight equal triangles, each of which has its angles at one of the points T, ρ, δ, &c., respectively. These triangles are divisible into two sets of twenty-four, such that the triangles of one set are interchangeable with one another, but not with those of the other set.

12. We shall in each class of the system adopt as axes of reference the three axes parallel to the edges of the cube, for they are either dyad, or tetrad, axes which remain at right angles to one another at all temperatures.

When the axes of reference are tetrad axes, the lines bisecting the angles between each pair are dyad axes, and therefore necessarily possible zone-axes (Chap. IX, Prop. 2). When the axes of reference are only dyad axes, the face-diagonals of the cube are the directions of zone-axes but not of dyad axes: for each of them is the intersection of a face of the cube with the possible plane containing the two triad axes which join the extremities of the face-diagonal to the centre of the cube. But in Art. 6, ii, it was shown that the radii to δ''', δ^4, δ^5 all made 90° with that through ρ. Hence the diagonal $O\rho$ of the cube is perpendicular to the possible face parallel to the radii through δ''', δ^4 and δ^5: this face is always taken as the parametral plane (111). But by a rotation of 120° about $O\rho$ the axes of

reference are interchanged, whilst the direction of the plane (111) remains the same. Hence it must meet the axes at equal distances from the origin; and

$$a = b = c \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This may also be proved from the equations to the normal $O\rho$ of the parametral plane, which are

$$a \cos T\rho = b \cos T'\rho = c \cos T''\rho.$$

But it was shown that $\quad T\rho = T'\rho = T''\rho.$

Hence $$\cos T\rho = \cos T'\rho = \cos T''\rho.$$

$$\therefore\ a = b = c.$$

We shall denote the pole (100) by A, (010) by A' and (001) by A'': they coincide with T, T', T'' of Fig. 225, respectively.

The points ρ are the positions of the poles of the faces of the octahedron, which we shall denote by the letter o. Further, we shall denote by $a_{,}$ a length measured on OY equal to a on OX, and by $a_{,,}$ the same length measured on OZ.

13. It follows that in the cubic system no element of the crystal varies with the substance, and that the angles between faces with known symbols must be fixed and constant. This can also be proved from the equations of the normal; for, if P is the pole (hkl), the equations are

$$\frac{a \cos AP}{h} = \frac{a_{,} \cos A'P}{k} = \frac{a_{,,} \cos A''P}{l}.$$

But the equal lengths, a, $a_{,}$, $a_{,,}$ can be each taken as unity or cancelled, when the equations become

$$\frac{\cos AP}{h} = \frac{\cos A'P}{k} = \frac{\cos A''P}{l} = \frac{1}{\sqrt{h^2 + k^2 + l^2}} \quad\ldots\ldots\ldots (2).$$

The last term is obtained as follows. Let each of the first three terms $= t$; then

$$ht = \cos AP, \text{ and } h^2t^2 = \cos^2 AP;$$
$$kt = \cos A'P, \quad ,, \quad k^2t^2 = \cos^2 A'P;$$
$$lt = \cos A''P, \quad ,, \quad l^2t^2 = \cos^2 A''P.$$

Adding the squared equations together, we have

$$t^2 (h^2 + k^2 + l^2) = \cos^2 AP + \cos^2 A'P + \cos^2 A''P = 1;$$

for, the axes being rectangular, $\cos^2 AP + \cos^2 A'P + \cos^2 A''P = 1.$

$$\therefore\ t = \frac{1}{\sqrt{h^2 + k^2 + l^2}}.$$

It is clear therefore that the arcs AP, $A'P$, $A''P$ can be computed for any values of h, k and l introduced into equations (2). The angles between any two faces whether of the same or of different forms, must consequently be fixed and constant. For, taking $P\,(hkl)$ and $Q\,(pqr)$ to be the poles of any two faces, we have for P the equations (2), and for Q the similar equations:

$$\frac{\cos AQ}{p} = \frac{\cos A'Q}{q} = \frac{\cos A''Q}{r} = \frac{1}{\sqrt{p^2+q^2+r^2}}.$$

But the general expression for the angle between two poles, given in Chap. XIII, Art. 36, is in this system

$$\cos PQ = \cos AP \cos AQ + \cos A'P \cos A'Q + \cos A''P \cos A''Q.$$

Introducing the value of $\cos AP$, $\cos AQ$, &c., from the equations of the two normals, we have

$$\cos PQ = \frac{hp+kq+lr}{\sqrt{(h^2+k^2+l^2)(p^2+q^2+r^2)}} \quad\text{.............(3).}$$

It is therefore only necessary to introduce on the right side of (3) the numerical values of the face-indices to obtain the cosine of the angle between any pair of faces.

From equations (2) and (3) all the angles corresponding to any particular cases can be deduced, as will be shown in later articles.

I. *Plagihedral class;* $\alpha\{hkl\}$.

14. The class having three tetrad axes, four triad axes and six dyad axes, but no other element of symmetry, we shall call the *plagihedral* class of the cubic system; and we shall use the symbol

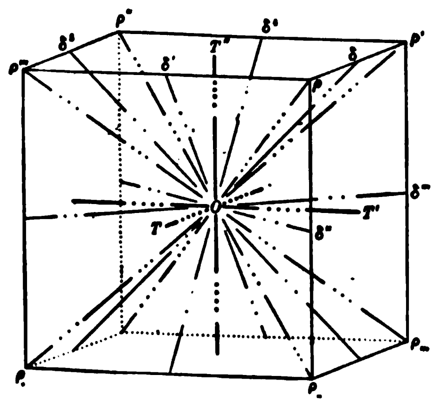


Fig. 226

$\alpha\{hkl\}$ to denote the general form which is a pentagonal icositetrahedron. This form is the only one which is peculiar to the class, and the Greek prefix will be omitted before the symbols of the special forms which are common to this and the next class.

15. *The cube*, {100}. The simplest form is that in which each face is parallel to two of the tetrad axes and therefore perpendicular to the third. If the vertical face (100) is turned through 90° three times successively about OT'' of Fig. 226, it is brought into the position of three other faces which have the symbols (010), ($\bar{1}00$), ($0\bar{1}0$), respectively. By similar rotations of 90° about OT', it is brought to the positions in which the faces have the symbols (001), ($\bar{1}00$), ($00\bar{1}$), respectively.

We have already seen (Art. 3) that successive rotations about OT' and OT'' are equivalent to a single rotation about one of the triad axes. Hence no other faces can belong to the form, which may be denoted by the symbol {100}, and includes the six faces:

$$100 \quad 010 \quad 001 \quad \bar{1}00 \quad 0\bar{1}0 \quad 00\bar{1} \ldots\ldots\ldots\ldots\ldots(\text{a}).$$

The edges, parallel to the tetrad axes, meet in sets of three at coigns ρ on one or other of the triad axes, and are each bisected at right angles by one of the dyad axes. In theoretical discussions the faces are, except when the contrary is expressly stated, supposed to be of equal dimensions. Such an ideal figure is shown in Fig. 226: since it serves as the basis for the construction of several important forms of the system, we shall, for convenience, denote its faces, edges and coigns as *cubic faces*, *cubic edges* and *cubic coigns*, respectively. The method of drawing the cube was described in Chap. VI, Arts. 9–12 and 22, 23. The faces are usually denoted by the letter a; and the poles by A, A', A'', &c.

16. *The octahedron*, {111}. The form {111}, of which the parametral plane is a face, is a regular octahedron, Fig. 227, with eight equal faces, each of which is an equilateral triangle:

$$\left.\begin{matrix} 111 & 1\bar{1}1 & \bar{1}\bar{1}1 & \bar{1}11 \\ 11\bar{1} & 1\bar{1}\bar{1} & \bar{1}\bar{1}\bar{1} & \bar{1}1\bar{1} \end{matrix}\right\} \ldots\ldots(\text{b}).$$

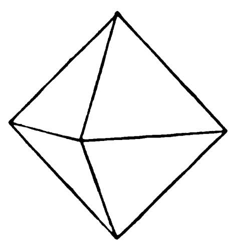


Fig. 227.

The coigns are at the points T, T', T'', &c., in Fig. 226 where the tetrad axes meet the cubic faces, so that the figure is readily drawn, when the cube has been projected by

either of the methods given in Chap. VI. The triad axes are perpendicular to pairs of parallel faces, and pass through their middle points; and the dyad axes are parallel to the edges. When the form is equably developed, each face is an equilateral triangle. In Nature the faces are apt, owing to the accidents attending the deposition of the crystalline matter, to be very unequally developed, as is shown in Fig. 228. The angles over the edges of adjacent faces of the regular figure are clearly all equal to the least angle between the triad axes, i.e. to 70° 32′; for it was proved in Art. 10 that $\rho\rho' = \rho'\rho'' = \&c. = 70° 32'$.

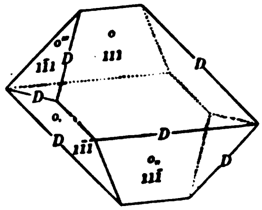


FIG. 228.

This angle can also be found from the equations (2) of the normal, which become for $o\,(111)$:

$$\cos Ao = \cos A'o = \cos A''o = 1 \div \sqrt{3} \ldots\ldots\ldots\ldots\ldots (4).$$

$$\therefore\ \tan^2 Ao = \sec^2 Ao - 1 = 3 - 1 = 2$$

$$\therefore\ \tan Ao = \sqrt{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

$$\therefore \wedge Ao = 54° 44', \text{ and } \wedge oo' = 180° - 2Ao = 70° 32'.$$

17. *The rhombic dodecahedron*, $\{110\}$. Another important form is that which has each of its faces perpendicular to one of the dyad axes. Each face is therefore parallel to two triad axes; for it was proved in Art. 6 that δ^4 of Fig. 225 is at 90° from ρ, and, similarly, it is at 90° from ρ'''. The face of which δ^4 is the pole is also parallel to the tetrad axis OT' and the dyad axis $O\delta'$, for the arcs δ^4T' and $\delta^4\delta'$ are also 90°: the face is equally inclined to the two other tetrad axes, for $\wedge\delta^4\bar{T} = \wedge\delta^4T'''$; it therefore makes equal intercepts on these axes of reference, and its symbol is $(\bar{1}01)$. The same relations hold for the faces perpendicular to each of the other dyad axes, and the form must consist of twelve similar and interchangeable faces, pairs of which are parallel, for they are parallel to both a tetrad and dyad axis.

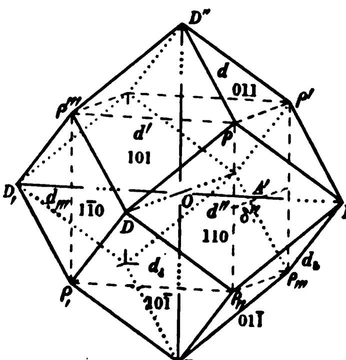


FIG. 229.

The form, Fig. 229, is most easily drawn by first constructing the cube, and taking the axes OA, OA', &c., through its middle point. On these axes points D, D', &c., are marked off; where $OD = 2OA$, $OD' = 2OA'$, &c. These points are then joined to the four adjacent cubic coigns ρ; and give the edges of the form. In the diagram the front edges of the cube are shown by lines of interrupted strokes. It is clear that a face, such as $D\rho D'\rho_{,,}$ contains the cubic edge $\rho\rho_{,,}$, parallel to the tetrad axis OD''; and that DD' lies in the face and bisects the line $\rho\rho_{,,}$ at δ''. For

$$A'D' = OA' = OD' \div 2, \text{ and } A'\delta'' = \rho\rho' \div 2 = OD \div 2.$$

Hence $$A'D' : A'\delta'' = OD' : OD.$$

The point δ'' therefore lies on DD' and bisects it.

Also, since OD is equal and parallel to $\rho\rho'$, the plane figure $OD\rho\rho'$ is a parallelogram. Therefore $D\rho$ is equal and parallel to $O\rho'$. The edges of the rhombic dodecahedron are therefore equal and parallel to the triad axes joining the centre to the coigns of the cube.

The form includes the twelve faces:

$$\left.\begin{matrix} 110 & \bar{1}10 & \bar{1}\bar{1}0 & 1\bar{1}0 \\ 011 & 01\bar{1} & 0\bar{1}1 & 0\bar{1}\bar{1} \\ 101 & \bar{1}0\bar{1} & \bar{1}01 & 10\bar{1} \end{matrix}\right\} \text{..................} \text{ (c)}.$$

The faces having their symbols in a row are parallel to and interchangeable about the tetrad axes OZ, OX and OY, respectively. Those which have their symbols in a column meet at the alternate coigns ρ, $\rho_{,,,}$, ρ'', $\rho_{,}$, respectively. Again, the angles between adjacent faces which meet in edges, such as $D\rho$, parallel to the triad axis are each of them 60° (Art. 6); the angles between alternate faces meeting at tetragonal coigns, D, D', &c., are each of them 90°.

In Nature the faces are often very unequally developed, and the crystals then sometimes simulate tetragonal crystals, as in Fig. 230, when four of the faces parallel to one of the tetrad axis are very largely developed in comparison with the remaining eight. In other cases they simulate rhombohedral crystals. The physical character of the faces will often enable the student to see that they are all homologous faces; and measurement on the goniometer will establish the fact that the only angles are 60° and 90°.

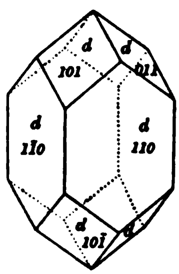


FIG. 230.

18. When the faces are parallel to only one of the dyad, or tetrad, axes we have special forms of three different types.

The tetrakis-hexahedron, $\{hk0\}$. Suppose a face of the form to be parallel to a tetrad axis, but not to any other axis of symmetry. As this is an axis of reference the corresponding index must be zero; and the face meets the two other tetrad axes at unequal but finite distances from the origin, or it will be parallel to a second tetrad axis, when it is a face of the cube; or to a dyad axis, when it is a face of the rhombic dodecahedron. Hence, one of the faces parallel to OA'' (say) is $(hk0)$; and the form $\{hk0\}$, Fig. 231, has twenty-four faces each of which is a similar isosceles triangle. On referring to Chap. XIV, Arts. 7, 22, 48, and 57, the reader will see that eight faces parallel to the principal (tetrad) axis and passing through the lines HK, &c., of Fig. 177 are associated together. The same relation holds in the form now under discussion, so that Fig. 177 applies also to this case. The difference in the form is due to the triad axes, which interchange the tetrad axis OZ with each of the axes OX and OY at right angles to it and to one another. Hence the eight faces parallel to OZ drawn through the lines HK, &c., of Fig. 177 are associated with two similar sets of eight faces parallel respectively to the axis OX and OY; the symbols of the two new sets of faces being given by constructing similar and equal ditetragons in each of the planes YOZ, ZOX.

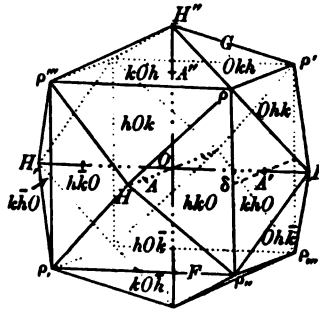


Fig. 231.

The form, Fig. 231, therefore consists of the following faces:

$$\left.\begin{array}{cccccccc} hk0 & kh0 & \bar{k}h0 & \bar{h}k0 & \bar{h}\bar{k}0 & \bar{k}\bar{h}0 & k\bar{h}0 & h\bar{k}0 \\ 0hk & 0kh & 0\bar{k}h & 0\bar{h}k & 0\bar{h}\bar{k} & 0\bar{k}\bar{h} & 0k\bar{h} & 0h\bar{k} \\ k0h & h0k & h0\bar{k} & k0\bar{h} & \bar{k}0\bar{h} & \bar{h}0\bar{k} & \bar{h}0k & \bar{k}0h \end{array}\right\} \ldots\ldots(\mathrm{d}).$$

The simplest way of constructing the form is first to draw the cube. The axes through the middle points A, A', &c., of the faces are then produced to points H, H', &c., equally distant from the origin, where $AH \div OA = A'H' \div OA' = \&c. = k \div h$. The lines joining H, H', H'', &c., to the four nearest coigns ρ of the cube give the equal sides of the faces, the third sides being the cubic edges $\rho\rho'$, $\rho'\rho''$, &c.

From the method of construction we may regard the form as the result of placing on each cubic face a like tetragonal pyramid; and hence its name. Further, the limits of the form are given: (i) by supposing the height of the pyramid to be nil; when the four faces meeting at a tetragonal coign, such as H, coalesce in the cubic face. The limiting form is then the cube $\{100\}$; for $k=0$, and two of the indices of each face are zero. (ii) The height cannot be made greater than OA, when the line HH' passes through δ, the point of bisection of the cubic edge $\rho\rho_{,,}$. At this limit the two faces $(hk0)$, $(kh0)$ coalesce in a face of the rhombic dodecahedron $\{110\}$, and $OH = OH' = 2OA$. Hence k is always less than h.

Four faces meet in a tetragonal coign at each of the points H, H', &c., and the edges from these points to the adjacent coigns of the cube are all like and interchangeable edges, the angles over which are equal and will be denoted by G. The angles between adjacent faces meeting in the edges $\rho\rho_{,,}$, &c., of the cube used in the construction are also all necessarily equal and will be denoted by F.

The symbols of the six faces meeting at a *ditrigonal* coign (p. 292) on the triad axis $O\rho$ are given in the first two columns of table **d**; and the pair of them in each row meet in one of the cubic edges forming the cubic coign ρ. It can also be seen from Fig. 231 that the faces which change places, when the crystal is turned through 120° about $O\rho$, are alternate faces. Rotation through 120° about this axis, counter-clockwise, brings H to H', H' to H'', and H'' to H; and interchanges the three faces given in the first column, as well as the three faces given in the second column; but not a face of the one column with a face of the second.

The indices in the symbols of each triad are in cyclical order (Art. 19), whilst the cyclical order of the one triad is the reverse of that in the second triad.

It will be noticed that adjacent faces meeting in a tetragonal coign, such as H, have symbols in which the indices are in reverse cyclical orders; for rotation through 90° about OX brings OY to OZ and OZ to $OY_{,}$, but leaves OX unchanged. Hence the face $(hk0)$ changes place with $(h0k)$. If the rotation is in the opposite direction, $(hk0)$ is brought to $(h0\bar{k})$.

The dyad axes each bisect a pair of opposite cubic edges; and rotation through 180° about one of them interchanges the pair of tetrad axes co-planar with the dyad axis and opposite ends of the

perpendicular tetrad axis. Thus by rotation about $O\delta$ the face $(hk0)$ changes place with $(kh0)$; and similarly $(h\bar{k}0)$ with $(\bar{k}h0)$.

If OG is the normal to the face $(hk0)$, we have, as in Chap. XIV, Art. 7,

$$\tan AOG = k \div h \quad\text{.........................}(6).$$

Further, alternate faces in a tetragonal zone, such as $(hk0)$, $(\bar{k}h0)$, are at 90° to one another, see Chap. XIV, Arts. 7 and 63.

Fig. 232 is a stereogram of the poles of the form $\{hk0\}$, from which the angles F and G can be easily found. Thus,

$$F = gg^1 = 2g\delta'' = 90° - 2Ag.$$

Hence,

$$\cos F = \cos gg^1 = \sin 2Ag$$
$$= 2\sin Ag \cos Ag = 2\tan Ag \cos^2 Ag$$
$$= \frac{2\tan Ag}{\sec^2 Ag} = \frac{2\tan Ag}{1+\tan^2 Ag}.$$

Replacing $\tan Ag$ by its value given in (6), we have

$$\cos F = \frac{2hk}{h^2+k^2} \quad\text{......}(7).$$

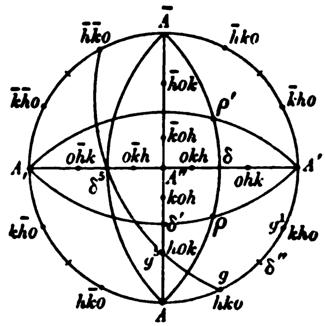


FIG. 232.

Again, in Fig. 232, the arc G between two adjacent poles, such as g and g^5, is the hypothenuse of an isosceles right-angled triangle gAg^5. Hence, by Napier's rules,

$$\cos G = \cos gg^5 = \cos Ag \cos Ag^5 = \cos^2 Ag.$$

But $$\cos^2 Ag = \frac{1}{1+\tan^2 Ag} = (\text{from }(6))\ \frac{h^2}{h^2+k^2}.$$

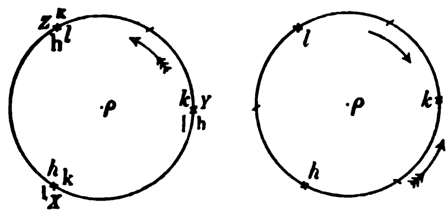
$$\therefore \cos G = \frac{h^2}{h^2+k^2} \quad\text{........................}(8).$$

The values of $\cos F$ and $\cos G$ can be easily found from the general expression (3).

The following are the angles for a few of the tetrakis-hexahedra of most frequent occurrence:

$\{hk0\}$	$Ag = 100 \wedge hk0$	$F = hk0 \wedge kh0$	$G = hk0 \wedge h0k$
{210}	26° 34′	36° 52′	36° 52′
{310}	18 26	53 8	25 51
{410}	14 2	61 56	19 45

19. The *cyclical orders* employed in Art. 18 can be explained by Figs. 233 (a) and (b). In the former a small circle may be



(a) Fig. 233. (b)

supposed to be described through the extremities X, Y, Z of the axes with ρ (the extremity of the triad axis) as centre. The indices h, k and l of the first face of a triad, interchangeable by rotations of 120° about $O\rho$, are attached to X, Y and Z. When now the crystal is rotated with the arrow through 120° about $O\rho$, any length on OX is transferred to the axis OY; and, similarly, a length on OY is transferred to OZ, and one on OZ to OX. The axes of reference themselves retain fixed directions in space, but the faces of the crystal and the axes of symmetry are moved into homologous positions. Hence the length $a \div h$ on OX is after rotation measured as $a_{,} \div h$ on OY; and, similarly, $a_{,} \div k$ on OY as $a_{,,} \div k$ on OZ, and $a_{,,} \div l$ on OZ as $a \div l$ on OX. The symbol of the face in its new position is therefore (lhk). The change in the order is indicated in Fig. 233 (a) by writing the indices against the corresponding axes outside the circle and in different type. A second rotation in the same direction produces similar changes; and the length $a \div l$ on OX is now transferred to OY, $a_{,} \div h$ on OY is now measured on OZ, and $a_{,,} \div k$ on OZ as $a \div k$ on OX. The symbol of the new face is (klh); and the change is shown in the diagram by letters of different type placed inside the circle. A third repetition brings the face back to its original position; and the reader can easily verify that a repetition of the process on the last arrangement of the indices gives the original set. The three faces connected together and interchangeable by rotation about a triad axis are hkl, lhk, and klh, the symbols of which have one cyclical order. If the indices in the symbols are taken in the reverse order, we have a second triad of faces interchangeable by rotation successively through 120° about the triad axis, the symbols of which are said to be in the reverse, or opposite, cyclical order.

If the indices alone are written on the circumference of the circle, Fig. 233 (*b*), and if, starting from each index in turn, they are taken in the direction of the feathered arrow, we obtain the symbols of the triad of faces in one cyclical order; viz. *hkl*, *klh*, *lhk*. If, however, starting from each index in turn, we take them in the reverse order, given by the unfeathered arrow, we obtain the triad of faces, the symbols of which are in the reverse, or opposite, cyclical order to the first; viz. *lkh*, *khl*, *hlk*.

The cyclical orders of the above symbols are a necessary consequence of the fact that rotation through 120° about a triad axis interchanges the axes OX, OY and OZ. They are of considerable assistance in verifying the correctness of the symbols of the faces of a form. Thus the symbols in each of the columns of (**d**) are those of triads, the faces of each of which can be interchanged by rotation about $O\rho$. The faces can be easily rearranged in triads such that the faces of each triad are interchangeable about any of the other triad axes; and the faces of each triad are in cyclical order.

A coign formed by three faces having their symbols in cyclical order is called a *trigonal* coign; one formed by two triads of faces the indices of which are the same numbers taken in opposite cyclical orders is called a *ditrigonal* coign.

20. If the faces of the form are parallel each to only one dyad axis, *i.e.* to an edge of the octahedron, we have two forms with parallel faces which can be constructed as follows. Two planes are drawn through each edge of the octahedron to meet on opposite sides of the origin that tetrad axis which is perpendicular to the octahedral edge at distances $ha \div l$. The number of octahedral edges being twelve, the forms have twenty-four faces: they differ in angles and in the shapes of the faces according as $h \gtrless l$.

The triakis-octahedron, $\{hhl\}$, $h > l$. Let, in Fig. 234, $H''AA'$, $H_{\prime\prime}AA'$ be two faces drawn through the octahedral edge AA' to meet OZ in H'', $H_{\prime\prime}$, where $OH'' = OH_{\prime\prime} = ha_{\prime\prime} \div l$—the points H being more remote from the origin than the points A. Through the similar edges $A'A''$, $A''A_{\prime}$, &c., pairs of like planes can be drawn to H and $\bar{H}$ on OX; and through $A''A$, $AA_{\prime\prime}$, &c., pairs of similar planes can be drawn to H' and $H_{\prime}$ on OY, where

$$OH = OH' = \&c. = ha \div l.$$

The faces $AA'H''$, $AA''H'$ pass through A, and meet the plane

YOZ in the lines $A'H''$, $A''H'$, respectively. These lines intersect at δ on the dyad axis $O\delta$, and the two faces meet in the edge $A\delta$.

Similarly, the faces $AA'H''$, $A'A''H$ have A' in common, and meet the plane ZOX in the lines AH'', $A''H$, respectively. The edge of intersection is therefore $A'\delta'$, δ' being a point on the dyad axis $O\delta'$ in which AH'' and $A''H$ intersect.

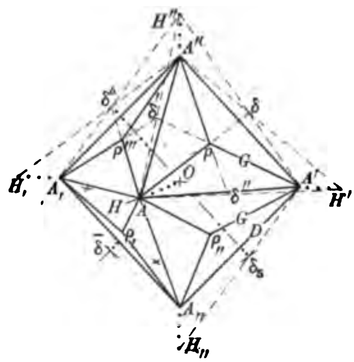


FIG. 234[1].

The two edges $A\delta$ and $A'\delta'$ are common to the face $AA'H''$, and therefore meet in a point ρ which must lie on the triad axis; for the three faces are interchanged by rotations of 120° about this axis, and must therefore meet at a point on it.

Similarly, the two faces $A'A''H$ and $A''AH'$ meet in the line $A''\delta''$ which also passes through ρ.

Again, rotation about the dyad axis $O\delta''$ must interchange X and Y, and equal positive and negative lengths on OZ. We therefore have the three faces meeting in the edges $A\rho_{,,}$, $A'\rho_{,,}$, $A_{,,}\rho_{,,}$; which can be drawn in the same way as those meeting at ρ. The edge AA' is at right angles to $O\delta''$ and is common to both the faces $AA'H''$, $AA'H_{,,}$.

The triads meeting at ρ''' and $\rho_{,}$ are obtained in a similar manner; and the four triads can be interchanged by successive rotations of 90° about OX. Four similar triads, having $\overline{A}$ for common ditetragonal coign, occupy the four octants behind the paper.

The figure may be regarded as the result of placing on each face of the octahedron a like trigonal pyramid having its base congruent with the octahedral face on which it is set. The form is known as the *triakis-octahedron*.

In Fig. 235 the face p'' drawn through $AA'H''$ of Fig. 234 has the intercepts $a : a_{,} : ha_{,,} \div l$. When transposed parallel to itself, the face may be given by $a \div h : a_{,} \div h : a_{,,} \div l$, and by the symbol

[1] To avoid complicating the diagram the lines of construction are not all shown. Several of the points on the dyad axes are indicated by crosses; only one dyad axis is drawn, and the tetrad axes are not continued through the origin.

(hhl). Similarly, the face p is $A'A''H$ of Fig. 234 and intercepts lengths on the axes in the ratios $a \div l : a_{,} \div h : a_{,,} \div h$; it is ($lhh$). In the same way it may be shown that p' is (hlh). The indices in the three symbols are in cyclical order; and, since two of the indices are equal, there is only one such arrangement.

The triads in the other octants only differ from the preceding triad inasmuch as one or more of the axes of reference are met on the negative side of the origin; consequently one or more of the indices change signs. The form consists of the following faces:

$$\left.\begin{matrix} lhh & hlh & hhl & \bar{l}hh & \bar{h}lh & \bar{h}hl \\ l\bar{h}h & h\bar{h}l & h\bar{l}h & \bar{l}\bar{h}h & \bar{h}\bar{h}l & \bar{h}\bar{l}h \\ l\bar{h}\bar{h} & h\bar{l}\bar{h} & h\bar{h}\bar{l} & \bar{l}\bar{h}\bar{h} & \bar{h}\bar{l}\bar{h} & \bar{h}\bar{h}\bar{l} \\ lh\bar{h} & hh\bar{l} & hl\bar{h} & \bar{l}h\bar{h} & \bar{h}h\bar{l} & \bar{h}l\bar{h} \end{matrix}\right\} \cdots\cdots\cdots\cdots \text{(e).}$$

As particular instances we have {221}, {331}, {332}, {443}.

In the stereogram, Fig. 236, the poles lie on the zone-circles in

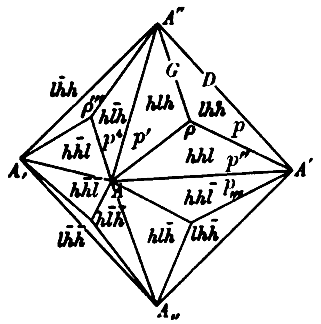


FIG. 235.

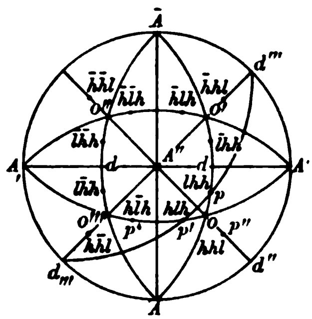


FIG. 236.

which the planes containing two triad axes, one tetrad and one dyad axis meet the sphere. From the construction it is clear that pairs of the faces are tautozonal with pairs of octahedral faces, and that a dyad axis is parallel to each of the edges which lie in the axial planes. It is clear also that a combination of the octahedron {111} with $\{hhl\}$, $h > l$, is obtained by drawing through points on $A\rho$, $A'\rho$, &c., equally distant from ρ, ρ', &c., lines parallel to the octahedral edges. The faces of $\{hhl\}$, therefore, bevel in pairs the edges of the octahedron; and in drawing the combination $\{hhl\}$ should be drawn first.

The limits of this form are given: (i) by taking the height of the trigonal pyramid above the octahedral face to be nil; when the

points H, H', H'', &c., coincide with A, A', A'', &c.: the three faces then coalesce in a face of $\{111\}$. (ii) By removing the points H, H', &c., to infinity; when $l = 0$. The two faces through each octahedral edge then coalesce in a plane parallel to that tetrad axis which is perpendicular to the edge: this plane is a face of the rhombic dodecahedron, $\{110\}$.

The angles, such as pp', $p'p''$, &c., over edges $A''\rho$, $A\rho$, &c., are clearly all equal and will be denoted by G. Similarly, the angles over octahedral edges, such as AA', &c., are all equal, and will be denoted by D.

We can obtain the inclination of $p\,(lhh)$, &c., to the axial poles A, A', &c., from equations (2). The angle Ap can also be obtained from the A. R. $\{Aopd\}$; or from the geometry of Fig. 237, which gives part of a section in the plane $XO\delta$ of Fig. 234. Now $OH = ha \div l$, $Od = OA' \cos 45^\circ = a \div \sqrt{2}$; and Op being the normal to the face, $\wedge\, AOp = \wedge\, OdH$.

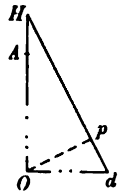


FIG. 237.

$$\therefore\ \tan AOp = \tan OdH = OH \div Od$$
$$= \frac{ha}{l} \div \frac{a}{\sqrt{2}} = \frac{h}{l}\sqrt{2} \quad\ldots\ldots\ldots\ldots(9).$$

Further, when $h = l$, we have seen that p becomes coincident with $o\,(111)$. $\therefore\ \tan Ao = \sqrt{2}$.

$$\text{Hence, } \tan Ap = \frac{h}{l}\tan Ao \quad\ldots\ldots\ldots\ldots (10).$$

The latter expression is identical with that obtained from the A. R. $\{Aopd\}$.

Taking the equations of the normals to p, p', p^4, we can find, by the general expression, the cosines of the angles G and D. Thus,

$$\text{for } p\,(lhh),\quad \frac{\cos Ap}{l} = \frac{\cos A'p}{h} = \frac{\cos A''p}{h} = \frac{1}{\sqrt{2h^2 + l^2}};$$

$$\text{for } p'\,(hlh),\quad \frac{\cos Ap'}{h} = \frac{\cos A'p'}{l} = \frac{\cos A''p'}{h} = \frac{1}{\sqrt{2h^2 + l^2}};$$

$$\text{for } p^4\,(h\bar{l}h),\quad \frac{\cos Ap^4}{h} = \frac{\cos A'p^4}{-l} = \frac{\cos A''p^4}{h} = \frac{1}{\sqrt{2h^2 + l^2}}.$$

Hence, by the general relation (3),

$$\cos G = \cos pp' = \frac{2hl + h^2}{2h^2 + l^2} \quad\ldots\ldots\ldots\ldots(11).$$

$$\cos D = \cos p'p^4 = \frac{2h^2 - l^2}{2h^2 + l^2} \quad\ldots\ldots\ldots\ldots(12).$$

The latter expression can easily be deduced from (9); for

$$D = 180° - 2Ap.$$

$$\therefore \cos D = -\cos 2Ap = \sin^2 Ap - \cos^2 Ap$$

$$= \frac{\sin^2 Ap - \cos^2 Ap}{\sin^2 Ap + \cos^2 Ap} = \frac{\tan^2 Ap - 1}{\tan^2 Ap + 1} = \frac{2h^2 - l^2}{2h^2 + l^2}.$$

The angle G may be obtained from the relations of the triangle $pp'o$ of Fig. 236, but the trigonometrical transformations are not so simple as those given for $\cos D$.

The following table gives the angles of some of the triakis-octahedra of most common occurrence:

$\{hhl\}$	$Ap = 100 \wedge lhh$	$D = hlh \wedge h\bar{l}h$	$G = lhh \wedge hlh$
{331}	76° 44′	26° 32′	37° 52′
{221}	70 32	38 57	27 16
{332}	64 46	50 29	17 20

21. *The icositetrahedron*, $\{hhl\}$, $h < l$. When $h < l$, the form is bounded by four-sided faces which are not parallelograms: it is therefore sometimes called the *trapezohedron*. As in the preceding form, three like faces meet at coigns on each of the triad axes, but they do not compose a pyramid having its base congruent with the face of the octahedron.

The faces t of the icositetrahedron $\{hhl\}$, Fig. 238, have the same general symbols as the triakis-octahedron, which only differ in that now the equal indices are each less than the third index. The form consists of:

$$\left.\begin{matrix} hhl & \bar{h}hl & \bar{h}\bar{h}l & h\bar{h}l \\ lhh & \bar{l}hh & \bar{l}\bar{h}h & l\bar{h}h \\ hlh & \bar{h}lh & \bar{h}\bar{l}h & h\bar{l}h \\ hh\bar{l} & \bar{h}h\bar{l} & \bar{h}\bar{h}\bar{l} & h\bar{h}\bar{l} \\ lh\bar{h} & \bar{l}h\bar{h} & \bar{l}\bar{h}\bar{h} & l\bar{h}\bar{h} \\ hl\bar{h} & \bar{h}l\bar{h} & \bar{h}\bar{l}\bar{h} & h\bar{l}\bar{h} \end{matrix}\right\} \ldots \text{(f)}.$$

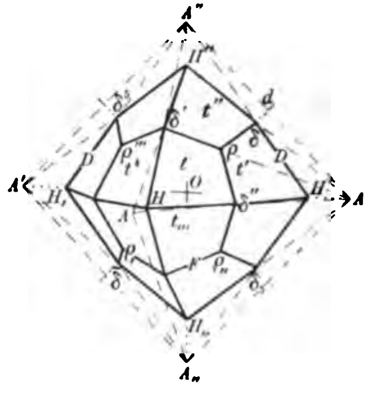


FIG. 238.

The edges in the planes XOY, YOZ, ZOX are the lines AH', $A'H$, $A'H''$, &c., in Fig. 238, which join the coigns of the octahedron to the points H, H', H'', &c., at distances $ha \div l$ on the tetrad axes. These lines intersect in points δ'', δ, &c., situated on

the dyad axes, and form similar and interchangeable edges $H\delta''$, $\delta''H'$, &c. Each of the edges meeting at a trigonal coign ρ join it to an adjacent point δ, and will, if continued, pass through a point A on the tetrad axis lying in the plane which contains the edge and the triad axis through ρ: these edges are similar and interchangeable; and the angles over them (denoted by F) are all equal to one another. The equal angles over each of the edges $H\delta''$, $H\delta'$, &c., will be denoted by D.

The expression for $\cos F$ will be of the same form as that for $\cos G$ of the previous Article, for the only difference is in the relative magnitudes of h and l. Hence, $\cos F = \dfrac{2hl+h^2}{2h^2+l^2}$ ... (13).

We can find $\cos D$ from the general relations (3); it is given by

$$\cos D = \frac{l^2}{2h^2+l^2} \quad\text{.........................(14).}$$

It may also be easily found from equations (2) and the right-angled spherical triangle tAt^4 of Fig. 239. For, since $\wedge\, tAt^4 = 90°$,

$$\cos tt^4 = \cos At \cos At^4 = \cos^2 At.$$

But from the equations of the normal

$$\frac{\cos At}{l} = \frac{\cos A't}{h} = \frac{\cos A''t}{h} = \frac{1}{\sqrt{2h^2+l^2}},$$

$$\therefore\ \cos D = \cos^2 At = \frac{l^2}{2h^2+l^2}.$$

Also, $$\tan At = \sqrt{\sec^2 At - 1} = \frac{h}{l}\sqrt{2} = \frac{h}{l}\tan Ao\text{...(15).}$$

The limits of the form are given by the limiting values of $ha \div l$. When this is zero, $h = 0$, and two indices of each face are zero. Each face is parallel to two axes of reference, and the limiting form is the cube. On the other hand, when the distance OH becomes OA, $h = l$, and the three indices become equal: the limiting form is the octahedron. Beyond this limit the form changes the shape of its faces and becomes the triakis-octahedron described in the last Article.

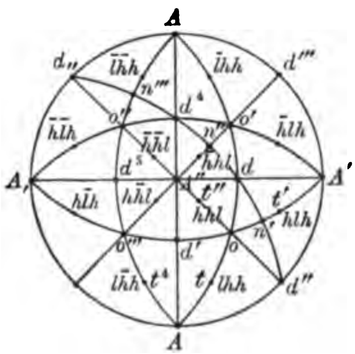


FIG. 239.

The icositetrahedron {112}, in which $l = 2h$, has its faces in sets of six, tautozonal with six faces of the rhombic dodecahedron {110}, as shown by the poles d and n in the stereogram, Fig. 239. Such a zone has a triad axis for zone-axis. A face of {112} truncates each of the edges of {110}; and, conversely, a face of the rhombic dodecahedron makes equal angles with pairs of the faces n which lie in a zone with it. In a combination of the forms, Fig. 240, the faces of {110} will be rhombuses replacing the coigns δ, and the edges will be parallel to the diagonals $H\rho$ of the faces n. If the faces {110} are large, the faces of {211} will be narrow planes truncating the edges of {110}. In constructing such a combination, it is best to draw {112} first and then to introduce the faces of {110}. The coigns δ must all be modified in a similar manner, and the desired lengths of $H\delta'$, &c., are readily determined by means of proportional compasses.

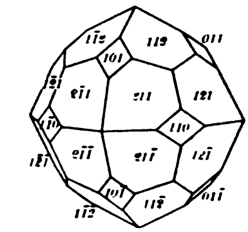


Fig. 240.

The forms of this series most commonly met with are {112} and {113}, the angles of which are:

$\{hhl\}$	$100 \wedge lhh$	$D = lhh \wedge l\bar{h}h$	$F = lhh \wedge hlh$
{112}	35° 16	48° 11′	33° 33′
{113}	25 14	35 6	50 29.

22. The special forms given in the preceding articles all have parallel faces and are geometrically centro-symmetrical. This is due to the faces in each form being parallel to one or more dyad axes. The forms therefore remain unchanged when a centre of symmetry is added to the axes of symmetry. The cube and rhombic dodecahedron are, as we shall see, common to all five classes of the cubic system; the four other forms are common to this and to two other classes of the system.

23. *The pentagonal icositetrahedron*, $\alpha\{hkl\}$. When the indices are all unequal finite integers, we obtain forms of the type shown in Fig. 241, in which each face is a pentagon. The form is called the *pentagonal icositetrahedron*, and its symbol is $\alpha\{hkl\}$. Owing to the axes of symmetry, three like and interchangeable faces meet at similar trigonal coigns at opposite ends of each of the triad axes,

and four faces at similar tetragonal coigns on each of the tetrad axes. Hence two like edges of each face meet at a point ρ on one of the triad axes, and also two similar edges at points, such as H'', on one of the tetrad axes. But the pair of edges meeting at ρ are always unlike those meeting at H''. Again the fifth (*single*) edge is perpendicular to, and bisected by, the dyad axis, and is always different from the four other edges. In Art. 19 we discussed the connection between the three faces which change places by rotations through 120° about a triad axis, and explained the cyclical arrangement of their symbols. It follows that, if (hkl) is one of the faces, the two others meeting at ρ are (lhk) and (klh).

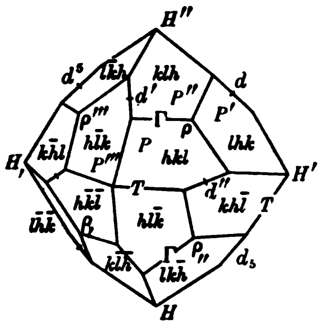


FIG. 241.

On the other hand, a rotation of 90° about a tetrad axis interchanges faces the symbols of which are in opposite cyclical orders, whilst the sign of one of the indices referring to the rotated tetrad axes is changed. Thus, rotation about OX brings $P(hkl)$ to $P'''(h\bar{l}k)$ on one side, and to $(hl\bar{k})$ on the other. The same relation holds for each of the tetrad axes, and consequently for faces arranged about OY and OZ. Two successive rotations of 90° in the same direction are the same as a single rotation of 180° about the same axis. But this only changes equal positive and negative lengths on each of the axes at right angles to the axis of rotation. Hence the fourth face meeting at H is $(h\bar{k}\bar{l})$; the cyclical order being the same as at first.

A face will be said to lie in that octant in which its pole lies in the corresponding stereogram of the form, such as Fig. 242; faces will be said to lie in *adjacent octants* when their poles occupy octants separated by one of the axial planes; and they will be said to occupy *alternate octants* when the poles lie in octants having in common only one of the axes of reference; and in *opposite octants* when the octants have only the origin in common.

It follows from the preceding discussion that the symbols of the faces in one octant are in the reverse cyclical order to those of the faces in the two adjacent octants and in the opposite octant; but that the triads in alternate octants have the same cyclical

order. The symbols of the faces can therefore be easily given; and the form $a\{hkl\}$ has the following twenty-four faces:

$$\left.\begin{array}{cccccc}
hkl & lhk & klh & kh\bar{l} & hl\bar{k} & lk\bar{h} \\
\bar{k}hl & \bar{h}lk & \bar{l}kh & \bar{h}k\bar{l} & \bar{l}h\bar{k} & \bar{k}l\bar{h} \\
\bar{h}\bar{k}l & \bar{l}\bar{h}k & \bar{k}\bar{l}h & \bar{k}\bar{h}\bar{l} & \bar{h}\bar{l}\bar{k} & \bar{l}\bar{k}\bar{h} \\
k\bar{h}l & h\bar{l}k & l\bar{k}h & h\bar{k}\bar{l} & l\bar{h}\bar{k} & k\bar{l}\bar{h}
\end{array}\right\}\ldots\ldots(g).$$

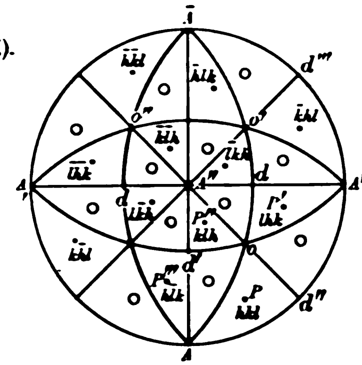


Fig. 242.

The four faces in each column are symmetrical with respect to OZ, and interchange places on successive rotations of 90° about this axis. The faces may also be arranged in tetrads symmetrical with respect to OX or to OY; and in pairs symmetrical with respect to each of the dyad axes. The form has three different angles between adjacent faces meeting in edges. These are: Γ between faces meeting at a trigonal coign, *e.g.* PP' or PP'' of Figs. 241 and 242; T between faces meeting at a tetragonal coign, *e.g.* PP'''; and Δ over the edge bisected at right angles by a dyad axis, *e.g.* $P'P'''$.

The equations of the normals to adjacent faces of the form are:

$$P\,(hkl),\qquad \frac{\cos AP}{h} = \frac{\cos A'P}{k} = \frac{\cos A''P}{l} = \frac{1}{\sqrt{h^2+k^2+l^2}};$$

$$P'\,(lhk),\qquad \frac{\cos AP'}{l} = \frac{\cos A'P'}{h} = \frac{\cos A''P'}{k} = \frac{1}{\sqrt{h^2+k^2+l^2}};$$

$$P''\,(klh),\qquad \frac{\cos AP''}{k} = \frac{\cos A'P''}{l} = \frac{\cos A''P''}{h} = \frac{1}{\sqrt{h^2+k^2+l^2}};$$

$$P'''\,(h\bar{l}k),\qquad \frac{\cos AP'''}{h} = \frac{\cos A'P'''}{-l} = \frac{\cos A''P'''}{k} = \frac{1}{\sqrt{h^2+k^2+l^2}}.$$

Hence, combining the normals in pairs, and introducing the corresponding values into equations (3) of Art. 13, we have:

$$\left.\begin{aligned}
\cos\Gamma &= \cos PP'' = \cos(hkl \wedge klh) = \frac{hk+kl+lh}{h^2+k^2+l^2};\\
\cos T &= \cos PP''' = \cos(hkl \wedge h\bar{l}k) = \frac{h^2}{h^2+k^2+l^2};\\
\cos\Delta &= \cos P''P''' = \cos(klh \wedge h\bar{l}k) = \frac{2hk-l^2}{h^2+k^2+l^2}.
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(16).$$

24. By the law of rational indices we know that, if (hkl) is a possible face, the parallel face $(\bar{h}\bar{k}\bar{l})$ is also possible; although the two faces cannot occur together as members of one form of this class, for there is no centre of symmetry. If then $(\bar{h}\bar{k}\bar{l})$ does occur, it belongs to a distinctly separate form which may be denoted by the symbol $a\{\bar{h}\bar{k}\bar{l}\}$, or $a\{lkh\}$. The reader will, on referring to table **g**, perceive that the triad of faces, in which all the indices are negative, have their symbols in the reverse cyclical order to those in which the indices are all positive: the symbols of faces in other opposite octants are also in reverse cyclical orders. As no limit has been placed on the relative values of h, k and l, except that they are finite and unequal, the relation is clearly general. Hence, the pentagonal-icositetrahedron $a\{lkh\}$, Fig. 243, consists of the twenty-four faces:

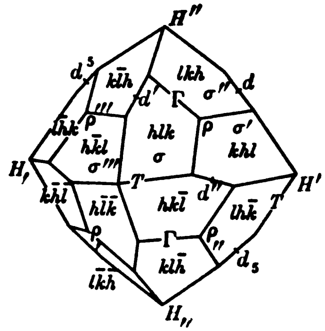


FIG. 243.

$$\left.\begin{array}{cccccc} lkh & hlk & khl & kl\bar{h} & hk\bar{l} & lh\bar{k} \\ \bar{k}lh & \bar{h}kl & \bar{l}hk & \bar{l}k\bar{h} & \bar{h}l\bar{k} & \bar{k}h\bar{l} \\ \bar{l}\bar{k}h & \bar{h}\bar{l}k & \bar{k}\bar{h}l & \bar{k}\bar{l}\bar{h} & \bar{h}\bar{k}\bar{l} & \bar{l}\bar{h}\bar{k} \\ k\bar{l}h & h\bar{k}l & l\bar{h}k & l\bar{k}\bar{h} & h\bar{l}\bar{k} & k\bar{h}\bar{l} \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots \text{(h)}.$$

It will be noticed that each of the faces in this form is parallel to one of those in the preceding form; and this relation necessarily entails that the faces are similar, and that the angles over the edges of a face are the same as those over the edges of the parallel face of $a\{hkl\}$. The two forms are complementary and enantiomorphous; for, as is evident from the figures, the two forms are reciprocal reflections of one another with respect to the axial planes.

The crystals of this class would therefore be expected to rotate the plane of polarization; but this has not been observed in any of the substances so far placed in the class.

The method of drawing the above forms will be best given at a later stage, when the drawing of the general forms characteristic of all classes of the system can be discussed at the same time.

table

25. The crystals of salt, $NaCl$; of sylvine, KCl; of ammonium chloride, NH_4Cl; provisionally of silver chloride, $AgCl$, and of cuprite, Cu_2O are placed in this class; although general forms, $\alpha\{hkl\}$, have only been observed on crystals of cuprite, and not one of the substances rotates the plane of polarization. The alkaline chlorides are included in the class, as a result of experiments on the corrosion of the crystal-faces.

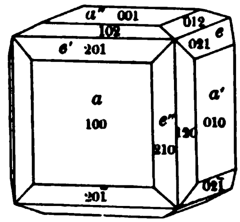


FIG. 244.

Salt is usually found in cubes; sometimes in cubes having their edges bevelled by faces of {210}, as shown in Fig. 244. Occasionally the crystals are octahedra; and these sometimes have their edges truncated by the faces of the rhombic dodecahedron {110}, Fig. 246. The corrosion-figures on the cubic faces indicate the presence of tetrad axes, and the absence of planes of symmetry.

Cuprite. The crystals are often single forms—the octahedron {111}, or the cube {100}, or the rhombic dodecahedron {110}. Combinations of two, or more, of these forms are also common; *e.g.* the cubo-octahedron, Fig. 245—{111} with {110}, Fig. 246—and {111} with {110} and {100}, Fig. 247.

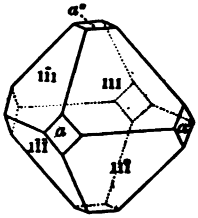


FIG. 245.

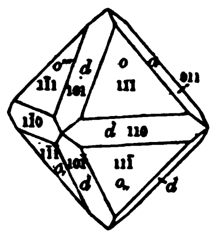


FIG. 246.

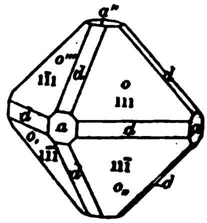


FIG. 247.

Fig. 247. A form $\alpha\{896\}$ was first observed by Professor Miers on crystals from Cornwall, like Fig. 248, (*Phil. Mag.* [v] XVIII, p. 127, 1884).

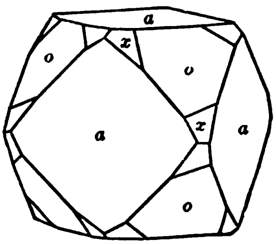


FIG. 248.

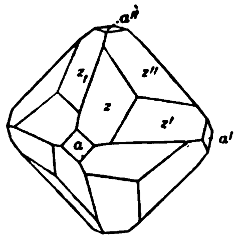


FIG. 249.

The forms shown in Fig. 248 are $a\{100\}$, $o\{111\}$, $x=\alpha\{896\}$. In other crystals the habit is different owing to the predominance of the faces

x and the smallness of the faces of the cube. Similar crystals, Fig. 249, to these last have been described by Dr. J. H. Pratt, in which the predominant faces z belong to $\alpha\{13\ 10\ 12\}$.

II. *Hexakis-octahedral class;* $\{hkl\}$.

26. If to the axes of symmetry characteristic of the last class a centre of symmetry is added, then nine planes of symmetry must also be present; for, by Chap. IX, Prop. 4, a plane of symmetry is perpendicular to each dyad and tetrad axis of a centro-symmetrical crystal.

The planes of symmetry perpendicular to the tetrad axes are parallel to the faces of the cube; and, being interchangeable by rotations of 120° about each of the triad axes, they are like planes: they will be called the *cubic* planes of symmetry Π, Π', Π''.

Each of the six planes of symmetry perpendicular to a dyad axis is parallel to two faces of the rhombic dodecahedron, and (Art. 17) contains one tetrad, one dyad and two triad, axes. They are called the *dodecahedral* planes of symmetry, and will be denoted by Σ: they intersect one another in sets of three in each of the triad axes (Chap. IX, Prop. 9), and the least angle between any pair of planes of a triad is 60°. They also intersect in pairs in each of the tetrad axes. Hence the tetrad axes are the lines in which four planes of symmetry meet, two of which are cubic planes and two are dodecahedral; the dyad axes are also the lines of intersection of a cubic with a dodecahedral plane of symmetry.

The planes parallel to the faces of the octahedron, being each perpendicular to one of the triad axes, cannot be planes of symmetry. For, in a centro-symmetrical crystal, the axis perpendicular to a plane of symmetry is one of even degree. Further, the faces of the octahedron are inclined to one another at angles of 70° 32′, and to the faces of the cube—now parallel to planes of symmetry Π—at angles of 54° 44′, which are neither of them angles possible between planes of symmetry (Chap. IX, Prop. 5).

The elements of symmetry present in crystals of this class are:

$$4\rho,\ 3T,\ 6\delta,\ C,\ 3\Pi \text{ and } 6\Sigma.$$

This is the greatest number, and most complex set, of elements of symmetry possible in any crystal.

27. It is clearly convenient to retain the three tetrad axes for axes of reference, and a face of the octahedron for parametral plane. Hence, as in the preceding class, $a = b = c$.

The equations of the normal $P\,(hkl)$ are, as before, those given in equations (2) Art. 13; and the angle between $P\,(hkl)$ and $Q\,(pqr)$ is given by the general expression (3).

28. The special forms of the last class all have their faces parallel in pairs to at least one dyad, or tetrad, axis; and are therefore unchanged by the introduction of a centre of symmetry. Hence, the following special forms are common to both classes:

1. The cube $\{100\}$, Art. 15.

2. The octahedron $\{111\}$, Art. 16.

3. The rhombic dodecahedron $\{110\}$, Art. 17.

4. The tetrakis-hexahedron $\{hk0\}$, Art. 18.

5. The triakis-octahedron $\{hhl\}$, $h > l$, Art. 20.

6. The icositetrahedron $\{hhl\}$, $h < l$, Art. 21.

These forms are easily seen to be geometrically symmetrical with respect to the planes Π and Σ, and need no fresh description: when they are forms of this class, Π and Σ are actual planes of symmetry.

29. *The hexakis-octahedron*, $\{hkl\}$. The addition of parallel faces to the general form $a\,\{hkl\}$ of the preceding class is clearly the same as the association of $a\,\{hkl\}$ with $a\,\{lkh\}$. Hence, the form $\{hkl\}$, Fig. 250, consists of the following forty-eight faces:

$$\left.\begin{array}{cccccc}
hkl & lhk & klh & lkh & hlk & khl \\
\bar{k}hl & \bar{h}lk & \bar{l}kh & \bar{k}lh & \bar{l}hk & \bar{h}kl \\
\bar{h}\bar{k}l & \bar{l}\bar{h}k & \bar{k}\bar{l}h & \bar{l}\bar{k}h & \bar{h}\bar{l}k & \bar{k}\bar{h}l \\
k\bar{h}l & h\bar{l}k & l\bar{k}h & k\bar{l}h & l\bar{h}k & h\bar{k}l \\
hk\bar{l} & lh\bar{k} & kl\bar{h} & kh\bar{l} & hl\bar{k} & lk\bar{h} \\
\bar{k}h\bar{l} & \bar{h}l\bar{k} & \bar{l}k\bar{h} & \bar{h}k\bar{l} & \bar{l}h\bar{k} & \bar{k}l\bar{h} \\
\bar{h}\bar{k}\bar{l} & \bar{l}\bar{h}\bar{k} & \bar{k}\bar{l}\bar{h} & \bar{k}\bar{h}\bar{l} & \bar{h}\bar{l}\bar{k} & \bar{l}\bar{k}\bar{h} \\
k\bar{h}\bar{l} & h\bar{l}\bar{k} & l\bar{k}\bar{h} & h\bar{k}\bar{l} & l\bar{h}\bar{k} & k\bar{l}\bar{h}
\end{array}\right\} \ldots\ldots\ldots\ldots\ldots(\text{i}).$$

It remains to prove that these faces are symmetrical in pairs with respect to the planes of symmetry Π and Σ. The reader will, on referring to Chap. XIV, Arts. 7 and 10, see that the rule connecting the symbols of a pair of faces, symmetrically placed with

respect to a plane of symmetry passing through one axis of reference and bisecting the angle between the two other axes, is that the indices referring to the latter axes are interchanged. Fig. 177 of that Article can be drawn in each of the axial planes; and the figures give the relations of the indices of the faces symmetrically placed with respect to those dodecahedral planes of symmetry which are perpendicular to the planes of the figures; for each plane Σ passes through one axis of reference and bisects the angle between the two other axes. Thus, (hkl) and (khl) are symmetrical with respect to that plane Σ through OZ which bisects the angle between the positive directions of the axes of X and Y: and so are, likewise, the several pairs, klh, lkh; $\bar{k}\bar{l}h$, $\bar{l}\bar{k}h$; $\bar{h}\bar{k}l$, $\bar{k}\bar{h}l$; and the several pairs of parallel faces which meet OZ on the negative side of the origin. We clearly have eight similar pairs of faces meeting in, and symmetrically situated about, each of the Σ planes. The same is true of pairs of incontiguous faces, such as $(\bar{k}hl)$ and $(h\bar{k}l)$. In taking distant pairs of faces, the student must be careful as to the signs of the axes which are reciprocal reflexions in the particular plane of symmetry.

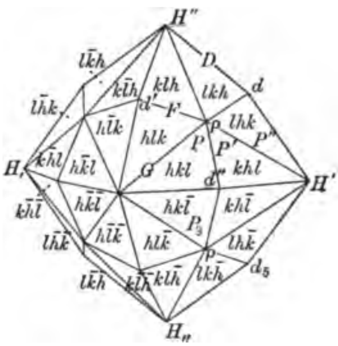


FIG. 250.

The faces can easily be arranged in pairs which meet in, and are symmetrically placed with respect to, each of the planes Π, Π′, Π″. Thus, hlk, $h\bar{l}k$; klh, $k\bar{l}h$; $hk\bar{l}$, $h\bar{k}\bar{l}$, &c., are symmetrical with respect to the plane Π′ which contains OX and OZ. The faces symmetrically placed with respect to this plane differ in the sign of the second index; for equal lengths on OY on opposite sides of the origin are reciprocal reflexions in this plane.

The six faces, given by the symbols in each row of table i, meet at a ditrigonal coign on one of the triad axes, and compose two triads such that the members of one triad with symbols in cyclical order are interchangeable amongst one another, but not with those of the other triad, the symbols of which are in the opposite cyclical order. The faces of the one triad are antistrophic in pairs to those of the other, for they are reciprocal reflexions each to each in dodecahedral planes Σ.

Again, the faces meet in sets of eight in ditetragonal coigns H, H', &c., where $OH = a \div h$ is the shortest intercept, and h is the greatest integer. The symbols of the eight faces, meeting the axis in the same point, have h in the same position. Thus, the faces meeting at H are: hkl, hlk, $h\bar{l}k$, $h\bar{k}l$, $h\bar{k}\bar{l}$, $h\bar{l}\bar{k}$, $hl\bar{k}$, $hk\bar{l}$. The remaining faces of table i can be arranged in similar sets of eight which meet at the other ditetragonal coigns.

Four faces meet at a coign on each of the dyad axes; thus, for example, hkl, khl, $kh\bar{l}$, $hk\bar{l}$ meet at d''.

Each face is a similar and equal scalene triangle, the edges of which join pairs of coigns of different character, and lie in the cubic and dodecahedral planes of symmetry: the angles over each of these edges are in all cases different. Twenty-four of the faces—those constituting geometrically the form $a\{hkl\}$—are interchangeable with one another. Similarly the remaining twenty-four faces are interchangeable with one another, for they have the same symbols as the faces of $a\{lkh\}$; and the faces of the second set are each of them parallel to a face of the first set. The faces of the two sets are likewise antistrophic, for a face of the one set is the reciprocal reflexion of a face of the second in one of the planes of symmetry.

The angles over the dissimilar edges may be denoted by D, F and G; and can be readily found from equation (3) by introducing into this equation the indices of two adjacent faces taken in pairs over dissimilar edges.

Thus, in Figs. 250 and 251,

$$P \text{ being } (hkl),\ \frac{\cos AP}{h} = \frac{\cos A'P}{k} = \frac{\cos A''P}{l} = \frac{1}{\sqrt{h^2+k^2+l^2}};$$

$$P' \ \text{,,}\ (khl),\ \frac{\cos AP'}{k} = \frac{\cos A'P'}{h} = \frac{\cos A''P'}{l} = \frac{1}{\sqrt{h^2+k^2+l^2}};$$

$$P'' \ \text{,,}\ (lhk),\ \frac{\cos AP''}{l} = \frac{\cos A'P''}{h} = \frac{\cos A''P''}{k} = \frac{1}{\sqrt{h^2+k^2+l^2}};$$

$$P_3 \ \text{,,}\ (hk\bar{l}),\ \frac{\cos AP_3}{h} = \frac{\cos A'P_3}{k} = \frac{\cos A''P_3}{-l} = \frac{1}{\sqrt{h^2+k^2+l^2}}.$$

Hence

$$\left.\begin{aligned} \cos D &= \cos PP_3 = \frac{h^2+k^2-l^2}{h^2+k^2+l^2};\\ \cos F &= \cos PP' = \frac{2hk+l^2}{h^2+k^2+l^2};\\ \cos G &= \cos P'P'' = \frac{h^2+2kl}{h^2+k^2+l^2} \end{aligned}\right\} \text{..........} \quad (16).$$

30. Crystals of the substances given in the following list belong to this class. In the table of common forms and combinations we shall denote the forms by the letters attached to the symbols: *a* {100}, *o* {111}, *d* {110}, *e* {210}, *n* {211}, *m* {311}, *p* {221}, *s* {321}, *t* {421}. When a single letter is given, it denotes that crystals consisting of the single form occur; when two or more letters are joined together, they denote a combination of the corresponding forms, that placed first being usually the predominant form. The same letters will be used in giving corresponding forms in other classes of cubic crystals. They are systematically used in Brooke and Miller's *Mineralogy* and also in Dana's *Mineralogy*.

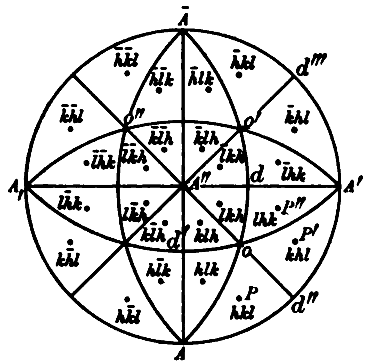


Fig. 251.

Substance.	*Chemical composition.*	*Forms and combinations.*
Silicon	Si	*o*; often twinned.
Iron	Fe	*o*.
Platinum	Pt	*a*.
Copper	Cu	*a*, *o*, *d*, *ao*, *ae*, *aod*, *aode*. Twins frequent.
Silver	Ag	*a*, *o*, *e*, *m*, *ao* (Fig. 252), *om*, *ode*. Twins frequent.
Gold	Au	*a*, *o*, *d*, *e*, *m*, *am*, *om*.
Lead	Pb	*o*, *ao*.
Galena	PbS	*a*, *o*, *ao*, *aod* (Fig. 254), *aodp*.
Argentite	Ag_2S	*a*, *o*, *d*, *ao*, *ad* (Fig. 255), *aodn*.
Arsenite	As_2O_3	*o*.
Senarmontite	Sb_2O_3	*o*.
Spinel	$MgO.Al_2O_3$	*o*, *d*, *od* (Fig. 253), *odm*, *odmp*. Often twinned.
Magnetite	$FeO.Fe_2O_3$	*o*, *d*, *a*, *ao*, *od*. Twins fairly common.
Fluor	CaF_2	*a*, *o*, *d*, *m*, *ao*, *ad*, *am*, *at*, *ap*. Often twinned.
Garnet	$(Ca, Fe)_3(Al, Fe)_2Si_3O_{12}$	*d*, *n*, *nd* (Fig. 261), *dne*, *dns*.
Analcime	$NaAl(SiO_3)_2.H_2O$	*n*, *an*.
Uraninite contains UO_2, UO_3, helium, &c.		*o*, *od*.

The crystals of the above substances generally consist of single forms—the cube, the octahedron, the rhombic-dodecahedron or the icositetrahedron {211}—or of simple combinations of the first three forms. In drawing simple combinations, such as Fig. 253, it is best to construct {110} first and then to introduce the faces of {111}. Fig. 254 might be constructed

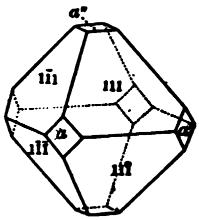


FIG. 252.

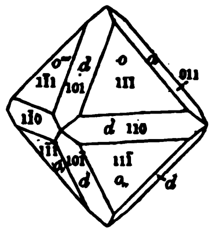


FIG. 253.

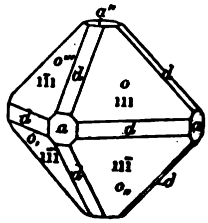


FIG. 254.

in the same way; but it would probably be simpler to draw {111} first, and then to introduce {100} and {110}.

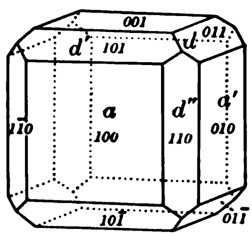


FIG. 255.

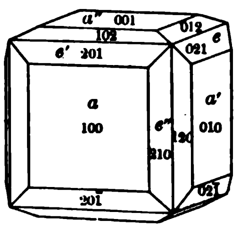


FIG. 256.

The combination shown in Fig. 256 is frequent on crystals of *copper*: the tetrakis-hexahedron {210} should be drawn first; the faces of the cube are then introduced by cutting off by proportional compasses equal lengths on each edge joining a tetragonal to a ditrigonal coign.

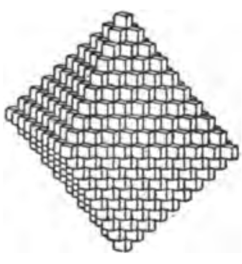


FIG. 257.

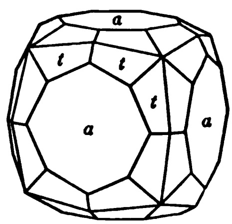


FIG. 258.

A similar combination occurs in crystals of *fluor*, but the faces bevelling the edges of the cube generally belong to the form {310}. In these crystals the relative dimensions of the faces of the two forms vary much. Fluor crystals have also been found in Cornwall which have the general appearance of octahedra, built up of small cubes in the manner shown in Fig. 257.

Fig. 255 represents a combination of the cube and rhombic dodecahedron {110} which often occurs on crystals of argentite, fluor, and other substances of this and of the other classes. Thus, for instance, it is not uncommon on crystals of cuprite.

In *analcime* the common form is *n* {211}. Crystals from a few localities have the habit shown in Fig. 259: in this combination the icositetrahedron

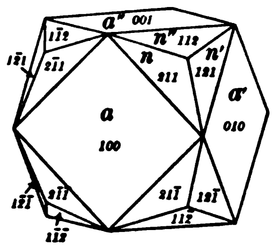


FIG. 259.

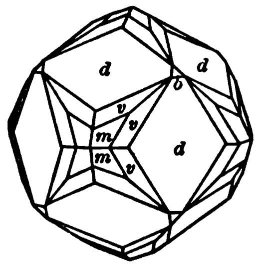


FIG. 260.

should be drawn first; from the coigns on the dyad axes the edges [*an*] are then drawn and complete the figure.

General forms such as *s* {321}, *t* {421}, &c., are rare; and especially so as simple forms. The form *t* {421} sometimes occurs associated with the cube on crystals of fluor, as is shown in Fig. 258. The form *t* should be first completely drawn, when the faces *a* of the cube are easily introduced.

Fig. 260 represents a crystal of *magnetite* showing a rare form *v* {531} associated with the rhombic dodecahedron {110}, the icositetrahedron *m* {311} and the octahedron *o* {111}. In drawing this combination, *v*

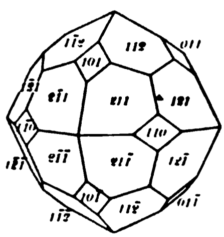


FIG. 261.

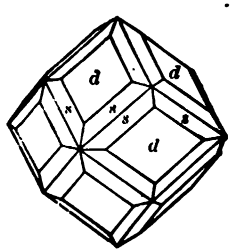


FIG. 262.

should be constructed first; then the faces m and d should be introduced, and lastly the faces o.

Garnet occurs frequently as simple rhombic dodecahedra {110}, and also as icositetrahedra n {211}. The two forms are also often associated together, and the relative dimensions of the faces of the two forms vary much; sometimes we get crystals resembling Fig. 261, in other cases the faces n appear as narrow planes truncating the edges of {110}. Fig. 262 represents a crystal showing the forms d {110}, and s {321}: in such a combination the form s should be drawn first. Fig. 263 shows a crystal in which the forms d {110}, s {321} and n {211} occur together: when the previous figure has been drawn, the introduction of the faces of n {211} is easy.

FIG. 263.

31. The two preceding classes have exhausted the cases in which three tetrad axes occur; and we shall now consider the other classes of the cubic system in which tetrad axes do not occur. There are three classes in which four triad axes, inclined to one another at the same angles as those in the preceding classes, are associated with three dyad axes parallel to the edges of a cube. In Chap. IX, Prop. 12, iii (c) and (d), it is shown that the only arrangements of several triad axes, not included in the two preceding classes, are those in which: (*1*) four like and interchangeable triad axes of *uniterminal* symmetry are given by the lines joining the centre to the coigns of a regular tetrahedron; and (*2*) four like axes are given by the four diagonals of a cube, but the facial development at adjacent coigns of the cube though similar is antistrophic, whilst that at alternate coigns is alone metastrophic.

It is indeed obvious from the definition of a triad axis that, if two triad axes are present inclined to one another at a finite angle, there must be at least two others; for a rotation of 120° both ways about one of them must bring the second into two new positions, which must be the directions of like triad axes. Having, however, established the proposition generally in Chap. IX, Prop. 12, we need not repeat the proof that there can only be four such axes, and that they occupy the positions just given.

We shall discuss the classes in the following order:—

III. The *tetrahedral* class, in which the four triad axes of case (*1*) are associated with three dyad axes, each joining the middle points of a pair of opposite edges of the tetrahedron, Fig. 264.

IV. The *dyakis-dodecahedral* class, in which the axes of symmetry of class III are associated with a centre of symmetry, and three planes of symmetry Π, each perpendicular to one of the dyad axes. This class is coextensive with case (*2*).

V. The *hexakis-tetrahedral* class, in which the axes of symmetry of class III are associated with six like planes of symmetry Σ, which intersect in sets of three in each of the triad axes.

III. *Tetrahedral class;* $\tau\{hkl\}$. Usually called tetartohedral

32. The crystals of this class have four similar triad axes, which join the middle point O of a tetrahedron, Fig. 264, to its coigns ρ', ρ''', &c., and which—when produced towards r, $r_{,}$, &c.—meet the faces opposite to the coigns at right angles in their middle points. For a rotation of 120° about each of the triad axes, such as, for instance, ρr, interchanges similar extremities of each of the three other axes, and therefore leaves the direction of the face $\rho'\rho'''\rho_{,,}$ through these extremities unchanged. Again, the lines joining the middle points of opposite edges of the tetrahedron are dyad axes, for rotations of 180° about any one of them interchange similar ends of the triad axes; *e.g.* a rotation of 180° about the axis δ interchanges the similar pairs of extremities r, $r_{,}$ and $\rho_{,,}$, ρ'''.

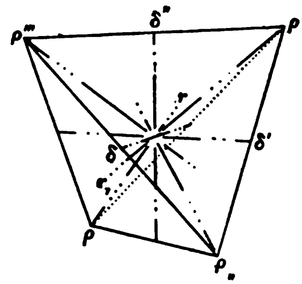


FIG. 264.

We shall now show that, when the opposite extremities of the triad axes are dissimilar, or similar but not interchangeable, the only axes of symmetry which can be associated with them are the three dyad axes bisecting opposite edges of the tetrahedron.

Let a sphere, Fig. 265, be described with centre at the middle point O of the tetrahedron, Fig. 264, and with radius $O\rho'$; and let the triad axis ρr be the diameter perpendicular to the paper. The similar extremities ρ', ρ''', $\rho_{,,}$ lie on diametral circles inclined to one another at angles of 120°, and are equally distant from the

central point r. Further, the great circles $\rho'\rho_{\prime\prime}$, $\rho'\rho'''$ are inclined to one another at an angle of 120°, for a rotation of this amount about $O\rho'$ interchanges $\rho_{\prime\prime}$ and ρ'''; and the angle $\rho_{\prime\prime}\rho'\rho'''$ is bisected by the great circle $r\rho'$: hence $\wedge r\rho'\rho_{\prime\prime} = \wedge r\rho'\rho''' = 60°$. The same is true of each of the angles at $\rho_{\prime\prime}$ and ρ'''. Again, the angles in which the diametral circle $r\rho'$ meets the great circle $\rho_{\prime\prime}\rho'''$ are right angles; and the arc $\rho_{\prime\prime}\rho'''$ is bisected at δ. Similarly, each of the arcs $\rho_{\prime\prime}\rho'$, $\rho'\rho'''$ is bisected at right angles in the points δ', δ'', respectively.

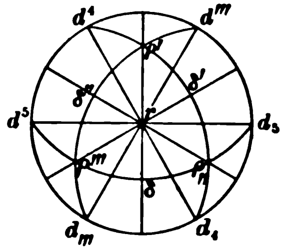


FIG. 265.

By Euler's theorem (Chap. IX, Art. 13) successive rotations of 120°, clockwise, about the diameters through $\rho_{\prime\prime}$ and ρ''' are equivalent to a single rotation clockwise about the diameter ρOr; and, since the external angle $\rho_{\prime\prime}r\delta' = 60°$, the least angle of rotation about r is 120°, and the diameter through r is a triad axis. Again, by the same theorem, successive rotations of 120°, clockwise, about the diameters through r and ρ' are equivalent to a rotation about the diameter through δ'; and, since the external angle $r\delta'\rho_{\prime\prime}$ of the triangle $r\delta'\rho'$ is 90°, the least angle of rotation is 180°, and the diameter is a dyad axis.

Successive rotations, counter-clockwise, of 180° about $O\delta'$ and of 120° about $O\rho_{\prime\prime}$ are equivalent to a single rotation of 240° about Or. But this rotation leaves the sphere in the position in which it would be left by a rotation of 120° about Or in the opposite direction. Hence, we have no new axis and no new rotation. Similarly, successive rotations of 120° about Or and 180° about $O\delta$, counter-clockwise, are equivalent to a rotation of 240° about $O\rho_{\prime\prime}$; we have then no new axis and no new rotation.

The class having only four triad axes and three dyad axes we shall call the *tetrahedral* class. The angles between the axes are fixed, and do not vary either with the substance or with the temperature. The angles at r and $\rho_{\prime\prime}$ in the right-angled spherical triangle $r\delta\rho_{\prime\prime}$ being 60°, we have:

$$\cos r\rho_{\prime\prime} = \cot \delta r\rho_{\prime\prime} \cot \delta\rho_{\prime\prime}r = \cot^2 60°;$$

$$\cos \delta\rho_{\prime\prime}r = \cos r\delta \sin \delta r\rho_{\prime\prime}, \quad \therefore \cos r\delta = \cot \delta\rho_{\prime\prime}r = \cot 60°;$$

$$\cos \delta r\rho_{\prime\prime} = \cos \delta\rho_{\prime\prime} \sin \delta\rho_{\prime\prime}r, \quad \therefore \cos \delta\rho_{\prime\prime} = \cot \delta r\rho_{\prime\prime} = \cot 60°.$$

Hence, $\wedge r\rho_{\prime\prime} = 70°\ 31{\cdot}7'$, $\wedge r\delta = \wedge \delta\rho_{\prime\prime} = 54°\ 44'$.

33. The dyad axes, being at right angles to one another, are the most convenient axes of reference, and we shall take OX to coincide with $O\delta$, OY with $O\delta'$, and OZ with $O\delta''$. We shall also take the face $\rho'\rho'''\rho_{,,}$ of Fig. 264 to be the parametral plane (111); and, as it is perpendicular to $\rho O r$, it will retain the same direction after each rotation of 120° about this axis. But a rotation of 120° about $\rho O r$ interchanges $O\delta$, $O\delta'$ and $O\delta''$, which must therefore be equal lengths. Hence the lengths intercepted by (111) on the axes are equal; and

$$a = b = c.$$

These equalities can also be proved from the equations of the normal Or to the face (111), which are

$$\frac{a \cos XOr}{1} = \frac{b \cos YOr}{1} = \frac{c \cos ZOr}{1}.$$

But
$$XOr = YOr = ZOr;$$
$$\therefore\ a = b = c.$$

We shall, as before, use A to represent the pole (100), A' (010) and A'' (001); and we shall denote a length a by $a_{,}$ when it is measured on OY, and by $a_{,,}$ when it is measured on OZ.

As the above axes of symmetry with their consequent rotations exist in all cubic crystals (Art. 1), being the least assemblage of axes of symmetry which can occur in any crystal having more than one triad axis, it follows that the same axes and parameters can be taken for crystals of all classes.

Further, the equations of the pole $P\,(hkl)$ are

$$\frac{\cos AP}{h} = \frac{\cos A'P}{k} = \frac{\cos A''P}{l} = \frac{1}{\sqrt{h^2 + k^2 + l^2}};$$

equations already established in Art. 13.

It follows also that expression (3) holds for the cosine of the angle between any two poles P and Q, whichever the class to which the crystal belongs.

34. *The tetrahedron*, $\tau\{111\}$. This form consists of the four faces: $111\ \ 1\bar{1}\bar{1}\ \ \bar{1}1\bar{1}\ \ \bar{1}\bar{1}1$ ....(j).

The form is most easily drawn by first constructing the cube by one of the methods given in Chap. VI. Opposite corners of each face, such as ρ', ρ''' of Fig. 226, are then joined so that three edges bounding each face cut off portions of the cube at alternate coigns. By tracing Fig. 226 on thin paper, and joining the pairs of points

ρ', ρ'''; ρ''', $\rho_{,,}$; $\rho_{,,}$, ρ'; &c., the student will form a tetrahedron $\tau\{111\}$ similar and similarly placed to Fig. 264. From the method of construction it is clear that each face is an equilateral triangle, for the edges are all diagonals of the faces of the cube.

If the pairs of coigns ρ, ρ''; ρ, $\rho_{,}$; &c., of Fig. 226 are joined, we obtain the complementary tetrahedron $\tau\{11\bar{1}\}$. This is tautomorphous with $\tau\{111\}$, for a rotation through 90° about one of the dyad axes—parallel to the edges of the cube—interchanges the two diagonals of the cubic face to which the axis is perpendicular, and adjacent coigns of the cube.

As was shown in Chap. III, Art. 2, the tetrahedron can be derived from the regular octahedron by omitting alternate faces of the latter, and extending the faces retained to intersect one another as shown in Fig. 266. Further, a geometrically equal and similar tetrahedron can be formed from the same octahedron by retaining the faces omitted in the first; and the faces of the one are parallel to those of the other. Hence the complementary tetrahedron can be represented either as $\tau\{11\bar{1}\}$ or by $\tau\{\bar{1}\bar{1}\bar{1}\}$; and comprises the faces:

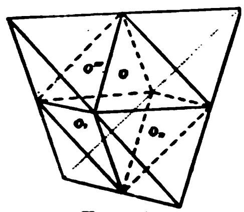


FIG. 266.

$$11\bar{1}\quad 1\bar{1}1\quad \bar{1}11\quad \bar{1}\bar{1}\bar{1}.$$

We shall use the letters o to denote the faces of $\tau\{111\}$, and generally the letters ω to denote those of the complementary form $\tau\{11\bar{1}\}$.

35. *The cube*, $\{100\}$. The cube, the faces of which are given in table **a**, Art. 15, is a possible form of this class. The faces are each parallel to two dyad axes, and must therefore occur in pairs of parallel faces which truncate opposite edges of the tetrahedron. For each face is perpendicular to a dyad axis, which bisects the angle between the normals to the two tetrahedral faces meeting in that edge through the middle point of which the axis passes.

Figs. 267 and 268 represent two possible crystals in which the cube is combined with a tetrahedron. In the first the tetrahedron $o=\tau\{111\}$ is the predominant form, and the faces of $\{100\}$ are narrow planes truncating the edges. In the second the cubic faces are large, and those of the tetrahedron $\tau\{11\bar{1}\}$ are small. In drawing such combinations the cube should be first constructed.

Equal lengths, measured from alternate coigns, are then cut off by proportional compasses on each cubic edge; and the pairs of points

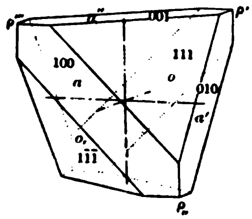


FIG. 267.

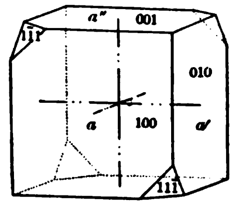


FIG. 268.

which give edges parallel to the face-diagonals are afterwards joined, and complete the figure.

36. *The rhombic dodecahedron*, {110}. If a face is drawn through an edge of the cube making equal angles with the cubic faces forming the edge, it must meet the two dyad axes perpendicular to that edge at equal distances from the origin. Suppose the face to be drawn through the upper horizontal cubic edge parallel to OY; then its symbol is (101). By a semi-revolution about OX this face is brought into the position of one through the lower horizontal cubic edge and meeting OZ at the same distance from the origin as the first, but on the negative side; its symbol is therefore $(10\bar{1})$. Again, a semi-revolution about OY brings these two faces into the positions of parallel faces. We therefore have the four tautozonal faces: 101, $10\bar{1}$, $\bar{1}0\bar{1}$, $\bar{1}01$; and the angles they make with one another are 90°. By rotations of 120° about one of the triad axes, the axis OY is brought in succession into the position of OZ and OX, and the four faces just given must be repeated in two sets of four similar faces passing through the cubic edges parallel to the axes of OZ and OX. The form so constructed is the rhombic dodecahedron {110}, which was fully described in Art. 17. As was there shown the edges are parallel to the triad axes; and Fig. 269 is most easily drawn by determining the

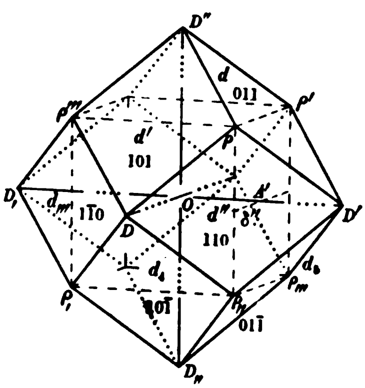


FIG. 269.

points D, D', &c., on the axes at distance $2a$ from the origin, and joining these points to the four nearest coigns of the cube.

As the elements of symmetry characteristic of this class occur in all classes of the cubic system, it follows that the cube and rhombic dodecahedron are common to all classes of the system.

37. *The dihedral pentagonal dodecahedron*, $\tau\{hk0\}$. When a face, drawn through a cubic edge is unequally inclined to the cubic faces through that edge, it meets the axes perpendicular to the edge at unequal distances from the origin. Thus, let the face be $H\rho\rho_{,,}$ of Fig. 270: its symbol is $(hk0)$, where h is greater than k. A semi-revolution about OX brings the face into the position $H\rho'''\rho_{,}$, in which it meets OY at the same distance on the negative side of the origin; and the new face is $(h\bar{k}0)$. Again, a semi-revolution about the dyad axis OZ, parallel to the two faces, brings them into positions parallel to their original ones. The new pair of faces therefore have the symbols $(\bar{h}\bar{k}0)$, $(\bar{h}k0)$.

Owing to the triad axes, this set of tautozonal faces is repeated in two other similar sets of four faces parallel, respectively, to OX and OY. We have also seen in Art. 19 that the faces interchangeable by rotations of 120° about a triad axis have their symbols in cyclical order. Hence $(hk0)$, $(0hk)$, $(k0h)$ constitute a triad meeting at an apex on $O\rho$, and the two new faces must belong to the form $\tau\{hk0\}$; which therefore consists of the twelve faces:

$$\left.\begin{matrix} hk0 & h\bar{k}0 & \bar{h}\bar{k}0 & \bar{h}k0 \\ 0hk & 0h\bar{k} & 0\bar{h}\bar{k} & 0\bar{h}k \\ k0h & \bar{k}0h & \bar{k}0\bar{h} & k0\bar{h} \end{matrix}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\text{(k)}.$$

On comparing this table with (d) of Art. 18, it will be seen that twelve of the faces of the tetrakis-hexahedron have been retained, viz. those which have their symbols in the same cyclical order. The form, Fig. 270, can be most easily drawn from Fig. 231. Of the four faces meeting at a tetragonal coign H, where $AH = A'H' = \&c. = ka \div h$, two only are retained which intersect in an edge $e'e_{,}$, parallel to the vertical axis. Similarly, through H', H'', &c., horizontal edges have to be drawn parallel respectively to the axes of X and Y. The above edges are often called the *cubic edges*, for they are parallel to those of the cube. The cubic coigns ρ remain coigns of $\tau\{hk0\}$, for three faces of Fig. 231 still meet at these coigns

in the new figure. It is, therefore, only necessary to find the points e, e', &c., to be able to complete the figure. The trace $H\delta'$ in the plane XOY of the face hkl, drawn to the middle point δ' of $pp_{,,}$ is produced in Fig. 270 to meet the horizontal cubic edge through H' in the point marked e'. Similarly, $H'\delta$ is prolonged to meet the cubic edge through H'' in e, and $H''\delta$ to meet the vertical cubic edge through H in e'; and so on for all the homologous points e. Each of these points is then joined to the two nearest origins of the cube, and the figure is completed.

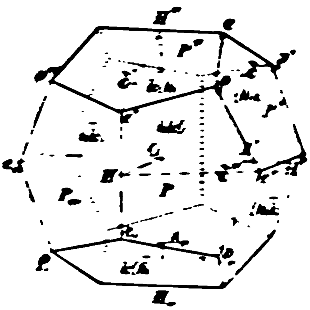


FIG. 270.

Each face is a similar and equal pentagon, four sides of which are like edges, whilst the fifth—the cubic edge—differs from the others. The form is generally called the pentagonal dodecahedron: but, when it has to be distinguished from the less common form described in Art. 43, it may be called the *dihedral pentagonal dodecahedron*: for the faces occur in pairs meeting in cubic edges, and also in pairs which are parallel. By a rotation of 120° about Op the three faces meeting at p are interchanged: and the edges pe, pe', pe'' must be similar, and the angles over them equal. Again, by rotation through 180° about the dyad axis OX, the origins p and p are interchanged: and the triad of edges meeting at p must be similar and equal to those meeting at p, and the angles over both sets of edges must be all equal. But the form consists of pairs of parallel faces: hence, the edges at the origin p' must be similar and parallel to those at $p_{,,}$ and the angles over them must be equal. Hence, the edges at p' are similar to those at p, and the angles over them are all equal. Therefore $p'e = pe = pe'' =$ &c. The same is true of each of the other faces. We shall denote the angle between pairs of faces meeting in an edge, such as pe, by C, and that over a cubic edge, such as $eHe_{,,}$ by D. The angles D and C can never be equal, as can be easily proved from expressions 17 and 18 [illegible] can be obtained either from the general expression [illegible] geometry of the figure.

$$OH' = ka - h, \text{ and } A\delta = a \text{ (see Art. 18)};$$

$$OH'\delta = A\delta - AH' = h - k.$$

But if A and P of Fig. 271 are the poles (100) and ($hk0$),

$$\therefore\ AP = 90° - OH\delta'' = 90° - OH'\delta,$$

$$\therefore\ \tan AP = \cot OH'\delta = k \div h.$$

This is the same expression as was shown in Chap. XIV, Art. 7, to hold for the angle Ag in a tetragonal zone.

But $\cos D = \cos 2AP = \cos^2 AP - \sin^2 AP = \dfrac{\cos^2 AP - \sin^2 AP}{\cos^2 AP + \sin^2 AP}$, since $\cos^2 AP + \sin^2 AP = 1$.

Dividing the numerator and denominator of the last ratio by $\cos^2 AP$, and replacing $\tan AP$ by its value $k \div h$, we have

$$\cos D = \frac{1 - \tan^2 AP}{1 + \tan^2 AP} = \frac{1 - k^2 \div h^2}{1 + k^2 \div h^2} = \frac{h^2 - k^2}{h^2 + k^2} \ldots\ldots (17).$$

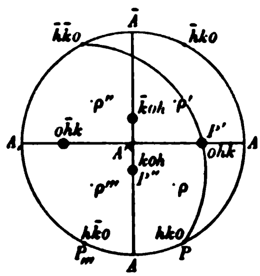


FIG. 271.

Again, in Fig. 271, which shows the poles P of the form $\tau\{hk0\}$, we have the right-angled triangle $PA'P'$; and

$\cos U = \cos PP' = \cos A'P' \cos A'P = \cos AP \sin AP$;

since $\qquad A'P' = AP$, and $A'P = 90° - AP$.

Transforming the equation, replacing $\sec^2 AP$ by $1 + \tan^2 AP$, and $\tan AP$ by $k \div h$, we have

$$\cos U = \cos^2 AP \tan AP = \frac{\tan AP}{\sec^2 AP} = \frac{\tan AP}{1 + \tan^2 AP} = \frac{hk}{h^2 + k^2} \ldots (18).$$

The following are the angles for a few of the forms of most common occurrence:

$\tau\{hk0\}$	$AP = 100 \wedge hk0$	$D = hk0 \wedge h\bar{k}0$	$U = hk0 \wedge k0h$
$\tau\{310\}$	18° 26′	36° 52′	72° 32·5′
$\tau\{210\}$	26 34	53 8	66 25
$\tau\{320\}$	33 41·4	67 22·8	62 31
$\tau\{430\}$	36 52	73 44	61 19.

38. Suppose the cubic edges through H, H', H'' to be drawn parallel to the axes OY, OZ and OX respectively, that is, in directions at right angles to those in the form $\tau\{hk0\}$. Let also new lines of construction $H\delta'$, $H'\delta''$, &c., be each prolonged to meet the cubic edges in the same axial plane in new points which may be denoted by ϵ', &c.; and let these points be joined, as before, to the nearest coigns ρ of the cube. The drawing represents the complementary form $\tau\{kh0\}$, the faces of which have their symbols in the

reverse cyclical order to those of $\tau\{hk0\}$. By rotation through 90° about one of the dyad axes the form $\tau\{kh0\}$ can be brought into a position geometrically identical with $\tau\{hk0\}$, and the two complementary forms are tautomorphous. The angles over corresponding edges are therefore equal for the same values of h and k.

39. We shall now show that the regular pentagonal dodecahedron of geometry is not an admissible form of this kind. For, in the regular polyhedron, the edges are all equal, and it may be regarded as the particular case of $\tau\{hk0\}$ in which the angles over all the edges become equal. Hence, $D=U$; and $\cos D=\cos U$.

$$\therefore \frac{h^2-k^2}{h^2+k^2}=\frac{hk}{h^2+k^2};$$
$$\therefore h^2-k^2=hk;$$
$$\therefore \left(\frac{h}{k}\right)^2-\frac{h}{k}=1;$$
$$\therefore \left(\frac{h}{k}\right)^2-\frac{h}{k}+\left(\frac{1}{2}\right)^2=\frac{5}{4};$$
$$\therefore \frac{h}{k}=\frac{1\pm\sqrt{5}}{2};$$

—a value involving a surd. The indices are therefore not rational; and the form is inadmissible as a particular case of this type of pentagonal dodecahedron.

40. By drawing pairs of faces through each edge of the tetrahedron to meet the two dyad axes, which do not intersect that edge, at distances $la\div h$ from the origin, we obtain special forms, corresponding to those similarly derived in Arts. 20 and 21 from the octahedron. The forms are known as the triakis-tetrahedron $\tau\{hhl\}$, when $h<l$; and as the deltoid dodecahedron $\tau\{hhl\}$, $h>l$.

The triakis-tetrahedron, $\tau\{hhl\}$, $h<l$. Suppose the tetrahedron $\rho'\rho_{,,}\rho'''\bar{\rho}$, Fig. 272, to be constructed, and let the axes intersect the edges at A, A', A'', &c., at distances a from the origin. Points L, L', L'', &c., are now taken on the axes at distances $la\div h$ from the origin. Then LL' is parallel to AA', and AA' is parallel to $\rho'\rho'''$. The face through L and $\rho'\rho'''$ must therefore pass through L', and the lines $L\rho'$, $L'\rho'''$ must both lie in the face. But this face meets the axes at distances $la\div h : la_{,}\div h : a_{,,}$, and a

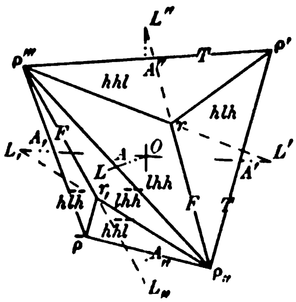


Fig. 272.

parallel face has the intercepts $a \div h : a_{,} \div h : a_{,,} \div l$; its symbol is therefore (hhl). Similarly, the face through L and $\rho'\rho_{,,}$ also passes through L'', and contains the lines $L\rho'$, $L''\rho_{,,}$: the intercepts on the axes are $la \div h : a_{,} : la_{,,} \div h$, and the symbol is (hlh). But the line $L\rho'$ is common to both faces, and is therefore one of the edges of the form. For the same reason, $L'\rho'''$, $L''\rho_{,,}$, &c., are all edges of the form. Hence the figure is drawn by joining each coign of the tetrahedron to the points L which are most distant from it. The figure consists of an equal and similar trigonal pyramid placed on each face of the tetrahedron, and having its base congruent with the tetrahedral face on which it stands. The form $\tau\{hhl\}$ consists of the twelve following faces:

$$\left.\begin{matrix} hhl & hlh & lhh \\ h\bar{h}\bar{l} & h\bar{l}\bar{h} & l\bar{h}\bar{h} \\ \bar{h}h\bar{l} & \bar{h}l\bar{h} & \bar{l}h\bar{h} \\ \bar{h}\bar{h}l & \bar{h}\bar{l}h & \bar{l}\bar{h}h \end{matrix}\right\} \text{........................ ....(1).}$$

The limits of the form are given: (i) by making $l = h$, when the points L coincide with the points A in the tetrahedral edges: the height of the pyramid is then nil; and the form coincides with the tetrahedron. (ii) The other limit is given by removing the points L to infinity. The index h must then become zero, and the face (lhh) becomes (100): the limiting form is therefore the cube.

The positions of the poles of $\tau\{hhl\}$ are given by the three poles t of Fig. 239 which lie in the alternate octants above the paper in which o and o'' lie; those below the paper would be given by circlets surrounding the corresponding poles in the two other octants. The form has two different angles between pairs of adjacent faces meeting in edges; viz. angles $F = lhh \wedge hlh$, over edges joining a trigonal coign ρ of the tetrahedron to the apex r of the pyramid, and angles T over tetrahedral edges.

If t is the pole (lhh), then, as was shown in Art. 21,

$$\tan At = h\sqrt{2} \div l = \frac{h}{l} \tan Ao \text{ ....(19).}$$

But, Figs. 239 and 272, $T = lhh \wedge l\bar{h}\bar{h} = 2At$.

$$\cos T = \cos 2At = \cos^2 At - \sin^2 At = \frac{1 - \tan^2 At}{1 + \tan^2 At} \text{ (see Art. 37),}$$

$$= \text{(from (19))} \frac{l^2 - 2h^2}{l^2 + 2h^2} \text{........................(20).}$$

The angle F is most easily obtained from expression (3), which gives

$$\cos F = \cos (lhh \wedge hlh) = \frac{2hl + h^2}{2h^2 + l^2} \quad\text{..............(21).}$$

Hence the angles for the following forms are:

$\tau\{lhh\}$	$At = 100 \wedge lhh$	$T = lhh \wedge l\bar{h}\bar{h}$	$F = lhh \wedge hlh$
$\tau\{311\}$	25° 14′	50° 29′	50° 29′
$\tau\{211\}$	35 16	70 32	33 33

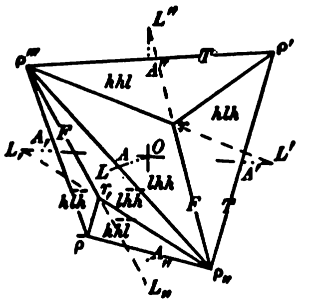


FIG. 272.

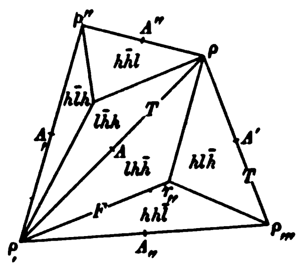


FIG. 273.

The complementary form $\tau\{h\bar{h}l\}$, Fig. 273, is tautomorphous, and offers no fresh characters. It can be constructed in a manner similar to $\tau\{hhl\}$; the tetrahedron $\tau\{11\bar{1}\}$ being taken as the auxiliary form.

41. *The deltoid dodecahedron,* $\tau\{hhl\}$, $h > l$. This form can be constructed from the auxiliary tetrahedron $\tau\{111\}$ in a manner similar to that employed in Art. 40 to give the triakis-tetrahedron. Let, in Fig. 274, the auxiliary tetrahedron be $\rho'\rho'''\rho_{\prime\prime}\bar{\rho}$; and on the axes OA, OA', OA'', &c., let the points L, L', L'', &c., be taken at distances $la + h$ from the origin. Since these points and the edges of the figure lie within the auxiliary tetrahedron, the axes and several other lines of construction are omitted, as they would unduly complicate the diagram. The line $L'L''$ is parallel to $A'A''$, and the latter is parallel to $\rho_{\prime\prime}\rho'''$. Similarly, LL', LL'', &c., are parallel to the opposite edges $\rho'\rho'''$, $\rho'\rho_{\prime\prime}$, &c., of the tetrahedron. Hence the face through L and $\rho'\rho'''$ also passes through L', and contains the lines joining L and L' to the opposite tetrahedral coigns ρ' and ρ'''. Similarly, the plane through L' and $\rho'''\rho_{\prime\prime}$ also contains L'' and the lines $L'\rho'''$, $L''\rho''$. The line $L'\rho'''$ is common to the two faces, and is an edge of the form. For the same reason

$L''\rho_{,,}$, $L\rho'$, &c., are similar edges of the form. They meet in triads at trigonal coigns r.

Again, the line $L\rho'''$ is common to the faces LrL' and $Lr_{,}L_{,,}$, for a face has to be drawn through L and $\rho'''\bar{\rho}$ which also passes through $L_{,,}$. The line $L\rho'''$ has therefore to be produced on the side remote from ρ''' to form the edge. The similar lines $L'\rho'$, $L''\rho'$, &c., have to be produced, each away from the tetrahedral coign; and these edges meet in triads at the trigonal coigns R.

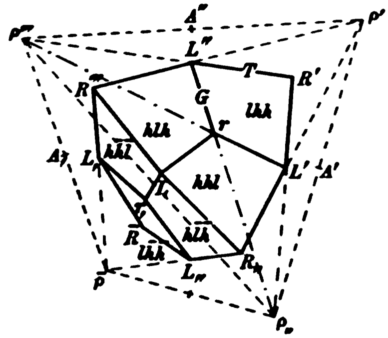


FIG. 274.

Each face is bounded by two different pairs of like edges, each pair of like edges meeting at a dissimilar trigonal coign, and each pair of unlike edges meeting at a coign on a dyad axis. Since the face approximates in shape to a delta, the form is called the *deltoid dodecahedron*. The general symbols of the faces are given in table l, but the relative magnitude of the equal indices to the third is altered, the latter being the least. We give the symbols of the faces of the particular case $\tau\{221\}$ in the following table:

$$\left.\begin{array}{cccc} 221 & 2\bar{2}\bar{1} & \bar{2}2\bar{1} & \bar{2}\bar{2}1 \\ 122 & 1\bar{2}\bar{2} & \bar{1}2\bar{2} & \bar{1}\bar{2}2 \\ 212 & 2\bar{1}\bar{2} & \bar{2}1\bar{2} & \bar{2}\bar{1}2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(\mathrm{m}).$$

The limits of the form are easily determined. i. The points L cannot be further from the origin than the points A: at this limit the three faces meeting at a coign r coalesce in the face of the tetrahedron $\tau\{111\}$ overlying them. ii. The limit in the other direction is attained when the points L, L', &c., reach the origin. But $OL = la \div h$, and at the limit $l = 0$: the face (lhh) then becomes (011). Or it may be stated thus. Each face is parallel to the plane

passing through a tetrahedral edge and the origin: this plane, containing an axis of reference and two triad axes is parallel to two faces of the rhombic dodecahedron {110}, which is the limiting form.

In a stereogram the poles lie on the zone-circles [Ao], &c., between the tetrahedral poles o and the adjacent poles d of the rhombic dodecahedron. Thus the poles p, which lie in the alternate octants containing o and o'' of Fig. 236, are those above the plane of the primitive; those below the paper would be given by circlets in the two remaining octants.

The form has two different angles, viz. $G = hlh \wedge hhl$ over edges joining the obtuse trigonal coigns r to the coigns L, and $T = hlh \wedge \bar{h}\bar{h}\bar{l}$ over edges joining the coigns L to the acute trigonal coigns R. The form {110} is the limiting case in which both these angles become equal to 60°. The expressions for the cosines of these angles are most easily found from (3), or they may be deduced from the geometry of the figure; they are:

$$\cos G = \cos (hlh \wedge hhl) = \frac{h^2 + 2hl}{2h^2 + l^2} \quad \text{(22)}.$$

$$\cos T = \cos (hlh \wedge \bar{h}\bar{h}\bar{l}) = \frac{h^2 - 2hl}{2h^2 + l^2} \quad \text{(23)}.$$

For the following particular cases we have:

$\tau\{hhl\}$	$A''P = 001 \wedge hhl$	$G = hlh \wedge hhl$	$T = hlh \wedge \bar{h}\bar{h}\bar{l}$
$\tau\{332\}$	64° 46′	17° 20·5′	97° 50′
$\tau\{221\}$	70 32	27 16	90 0
$\tau\{331\}$	76 44	37 52	80 55

42. If the complementary tetrahedron $\tau\{11\bar{1}\}$ is taken, and the coigns joined to the points L, L', &c., in the manner described in Art. 41, we obtain the complementary deltoid dodecahedron $\tau\{hh\bar{l}\}$, Fig. 275, the faces of which are each parallel to one of the faces of $\tau\{hhl\}$. The two forms have therefore equal angles; they are also tautomorphous, for a rotation of 90° about one of the dyad axes will bring the faces of one of them into the same position as those of the other.

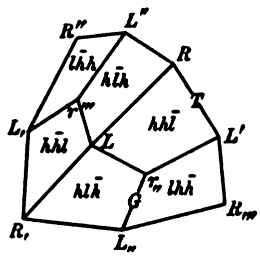


FIG. 275.

43. *The tetrahedral pentagonal dodecahedron*, $\tau\{hkl\}$. When the faces of the form occupy a general position, so that h, k and l are all unequal finite numbers, the form is bounded by twelve similar and interchangeable pentagons. The faces are grouped round each triad axis in sets of three, such that those meeting at opposite ends of each triad axis compose dissimilar trigonal coigns, which are respectively denoted by ρ and R in Figs. 276 and 278. The three faces meeting at a trigonal coign are similar and interchangeable, and have their symbols in cyclical order: thus, the faces meeting at ρ in Fig. 276 are (hkl), (lhk) and (klh). The faces are also grouped in pairs, each of which meets in an edge—the single edge—passing through one of the points H, where OH is the least intercept on the axes: each of these edges is bisected by, and at right angles to, the axis through the point. Thus the face (hkl) is repeated in a face $(h\bar{k}\bar{l})$; and, h being the greatest index, these are adjacent faces intersecting in a line at right angles to OX: when the form is, like the figure, equably developed, the edge will also be bisected at H. But $(h\bar{k}\bar{l})$ also meets $(kl h)$, and they are two faces of a triad meeting at a trigonal coign R'''. The third face must have its indices in the same cyclical order, and must meet OY at the point $H_{,}$, for the triad axis OR''' interchanges the three points H, $H_{,}$ and H'': hence the face has the symbol $(\bar{l}\bar{h}k)$. The same process can be repeated with respect to each set of faces; so that the form $\tau\{hkl\}$ consists of the faces:

$$\left.\begin{matrix} hkl & h\bar{k}\bar{l} & \bar{h}\bar{k}l & \bar{h}k\bar{l} \\ lhk & l\bar{h}\bar{k} & \bar{l}\bar{h}k & \bar{l}h\bar{k} \\ klh & k\bar{l}\bar{h} & \bar{k}\bar{l}h & \bar{k}l\bar{h} \end{matrix}\right\}\ldots\ldots(\text{n}).$$

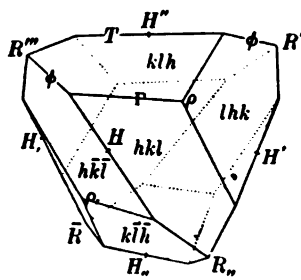


FIG. 276.

On referring to table g, of Art. 23, the reader will perceive that geometrically the form consists of that half of the faces of $a\{hkl\}$, which meet at coigns ρ in alternate octants, and have their symbols in the same cyclical order. The second half of the faces of $a\{hkl\}$ would give rise to a similar pentagonal dodecahedron $\tau\{lk\bar{h}\}$, which is tautomorphous with $\tau\{hkl\}$; for the faces of $a\{hkl\}$ in adjacent octants are interchangeable by a rotation of 90° about each of the axes of reference. Again, it was seen in Art. 29 that $a\{hkl\}$ consists geometrically of one-half of the faces

in the form $\{hkl\}$. Hence $\tau\{hkl\}$ has one-fourth of the number of faces in this latter form, and was formerly regarded as a tetartohedral form; the poles of the form are shown in Fig. 277.

The five edges of each face consist of a pair of like edges meeting at a trigonal coign ρ, of a second pair of like edges meeting at a trigonal coign R, and of the single edge passing through the point H on the axis of reference. The edges meeting at ρ and R are never similar to one another or to the single edge; and the angles over the dissimilar edges are always different. Hence we have three different angles between adjacent faces, which we shall denote by Γ, ϕ and T; Γ being $hkl \wedge klh$, ϕ being $klh \wedge h\bar{k}\bar{l}$, and T being $hkl \wedge h\bar{k}\bar{l}$. Expressions for the cosines of these angles can be easily obtained from expression (3), and are:

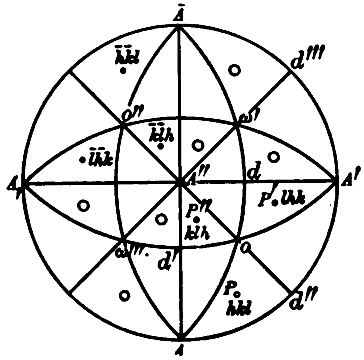


FIG. 277.

$$\left.\begin{aligned}\cos\Gamma &= \cos(hkl \wedge klh) = \frac{hk + kl + lh}{h^2 + k^2 + l^2},\\ \cos\phi &= \cos(klh \wedge h\bar{k}\bar{l}) = \frac{hk - kl - lh}{h^2 + k^2 + l^2},\\ \cos T &= \cos(hkl \wedge h\bar{k}\bar{l}) = \frac{h^2 - k^2 - l^2}{h^2 + k^2 + l^2}\end{aligned}\right\}\dots\dots\dots\dots (24).$$

44. From $\alpha\{lkh\}$ of Art. 24 we can also obtain two similar pentagonal dodecahedra, $\tau\{lkh\}$ and $\tau\{hk\bar{l}\}$; of which the latter, shown in Fig. 278, has the faces:

$$\left.\begin{matrix} hk\bar{l} & h\bar{k}l & \bar{h}kl & \bar{h}\bar{k}\bar{l} \\ lh\bar{k} & l\bar{h}k & \bar{l}hk & \bar{l}\bar{h}\bar{k} \\ kl\bar{h} & k\bar{l}h & \bar{k}lh & \bar{k}\bar{l}\bar{h} \end{matrix}\right\}\dots\dots(o).$$

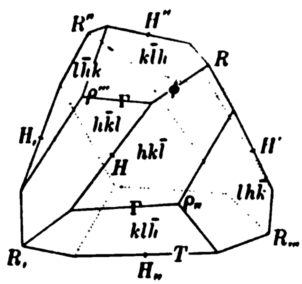


FIG. 278

The two dodecahedra, $\tau\{lkh\}$ and $\tau\{hk\bar{l}\}$, are tautomorphous, for each of them consists geometrically of those faces of $\alpha\{lkh\}$ which are situated

in alternate octants; and all the faces of $\alpha\{lkh\}$ can be interchanged by successive rotations of 90° about any of the axes of reference. But neither $\tau\{hk\bar{l}\}$ nor $\tau\{lkh\}$ can be brought into a position of congruence with $\tau\{hkl\}$ or $\tau\{lk\bar{h}\}$. The dodecahedron $\tau\{hk\bar{l}\}$, Fig. 278, may be said to be the inverse of $\tau\{hkl\}$ of Fig. 276; and vice versâ. For, the numerical values of the indices being the same, the faces of the one are parallel to those of the other; and parallel faces can only be interchanged by the axes of symmetry of this class, when the former are parallel to one of the dyad axes. Again, $\tau\{hkl\}$ and $\tau\{hk\bar{l}\}$, are enantiomorphous, for the faces of the one are reflexions of those of the other in the axial planes. Crystals of this class may be expected to rotate the plane of polarization of a beam of plane-polarised light traversing them: this has been established for crystals of sodium chlorate and sodium bromate, but not for all the substances placed in the class.

45. If the regular pentagonal dodecahedron of geometry can be a particular case of this form, the angles Γ, ϕ and T must be all equal. Equating the expressions for $\cos\Gamma$ and $\cos\phi$ given in (24), we have

$$hk+kl+lh=hk-kl-lh;$$

$$\therefore (h+k)\,l=0.$$

This equation is satisfied if $l=0$, or if $h+k=0$. The first indicates that the symbol must be $\tau\{hk0\}$, a case already discussed in Art. 39. When $h+k=0$, the form has the symbol $\tau\{h\bar{h}l\}$, and is a triakis-tetrahedron (Art. 40) or a deltoid dodecahedron (Art. 41), and the faces are not pentagons.

It follows that the regular pentagonal dodecahedron is not admissible as a form of this type.

46. Crystals of the following substances belong to this class:

Substance.	*Chemical composition.*	*Forms and combinations.*
Barium nitrate	$Ba(NO_3)_2$	*oω, aoω, aoωt*, &c.
Strontium nitrate	$Sr(NO_3)_2$	*oω, oωa.*
Lead nitrate	$Pb(NO_3)_2$	*oω, oωae* $=\tau\{210\}$, *oωaep* $=\{10\ 5\ 6\}$.
Sodium chlorate	$NaClO_3$	*adop* $=\tau\{210\}$, Fig. 280(*a*); *adop* $=\tau\{120\}$, Fig. 280(*b*); *o*.
Sodium bromate	$NaBrO_3$	*aoωd.*
Sodium uranyl acetate	$NaUO_2(C_2H_3O_2)_3$	*od.*
Sodium sulphantimonate	$Na_3SbS_4 . 9H_2O$	*oωde, oωde,* $=\tau\ \{12\ 0\}$.
Sodium strontium arsenate	$NaSrAsO_4 . 9H_2O$	*aωde, aωde,*.

Barium nitrate. Fig. 279 represents a plan on the plane YOZ of a crystal showing an exceptionally large number of different forms; namely: $a\{010\}$, $o=\tau\{1\bar{1}1\}$, $l=\tau\{131\}$, $t=\tau\{214\}$, $n=\tau\{351\}$, $h=\tau\{2\bar{1}4\}$, $s=\tau\{2\bar{1}1\}$.

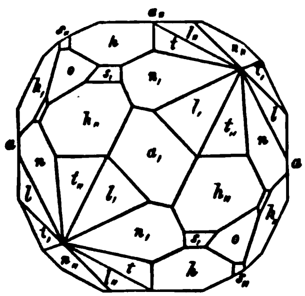


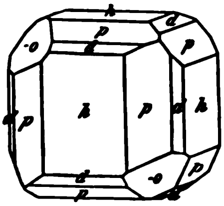
FIG. 279.

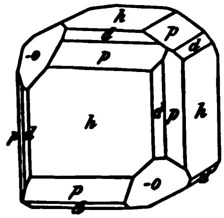
Dr. Wulff (Groth's *Zeitsch. f. Kryst. u. Min.*, IV, 122, 1879) has exhaustively studied the crystals of barium, strontium and lead nitrates. He observed that when crystals of barium nitrate are deposited from pure aqueous solutions the cube predominates. The cubic coigns are modified by faces of the complementary tetrahedra of unequal size, and of which the largest were taken to be $\tau\{1\bar{1}1\}$: the crystals also showed sometimes faces of $\tau\{120\}$, and sometimes elongated triangular faces of a tetrahedral pentagonal dodecahedron λ. The positions of the faces λ could not be fully determined, but they approached those of $\tau\{421\}$. When sodium nitrate is present in the solution, crystals are deposited which resemble regular octahedra, having their coigns modified by faces of the cube $\{100\}$ and by very small faces of the forms λ and $\tau\{120\}$. From solutions containing also potassium nitrate and sugar, he obtained more complicated crystals in which the faces of the cube and tetrahedra were absent: the predominant form was $\tau\{211\}$, and the forms λ, $\tau\{120\}$ and $\tau\{2\bar{2}1\}$ appeared as small faces modifying the acute coigns of the triakis-tetrahedron. In all the above crystals $\tau\{120\}$ and the tetrahedral pentagonal dodecahedron λ occupied the same relative positions, so that the λ faces were nearer in position to the faces of the complementary form $\tau\{210\}$ than to those of $\tau\{120\}$.

The crystals manifest a weak, but anomalous, double refraction, and no rotation of the plane of polarization has been established. Crystals of *strontium nitrate* and of *lead nitrate* have very nearly the same characters: they also manifest anomalous double refraction but no rotation of the plane of polarization.

Sodium chlorate. When obtained from aqueous solutions at ordinary temperatures, the crystals are cubes $h\{100\}$, having their edges and coigns modified: (i) by narrow faces $d\{110\}$, $p=\tau\{210\}$ and tetrahedra $o=\tau\{1\bar{1}1\}$, Fig. 280 (a); or (ii) by $d\{110\}$, $p=\tau\{120\}$, and $o=\tau\{1\bar{1}1\}$, Fig. 280 (b). The complementary pentagonal dodecahedra, $\tau\{210\}$ and $\tau\{120\}$, never occur together on the same crystal. The crystals represented by Fig. 280 (a), rotate the plane of polarization of a plane-polarised beam to the left; *i.e.* to an observer receiving the light transmitted by a plate of the crystal the rotation is counter-clockwise. The crystals shown in Fig. 280 (b) rotate the plane of polarization to the right; *i.e.* under similar circumstances to those of the first case the rotation is clockwise. This rotatory power is

possessed by all simple (*i.e.* untwinned) crystals of the substance, and is invariably associated with the arrangement of the faces given above.





(*a*) FIG. 280. (*b*)

The rule connecting the direction of rotation with the facial development may be given as follows. The cube being placed in the usual position with the tetrahedron as $\tau\{1\bar{1}1\}$, then in a lævogyral crystal the pentagonal dedecahedron is $\tau\{210\}$, and its vertical face (210) is to the left of (110); in a dextrogyral crystal the form is $\tau\{120\}$, and its vertical face (120) is to the right of (110).

General forms, *i.e.* forms having finite unequal indices, have never been observed. But we may anticipate that, if such forms are discovered, lævogyral crystals will have $\tau\{hkl\}$ and $\tau\{hk\bar{l}\}$ in which $h>k>l$; for in these forms the limit, when $l=0$, is a pentagonal dodecahedron, $\tau\{hk0\}$, having the vertical face to the left of (110). Dextrogyral crystals may, on the other hand, be expected to show $\tau\{khl\}$ and $\tau\{kh\bar{l}\}$, the limiting form $\tau\{kh0\}$ having its vertical face to the right of (110).

The triad axes in untwinned crystals have been found to be pyro-electric axes (Friedel and Curie; *Bull. Soc. franç. de Min.* VI, p. 191, 1883).

IV. *Dyakis-dodecahedral class;* $\pi\{hkl\}$.

47. If a centre of symmetry is added to the assemblage of axes of symmetry characteristic of the last class, there must also be three planes of symmetry, each perpendicular to one of the dyad axes. These planes are parallel to the faces of the cube having its edges parallel to the dyad axes, and are called the cubic planes of symmetry, Π. The elements of symmetry of the class may be represented by:

$$4\rho,\ 3\delta,\ C,\ 3\Pi.$$

The arrangement of axes and parametral plane (111) of the last class is clearly not altered; and, as before,

$$a = b = c.$$

The general equations (2) and (3) of Art. 13 will therefore hold for the angular relations of faces in this class.

crystal is turned through 120° about the axis through the coign. The four edges meeting at a point H on an axis of reference consist of two dissimilar pairs, one pair of like edges lying in the same axial plane. Hence the crystal has three different angles between adjacent pairs of faces; viz. $\Gamma = hkl \wedge klh$ over edges converging to a triad coign, $D = hkl \wedge hk\bar{l}$ over the longer edges lying in the axial planes, and $W = hkl \wedge h\bar{k}l$ over the shorter edges lying in the axial planes. Expressions (25) for the cosines of these angles are easily obtained from (3); they are:

$$\left.\begin{aligned} \cos\Gamma &= \cos(hkl \wedge klh) = \frac{hk + kl + lh}{h^2 + k^2 + l^2}, \\ \cos D &= \cos(hkl \wedge hk\bar{l}) = \frac{h^2 + k^2 - l^2}{h^2 + k^2 + l^2}, \\ \cos W &= \cos(hkl \wedge h\bar{k}l) = \frac{h^2 - k^2 + l^2}{h^2 + k^2 + l^2}. \end{aligned}\right\} \quad \text{............(25).}$$

The values of the angles for some of the forms most commonly met with are:

$\pi\{hkl\}$	$AP = 100 \wedge hkl$	Γ	D	W
$\pi\{321\}$	36° 42′	38° 13′	31° 0′	64° 37′
$\pi\{421\}$	29 12	48 11	25 13	51 45
$\pi\{531\}$	32 19	48 55	19 28	60 56

51. By the law of rational indices, the faces having their symbols in the reverse cyclical order to those in $\pi\{hkl\}$ are possible, but they cannot occur as a part of this form. They however constitute a complementary form $\pi\{lkh\}$, Fig. 283, which can by a rotation of 90° about any one of the axes of reference be brought into a position identical with that of $\pi\{hkl\}$; for it was seen, in Art. 24, that a rotation of 90° about an axis of reference interchanges faces having their symbols in opposite cyclical orders. It follows, therefore, that the two forms are tautomorphous; and that the angles over corresponding edges are, for the same values of the indices, equal.

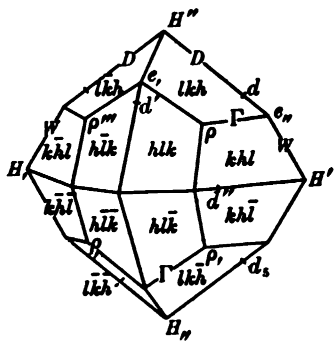


FIG. 283.

52. **Crystals of the following substances belong to this class:**

Substance.	*Chemical composition.*	*Forms and combinations.*
Pyrites	FeS_2	*a*, *o*, *e* = π{210}, {331}, *as* = π{321}, [*aes*, &c.
Cobaltine	CoAsS	*a*, *o*, *e*, *ae*, *oe*, *aoe*.
Smaltine	$CoAs_2$	*a*, *ao*, forms π{*hk*0} rare.
Sperrylite	$PtAs_2$	*a*, *ao*, *aeo*.
Skutterudite	$CoAs_3$	*on* = {211}, *ond*.
Hauerite	MnS_2	*o*, *oe*.
Potash Alum	$K_2Al_2(SO_4)_4 . 24H_2O$	*o*, *a*, *oa*, *oda*, *oe*.
also the isomorphous salts of ammonium, &c.		
Stannic iodide	SnI_4	*oe*.

Pyrites. This very common mineral is found in crystals which are generally single forms or simple combinations. The cube is common; and the faces are often striated, as shown in Fig. 284. The striæ on adjacent faces meeting in a coign are parallel respectively to the edges meeting in it, so that a rotation of 120° about the triad axis interchanges the striæ. Another common form is the octahedron {111}, the faces of which are usually smooth and bright: occasionally they each have three sets of striæ, parallel to the lines in which the face would intersect the adjacent faces of π{210}. The pentagonal dodecahedron π{210} is fairly common; and the faces are usually striated: the crystals can be divided into two groups, in one of which the striæ are parallel to the cubic edges, in the other they are perpendicular to these edges. Rose discovered that the crystals of the two groups are thermo-electrically active; and J. Curie (*Bull. Soc. franç. de Min.* VIII, 127, 1885) has shown that the crystals striated parallel to the cubic edges are more positive than antimony, whilst those striated perpendicular to the cubic edges are about as negative as bismuth. Different portions of the same face are sometimes striated in directions at right angles to one another, and these portions are thermo-electrically different. A combination of the cube and π{210}, which is fairly common, is shown in Fig. 285: in such crystals each face of the cube is usually striated parallel to those faces of π{210} which are inclined to it at the least angle; thus (100) would be striated vertically, *i.e.* parallel to [*ae″*]. The relative dimensions of the two forms

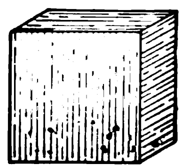


FIG. 284.

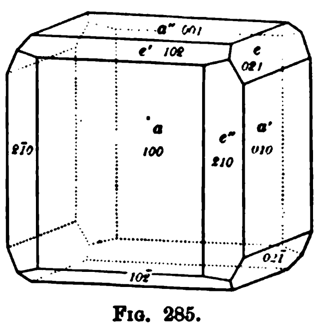


FIG. 285.

vary much. In drawing such a combination, the form $\pi\{210\}$ should be first completed, when the faces of the cube are easily introduced.

The combination of the octahedron {111} with $\pi\{210\}$ (slightly developed) shown in Fig. 286 is sometimes found in crystals of pyrites, but it is more frequent in crystals of cobaltine. The crystals of this latter mineral are also often found with the habit given in Fig. 285.

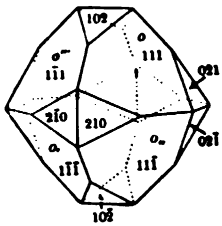


Fig. 286.

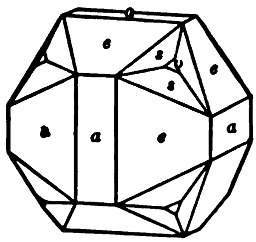


Fig. 287.

Fig. 287 represents a combination not infrequently observed in crystals of pyrites from Elba. The forms are $a\{100\}$, $o\{111\}$, $e=\pi\{210\}$, and $s=\pi\{321\}$. The dyakis-dodecahedron $\pi\{321\}$ also occurs as a single form in crystals from Elba. In drawing the combination it is best to construct $\pi\{321\}$ first, the octahedral faces can then be introduced; and the edges [*se*] are parallel to adjacent edges [*so*] (*o*, *s*, *e* being tautozonal), so that the figure is quickly completed.

V. *Hexakis-tetrahedral class;* $\mu\{hkl\}$.

53. To the assemblage of four triad and three dyad axes of Class III six planes of symmetry, Σ, may be added, which intersect in sets of three in each of the triad axes and in pairs in each of the dyad axes. The planes are parallel to pairs of faces of the rhombic dodecahedron, and are called the *dodecahedral* planes of symmetry.

The forms are not centro-symmetrical, for each of the planes of symmetry would in that case be associated with a dyad axis perpendicular to it, and we should have four axes of symmetry in a plane, inclined to one another at angles of 45°. The axis of symmetry perpendicular to this plane would then be a tetrad axis (Chap. IX, Prop. 11); and the elements of symmetry would be those characteristic of Class II. The triad axes are uniterminal, and have been shown to be pyro-electric axes in crystals of blende and boracite: the pyro-electricity can be tested by a simple method invented by MM. Friedel and Curie (*Bull. Soc. franç. de Min.* VI, p. 191, 1883).

The elements of symmetry of the class may be represented by: 4ρ, 3δ, 6Σ.

54. The cube, the rhombic dodecahedron and the tetrahedron of Class III belong also to this class. It is easy to see that these forms are geometrically symmetrical with respect to the planes of symmetry. As, moreover, the planes of symmetry are parallel to possible faces, viz. to those of the rhombic dodecahedron, we can in this class prove that the triad axes are possible zone-axes; for they are parallel to the edges of this form. The faces of the tetrahedron are also possible, for the edges are parallel to the normals of the planes of symmetry; and, as we saw in Chap. IX, Prop. 1, these normals are possible zone-axes.

The dyad axes are selected as axes of reference, and a face of the tetrahedron as parametral plane (111). Hence, as before, $a = b = c$; and the analytical relations established in Art. 13 hold for crystals of this class.

55. Again, the triakis-tetrahedron and the deltoid dodecahedron of Arts. 40 and 41 belong to this class. The student will perceive that Figs. 272, 273 and 274 are symmetrical with respect to planes passing each through an axis of reference and a triad axis. For instance, the plane through OL and Or of Fig. 288, identical geometrically with Fig. 272, bisects the planes $\rho_{,,}r\rho'''$, $\rho_{,,}r_{,}\rho'''$, and contains the edges $r\rho'$, $r_{,}\bar{\rho}$ formed by faces which are reciprocal reflexions in the plane. It is clear that the plane of symmetry repeats the point L' in L'', and vice versâ. Hence, the one face being (hlh), the other is (hhl). We have therefore the triad (hhl), (lhh), (hlh) in the first octant, symmetrical in pairs with respect to the planes of symmetry through Or and each of the axes OL, OL', OL''. Further, the second plane of symmetry through OL also passes through $\rho_{,,}\rho'''$, and the above triad of faces is repeated over the plane in a similar triad meeting at $r_{,}$. But this is the triad obtained by a rotation of 180° about the axis OX. Hence no new faces are introduced. The figures, having been fully discussed in Arts. 40 and 41, need no fresh description. When they are forms of this class, the symbols $\mu\{hhl\}$ are used. The symbols of the faces in $\mu\{hhl\}$ are given in

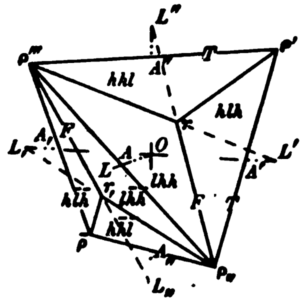


FIG. 288.

table l, p. 320: for the triakis-tetrahedron $h < l$, and for the deltoid dodecahedron $h > l$.

56. *The tetrakis-hexahedron*, $\{hk0\}$. In Chap. XIV, Art. 7 it was shown that a face $(hk0)$ parallel to a dyad axis, in which two planes of symmetry intersect at right angles, these angles being bisected by two dyad axes perpendicular to the first axis, is repeated in seven other faces. Their traces in the axial plane perpendicular to the first dyad axis are given by the lines forming the ditetragon of Fig. 177. Hence the faces, $hk0$, $kh0$, $k\bar{h}0$, $h\bar{k}0$, $\bar{h}\bar{k}0$, $\bar{k}\bar{h}0$, $k\bar{h}0$, $h\bar{k}0$, occur together in any form which has such an arrangement of elements of symmetry. The triad axes interchange the axes OX, OY and OZ; and at the same time any faces symmetrically related to them. Hence, there must be two other similar sets of eight faces parallel, respectively, to OX and OY and symmetrical with respect to the pairs of planes Σ through these axes. We, therefore, obtain a twenty-four faced figure, Fig. 289, geometrically identical with Fig. 231 described in Art. 18. The planes of symmetry pass each through a set of edges, such as $\rho\rho'$, $H\rho$, $H\rho_{,}$ of that figure. The Greek prefix is dropped before the symbol $\{hk0\}$ of the form, which includes the faces given in table d. The form is therefore common to Classes I, II and V.

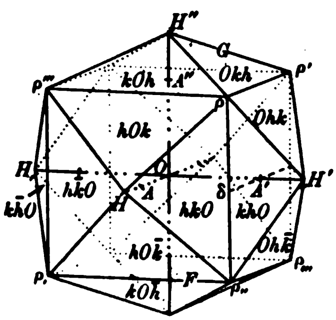


FIG. 289.

57. *The hexakis-tetrahedron*, $\mu\{hkl\}$. A face (hkl) having any general position, *i.e.* h, k and l being finite and unequal numbers, is associated with five other faces, the symbols of which are obtained by taking the indices in the two opposite cyclical orders. For rotations of 120° about the triad axis bring the face to the positions given by (lhk) and (klh). Again, the faces are arranged in pairs equally inclined to each of the three planes of symmetry traversing the octant which contains the face. Thus (hkl) being one of the faces, (khl) is its reciprocal reflexion in the plane of symmetry through OZ and bisecting the angle between the axes of X and Y: similarly, (lhk) and (hlk), (klh) and (lkh) are pairs of faces which are reciprocal reflexions in the same plane. But (khl), (hlk), (lkh)

are in cyclical order—that opposed to the first; and there can be no other homologous faces meeting at the same coign on the triad axis. The six faces meeting at a ditrigonal coign in one octant are:

$$hkl \quad lhk \quad klh \quad khl \quad lkh \quad hlk.$$

These six faces are repeated in a similar set of six faces by each of the dyad axes forming the edges of the octant; and the four sets occupy alternate octants. Now rotation of 180° about an axis of reference changes the signs of the indices on the two other axes but not their order. The form $\mu\{hkl\}$, Fig. 290, includes therefore the faces:

$$\left.\begin{array}{llllll} hkl & lhk & klh & lkh & hlk & khl \\ h\bar{k}\bar{l} & l\bar{h}\bar{k} & k\bar{l}\bar{h} & l\bar{k}\bar{h} & h\bar{l}\bar{k} & k\bar{h}\bar{l} \\ \bar{h}k\bar{l} & \bar{l}h\bar{k} & \bar{k}l\bar{h} & \bar{l}k\bar{h} & \bar{h}l\bar{k} & \bar{k}h\bar{l} \\ \bar{h}\bar{k}l & \bar{l}\bar{h}k & \bar{k}\bar{l}h & \bar{l}\bar{k}h & \bar{h}\bar{l}k & \bar{k}\bar{h}l \end{array}\right\} \text{..........} (q).$$

The poles of the faces lie in sets of six on small circles

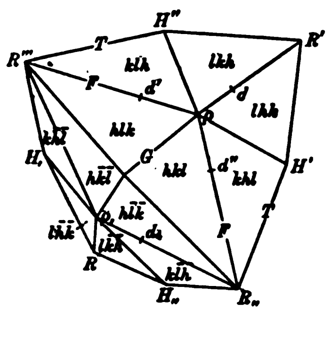


FIG. 290.

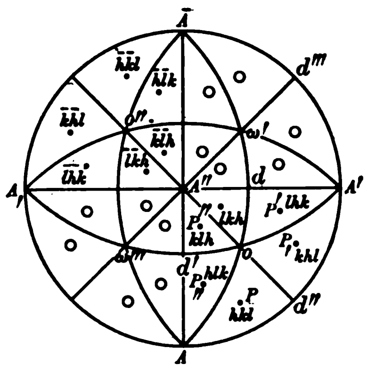


FIG. 291.

surrounding the tetrahedral poles o situated in alternate octants: they are shown in the stereogram Fig. 291.

The form may be regarded as the result of placing on each face of the tetrahedron a similar ditrigonal pyramid, and hence it is called the *hexakis-tetrahedron*. The faces are all equal and similar scalene triangles, and the form has three different angles, viz. those over the dissimilar edges of each face: we shall denote them by

the letters, F, G and T. Expressions for their cosines are obtained from (3); they are

$$\left.\begin{aligned}\cos F &= \cos(hkl \wedge khl) = \frac{2hk + l^2}{h^2 + k^2 + l^2},\\ \cos G &= \cos(hkl \wedge hlk) = \frac{h^2 + 2kl}{h^2 + k^2 + l^2},\\ \cos T &= \cos(hkl \wedge h\bar{l}\bar{k}) = \frac{h^2 - 2kl}{h^2 + k^2 + l^2}.\end{aligned}\right\} \qquad (26).$$

For the following particular cases, the angles are:

$\mu\{hkl\}$	F	G	T
$\mu\{321\}$	21° 47′	21° 47′	69° 5′
$\mu\{421\}$	17 45	35 57	55 9
$\mu\{531\}$	27 40	27 40	57 7

58. By raising similar ditrigonal pyramids, one on each face of the tetrahedron $\mu\{1\bar{1}1\}$, we obtain a complementary hexakis-tetrahedron $\mu\{h\bar{k}l\}$, Fig. 292, the faces of which are parallel respectively to those of $\mu\{hkl\}$. The two forms are tautomorphous, for a rotation of 90° about one of the axes of reference interchanges faces in adjacent octants. The angles over the corresponding edges are equal for the same numerical values of the indices.

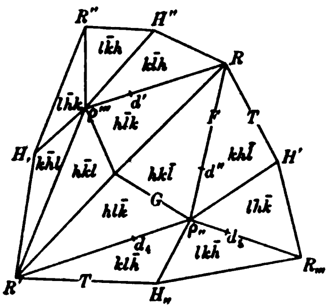


Fig. 292.

59. Crystals of the following substances belong to this class:

Substance.	*Chemical composition.*	*Forms and combinations.*
Diamond	C	*o*ω, *d*, *ad*, *p* {221}, *i* {430}, &c.
Blende	ZnS	*d*, *a*, *a*ω, *ao*ω, *o*ω, *dao*, *am*, *dm*, &c.
Fahlerz	$3(Cu_2, Fe)S.(Sb, As)_2S_3$	*o*, *oa*, *od*, *n*, *on*, &c.
Boracite	$Mg_7B_{16}O_{30}Cl_2$	*o*, *oa*, *ao*ω, *ao*ω*dv* = $\mu\{531\}$.
Helvine	$(Be, Mn, Fe)_7Si_3O_{12}S$	*o*ω, *o*ω*d*, *ond*, *ons*.
Haüyne	$Na_2Ca(NaSO_4Al)AlSi_3O_{12}$	*o*ω, *da*.
Eulytine	$Bi_4Si_3O_{12}$	*n*, *nan*,*o*.

Diamond. This mineral is sometimes found in apparently regular octahedra, which have their edges replaced by grooves in the way shown in Fig. 293. Another apparently regular octahedron is shown in

Fig. 294, the edges of which are also replaced by grooves; but in this case the sides of the grooves are made by rounded faces belonging to an

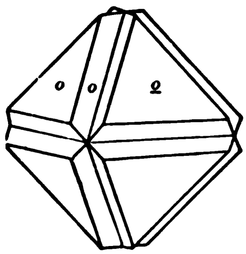


FIG. 293.

FIG. 294.

indeterminate form $\mu\,\{hkl\}$. A few crystals have been observed which are hexakis-tetrahedra, the faces being too rounded for the angles to be measured: occasionally the obtuse coigns of these hexakis-tetrahedra are modified by faces of the tetrahedron.

Blende. The crystals are sometimes combinations of $o = \mu\,\{111\}$ with $\omega = \mu\,\{1\bar{1}1\}$, the faces of the latter being slightly developed, and often dull and pitted, whilst the faces o are large, bright, and striated. These two forms are often associated with the cube, the faces of which appear as narrow truncations of the edges of $\mu\,\{111\}$ when this form predominates. But the most common and prominent form is the rhombic dodecahedron; its faces being parallel to perfect cleavages. This form is usually associated with faces of the cube and of the tetrahedron $\mu\,\{111\}$: it is also frequently associated with elongated triangular and rounded faces of a form $\mu\,\{hhl\}$, $h<l$, modifying those edges which meet the trigonal coigns in alternate octants. Were the form $\mu\,\{113\}$, we should have the crystal shown in Fig. 295.

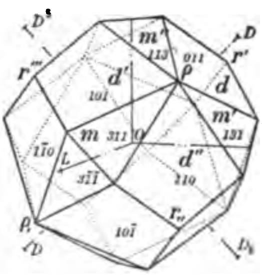


FIG. 295.

Fig. 295 is made by first drawing $\{110\}$ (Art. 17). On each of the axes of reference points L are marked off, where $OL = 3OA$; and the lines DD, D^5D_5, &c., are drawn through the origin parallel to the edges of the tetrahedron. They give the traces of planes through the origin parallel to those faces of $\{110\}$ which are perpendicular to the axial plane containing DD, D^5D_5, &c. Drawing the faces of $\{311\}$ through the points L, their traces on the axial planes will be $A'L''$, $A'L_{\prime\prime}$, &c.: these traces meet DD, D^5D_5, &c., in the points D, D_5, &c., marked on them; and, Od being the semi-diagonal of the cubic face, $OD = OD_5 = \&c. = 3Od \div 2$. The edges $[\rho e]$ of intersection of the two forms are given by joining the points L on one axis to the points D situated in the perpendicular axial plane. Thus, the edge $\rho e = [011,\ 131]$ is parallel to the line

LD_8, the edge [110, 113] is parallel to LD^8; and so on. The edge [311, $3\bar{1}\bar{1}$] is then drawn parallel to the edge of the tetrahedron $\mu\{111\}$ intersecting the axis OL on the same side of the origin; and similarly for all the homologous edges.

Fahlerz. The habit is markedly tetrahedral owing to the predominance of $\mu\{111\}$, which sometimes occurs alone. The tetrahedron is sometimes

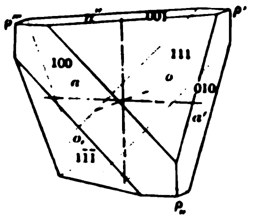


Fig. 296.

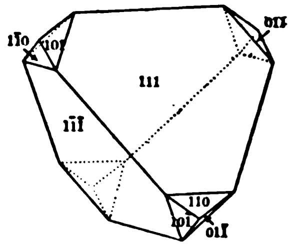


Fig. 297.

associated with narrow faces of {100}, Fig. 296; sometimes with those of {110}, Fig. 297; and sometimes with $\mu\{211\}$, Fig. 298. Several of these subordinate forms often occur together, modifying the edges and coigns of $\mu\{111\}$. Occasionally the complementary tetrahedron $\mu\{11\bar{1}\}$ is present, the faces being usually small.

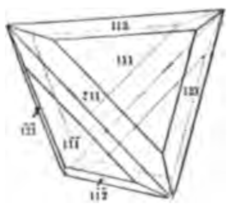


Fig. 298.

In drawing Fig. 297 the tetrahedron $\mu\{111\}$ is first constructed. Equal lengths on all the edges, measured from the coigns, are then cut off by proportional compasses. If the points so obtained are joined in pairs by lines parallel to the opposite sides, a combination of $\mu\{111\}$ with $\mu\{11\bar{1}\}$ is made. On each of the faces of the latter a trigonal pyramid is raised by drawing through the coigns lines parallel to the edges of the rhombic dodecahedron, the directions of which are obtained in the manner given in Art. 17.

Boracite. These crystals are found in tetrahedra $\mu\{111\}$, occasionally alone and occasionally with their edges truncated by narrow planes of

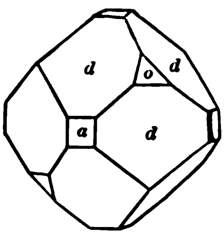


Fig. 299.

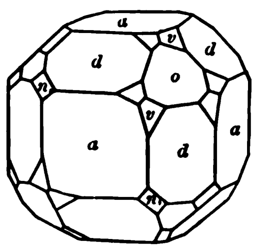


Fig. 300.

{100}, Fig. 296. In some crystals, Fig. 299, the rhombic dodecahedron {110} predominates, its coigns being modified by faces of the cube {100} and of the tetrahedron μ {111}. A common form is that in which the cube predominates, its edges being truncated by {110}, and its coigns being modified by one or both tetrahedra. A crystal of this habit showing also forms $n_{,} = \mu\,\{21\bar{1}\}$ and $v = \mu\,\{531\}$ is represented in Fig. 300.

To draw the general forms {*hkl*}, μ {*hkl*}, π {*hkl*}, a {*hkl*} *and* τ {*hkl*}.

60. {*hkl*}. The drawing of the hexakis-octahedron {*hkl*} is made the basis for that of the general forms of the other classes. On each of the axes three points H, K and L are determined, as shown in Fig. 301, where $OH = a \div h$, $OK = a \div k$, $OL = a \div l$; points on OY being distinguished by single dashes, points on OZ by double dashes. When these lengths are very short, all three are multiplied by some convenient number r. Thus in drawing {321}, it is best to take $OH = 2a$, $OK = 3a$, $OL = 6a$, each being six times the length given by the face-symbol.

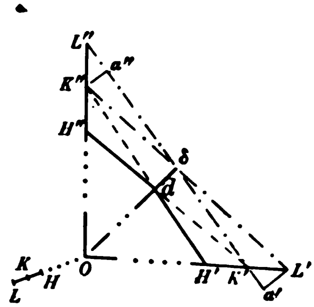


Fig. 301.

Assume OH to be the least intercept and OL the greatest, and therefore $h > k > l$. Then eight faces, having symbols in which h occupies the same place, meet in each point H. Pairs of lines are drawn crosswise in each axial plane joining a point L on one axis to a point K on the other; *e.g.* $L'K''$ and $K'L''$ in Fig. 301. These pairs intersect in points δ, δ', δ'', &c., on the dyad axes lying in their plane. The lines $H\delta$, $H'\delta'$, $H''\delta''$, &c., give the edges $H\rho$, $H'\rho$, &c., of Fig. 302. For the face (*hkl*) passes through $HK'L''$, and (*hlk*) through $HL'K''$; and similarly for other pairs of the faces.

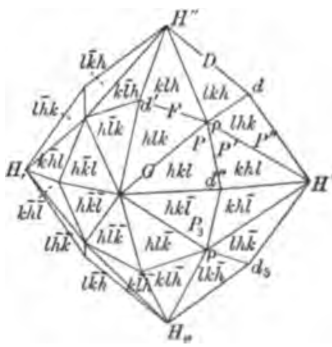


Fig. 302.

The coigns ρ on the triad axes being now determined, it only remains to find the points d at which two pairs of faces, such as (hkl) and (khl), $(hk\bar{l})$ and $(kh\bar{l})$, meet the respective dyad axis. These points are found by joining crosswise in pairs the points H on one axis to the points K' and K'' on the others, and the points K to the points H', H'', &c. Thus, in Fig. 301, $H'K''$ and $K'H''$ intersect in d on the dyad axis in their plane, and the line Ld coincides with the edge ρd of Fig. 302. The figure is now completed by joining the points H to the adjacent points d.

61. $\mu\{hkl\}$. Geometrically this form consists of the faces of $\{hkl\}$ which occupy alternate octants in sets of six. Hence the coigns ρ, ρ'', $\rho_{,}$, $\rho_{,,,}$, and the edges ρH, $\rho H'$, ρd, $\rho d'$, &c., remain coigns and edges of $\mu\{hkl\}$. But at the points d only two faces meet, and the edges $L\rho d$, $L'\rho d'$, $L''\rho d''$, &c., have to be all prolonged to meet the triad axis traversing the adjacent octants at the points R', R''', &c., of Fig. 303. The figure is completed by joining each of the points H to the adjacent coigns R.

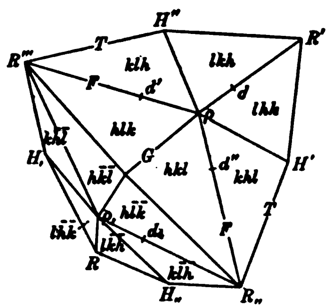


FIG. 303.

62. $\pi\{hkl\}$. All the coigns H and ρ of $\{hkl\}$, Fig. 302, remain coigns of the dyakis-dodecahedron $\pi\{hkl\}$; for four faces meeting at each coign H are common to both forms, and of the six faces, meeting at each ditrigonal coign of the former, a triad having their symbols in cyclical order remain in the new form. Further, one pair of the edges in each axial plane meeting at coigns H remain edges of $\pi\{hkl\}$. Thus, the edge $H'd$ of Fig. 302 and the corresponding edge joining H' to a point $K_{,,}$ on $OZ_{,}$ remain edges of Fig. 304: they have to be prolonged to meet the edges $[klh, \bar{k}lh]$ and $[kl\bar{h}, \bar{k}l\bar{h}]$ at new coigns e. These new edges are easily constructed; for they are

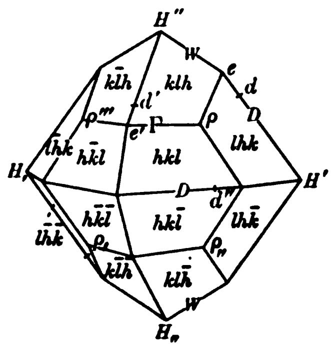


FIG. 304.

the lines joining the pairs of points H'', L'; &c. The homologous points e, e', e'', &c., being determined, fix the directions of the edges $[\rho e]$, $[\rho e']$, &c.

The complementary form $\pi\{lkh\}$, Fig. 305, is drawn in a similar manner; the edges Hd', $H''d$, $H'd''$, &c., being prolonged to meet new edges $H''L = [lkh, \bar{l}\bar{k}h]$, &c., in homologous points $e_{,}$, $e_{,,}$, &c. These are then joined to the adjacent points ρ, so completing the drawing.

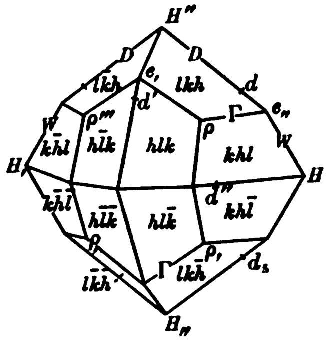


FIG. 305.

63. $\alpha\{hkl\}$. The coigns H and ρ of Fig. 302 remain coigns of $\alpha\{hkl\}$, and for the same reasons as were given in Art. 62 for the retention of these coigns in $\pi\{hkl\}$; but the edges of Fig. 302 have all to be replaced by new ones. The triads of faces of $\alpha\{hkl\}$, Fig. 306, meeting at ρ, ρ'', $\rho_{,}$ and $\rho_{,,,}$ in the first[1] and alternate octants are the same as those of $\pi\{hkl\}$ which lie in the same octants. Hence the edges ρe, $\rho e'$, $\rho e''$ of Fig. 304, and their homologues in the alternate octants are common to both forms. In the adjacent octants the faces of $\alpha\{hkl\}$ are the same as those of $\pi\{lkh\}$ which meet at the coigns ρ', ρ''', $\rho_{,,}$, $\bar{\rho}$ of Fig. 305; and the edges in these octants are the edges of Fig. 306 which meet at these coigns.

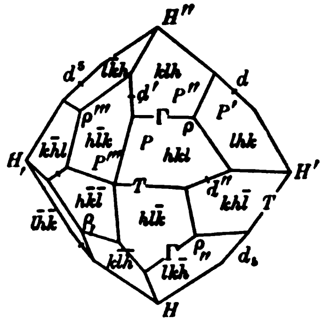


FIG. 306.

The edges of $\alpha\{hkl\}$ meeting at a tetragonal coign H are found by joining H to four points α, which are the intersections of alternate sides of the ditetragon formed by joining the points K and L in the axial plane perpendicular to the axis through H. Thus, in Fig. 301, the face (hkl) meets the plane YOZ in the trace $K'L''$; the face $(hl\bar{k})$ meets the same plane in a trace $L'K_{,,}$, where $OK_{,,}$ is $a_{,,} \div \bar{k}$ measured on $OZ_{,}$. These traces meet at α', and the edge $[hkl, hl\bar{k}]$ is the line joining H to α'. Similarly, the edge

[1] By *first octant* is meant that in which all the indices of a face are positive.

$[hkl, h\bar{l}k]$ is the line joining H to α'', where α'' is the intersection of $K'L''$ and $L_{,}K''$. The construction of these edges is the same as that of the polar edges of $\alpha\{hkl\}$ of Chap. XIV, Art. 52. Similar points are found in each of the other axial planes and joined to a point H on the perpendicular axis. The lines $H\alpha'$, $H\alpha''$, &c., meet the edges from $\rho_{,,}$, ρ, &c., which bound the same faces at new coigns; and the adjacent pairs of these coigns, being now joined, give the edges which are bisected at right angles by the dyad axes at the points d'', d, d', &c.

64. $\tau\{hkl\}$. This form consists geometrically of twelve of the faces of each of the three last forms, viz. of those triads which have their symbols in one cyclical order and lie in alternate octants. Hence the edges ρe, $\rho e'$, &c., situated in alternate octants of Figs. 304 and 306, are common also to the pentagonal dodecahedron, Fig. 307. Through the point H, common to (hkl) and $(h\bar{k}\bar{l})$, a line is drawn parallel to the trace $K'L''$, Fig. 301, in the perpendicular axial plane; and the similar edges through each of the points H', H'', &c., are obtained in a like manner. These edges meet the edges $\rho e'$, &c., in new coigns which have to be joined to trigonal coigns R''', R', &c., on the triad axes traversing the adjacent octants. These points R''', &c., are the same as the coigns so marked in Fig. 303. For of the six faces meeting at R''' in this latter figure, three remain in $\tau\{hkl\}$ and must in Fig. 307 meet at the same point: the triad is (klh), $(h\bar{k}\bar{l})$, $(\bar{l}\bar{h}k)$. The same is true of each of the other trigonal coigns R. The figure can then be completed.

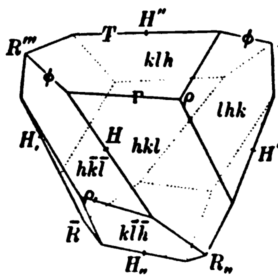


Fig. 307.

65. To a reader familiar with analytical geometry, a simpler method is to find the lengths $O\rho$, OR, Od, &c. in terms of the corresponding lines of the cube, Fig. 224; and then to mark off the lengths on these lines by proportional compasses. Thus, if Or is the semi-diagonal of the cube giving the parameters, then

$$O\rho = Or \div (h+k+l)\,;\ OR''' = Or''' \div (h+k-l)\,;\ \&c.$$

And if Od is one-half of the face-diagonal, $Od = Od \div (h+k)$.

CHAPTER XVI.

THE RHOMBOHEDRAL SYSTEM.

1. THIS system includes the seven classes of crystals which have a single triad axis; and we shall assume that this axis is in all classes a possible zone-axis perpendicular to a possible face. This can be proved to be the case in five of the classes; but it cannot be established as a deduction from the law of rational indices in classes I and II, in which the triad axis is either the only element of symmetry, or is associated with only a centre of symmetry. Owing to the unique character of the triad axis, it is often called the *principal axis*: it coincides in direction with the optic axis. The classes are:

I. The *acleistous trigonal* (*tetartohedral*, *trigonal-pyramidal*) class, the crystals of which have a triad axis but no other element of symmetry.

II. The *diplohedral trigonal* (*rhombohedral*, *parallel-faced hemihedral*) class, in which the triad axis is associated only with a centre of symmetry.

III. The *scalenohedral* (*ditrigonal-scalenohedral*, *rhombohedral-holohedral*) class, in which the elements of symmetry of class II are associated with three like planes of symmetry intersecting in the triad axis at angles of 60°, and with three like dyad axes, each perpendicular to one of the planes of symmetry and therefore to the triad axis.

IV. The *trapezohedral* (*trigonal-trapezohedral*) class, in which the triad axis is associated only with three like dyad axes making right angles with it and 120° with one another: the opposite ends of each dyad axis are dissimilar.

V. The *acleistous ditrigonal* (*hemimorphic*, *ditrigonal-pyramidal*)

class, in which the triad axis is associated only with three like planes of symmetry intersecting in it at 60° to one another.

VI. The *trigonal bipyramidal* class, in which the triad axis is associated with one plane of symmetry perpendicular to it.

VII. The *ditrigonal bipyramidal* class, in which the elements of symmetry of class VI are associated with three like planes of symmetry intersecting in the axis at 60° to one another: the intersections of these planes with the plane of symmetry perpendicular to the triad axis are dyad axes, opposite ends of which are dissimilar.

2. In a crystal having a single triad axis those faces which are not perpendicular to it occur in sets of three; and, when the crystal is turned through 120° about the axis, the three faces of each set change places. The face which is perpendicular to the principal axis must be that of a pinakoid or pedion, according as the facial development at opposite ends of the axis is similar or dissimilar: this face is often called the *base*; for in drawings of rhombohedral crystals the principal axis is always placed vertically.

When the faces are parallel to the triad axis, they occur in sets of three tautozonal planes inclined to one another at angles of 120°, and in some classes compose a *trigonal prism*. When, further, a centre of symmetry, or planes of symmetry through the axis, or dyad axes perpendicular to it, occur in the crystal, the three faces are, in general, associated with other similar triads tautozonal with the first triad; and the form is either a *hexagonal* or a *dihexagonal prism*. Their zone will be called a *hexagonal zone*, for a face can be found in it at 60° to any other of its faces (Chap. IX, Art. 12).

When the triad axis is the only element of symmetry, and the faces are inclined to the axis at finite angles, other than 90°, they compose an *acleistous trigonal pyramid*, such as that shown in Fig. 308, having its apex in the triad axis. Such a pyramid cannot occur alone, and it must in any crystal be associated with a pedion, as in

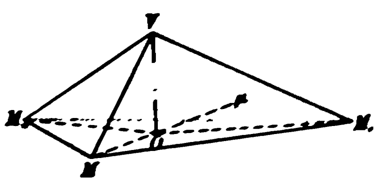


Fig. 308.

Fig. 308; or with another acleistous trigonal pyramid having its apex at the opposite extremity of the principal axis.

CHAPTER XVI.

THE RHOMBOHEDRAL SYSTEM.

1. This system includes the seven classes of crystals which have a single triad axis; and we shall assume that this axis is in all classes a possible zone-axis perpendicular to a possible face. This can be proved to be the case in five of the classes; but it cannot be established as a deduction from the law of rational indices in classes I and II, in which the triad axis is either the only element of symmetry, or is associated with only a centre of symmetry. Owing to the unique character of the triad axis, it is often called the *principal axis*: it coincides in direction with the optic axis. The classes are:

I. The *acleistous trigonal* (*tetartohedral*, *trigonal-pyramidal*) class, the crystals of which have a triad axis but no other element of symmetry.

II. The *diplohedral trigonal* (*rhombohedral*, *parallel-faced hemihedral*) class, in which the triad axis is associated only with a centre of symmetry.

III. The *scalenohedral* (*ditrigonal-scalenohedral*, *rhombohedral-holohedral*) class, in which the elements of symmetry of class II are associated with three like planes of symmetry intersecting in the triad axis at angles of 60°, and with three like dyad axes, each perpendicular to one of the planes of symmetry and therefore to the triad axis.

IV. The *trapezohedral* (*trigonal-trapezohedral*) class, in which the triad axis is associated only with three like dyad axes making right angles with it and 120° with one another: the opposite ends of each dyad axis are dissimilar.

V. The *acleistous ditrigonal* (*hemimorphic*, *ditrigonal-pyramidal*)

class, in which the triad axis is associated only with three like planes of symmetry intersecting in it at 60° to one another.

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VII. The *ditrigonal bipyramidal* class, in which the elements of symmetry of class VI are associated with three like planes of symmetry intersecting in the axis at 60° to one another: the intersections of these planes with the plane of symmetry perpendicular to the triad axis are dyad axes, opposite ends of which are dissimilar.

2. In a crystal having a single triad axis those faces which are not perpendicular to it occur in sets of three; and, when the crystal is turned through 120° about the axis, the three faces of each set change places. The face which is perpendicular to the principal axis must be that of a pinakoid or pedion, according as the facial development at opposite ends of the axis is similar or dissimilar: this face is often called the *base*; for in drawings of rhombohedral crystals the principal axis is always placed vertically.

When the faces are parallel to the triad axis, they occur in sets of three tautozonal planes inclined to one another at angles of 120°, and in some classes compose a *trigonal prism*. When, further, a centre of symmetry, or planes of symmetry through the axis, or dyad axes perpendicular to it, occur in the crystal, the three faces are, in general, associated with other similar triads tautozonal with the first triad; and the form is either a *hexagonal* or a *dihexagonal prism*. Their zone will be called a *hexagonal zone*, for a face can be found in it at 60° to any other of its faces (Chap. IX, Art. 12).

When the triad axis is the only element of symmetry, and the faces are inclined to the axis at finite angles, other than 90°, they compose an *acleistous trigonal pyramid*, such as that shown in Fig. 308, having its apex in the triad axis. Such a pyramid cannot occur alone, and it must in any crystal be associated with a pedion, as in Fig. 308; or with another acleistous trigonal pyramid having its apex at the opposite extremity of the principal axis.

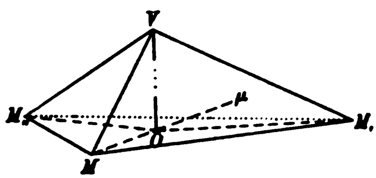


FIG. 308.

3. The simplest crystal possible in this system consists therefore of a trigonal pyramid having the pedion for base. When the faces of the pyramid are equably developed, they are isosceles triangles, and the base is an equilateral triangle having its centre in the axis. Such a crystal may be represented by the upper or lower half of Fig. 309; and suffices to give a set of axes of reference and parameters which will serve for every class of the system. We select for the axes of X, Y and Z three lines parallel to the similar and interchangeable edges of any trigonal pyramid which meet at an apex on the triad axis; and we select the base (pedion or face of the pinakoid) for the parametral plane (111).

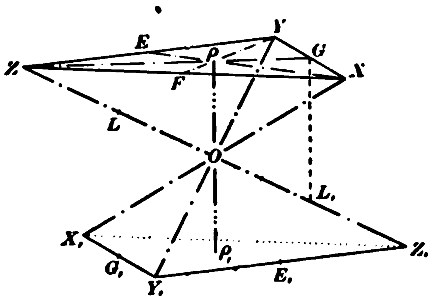


Fig. 309.

The set of axes, Fig. 309, resembles the legs of a cross-legged stool, the seat of which is an equilateral triangle and serves for the parametral plane (111). If X, Y and Z are the angular points of the triangular base, and if the model of the axes is rotated through 120° about the principal axis $O\rho$, the axis OX takes the place of OY, OY that of OZ, and OZ that of OX without disturbing the apparent position of the model or the direction of the parametral plane. Hence the lines ρX, ρY and ρZ are all equal; and so are also the angles $XO\rho$, $YO\rho$, $ZO\rho$. But the angles $O\rho X$, $O\rho Y$, $O\rho Z$ are all equal to 90°; for the parametral plane is perpendicular to the triad axis. Hence $OX = OY = OZ$; *i.e.* if the pedion, or face of the pinakoid, is taken as parametral plane (111), the parameters are all equal. Further,

$$\wedge X\rho Y = \wedge Y\rho Z = \wedge Z\rho X = 120^\circ;$$

and the sides XY, YZ, ZX are all equal.

The above relations resemble those established in the cubic system, and for the following reason. In both systems three like and interchangeable lines are taken as axes, and a face perpendicular to the triad axis, which interchanges the axes of reference, is taken to be (111): the parameters on these axes are therefore equal, and may be taken to be unity. The two cases differ, however, in a very important respect. In the cubic system the axes are at 90° to one

another and are themselves axes of symmetry of even degree, and the angles between them remain fixed and constant at all temperatures. In the system now under consideration the axes are not axes of symmetry; for, if they were, it is clear we should have more than one triad axis. The angles between them are not 90°, and vary with the temperature; and so do the angles the pyramid-edges make with the triad axis. Even if, under certain special circumstances, a set of pyramid-edges were found at right angles to one another, they could not remain so at all temperatures; for this would require the coefficient of expansion along the triad axis to be the same as that in directions at right angles to it, which is not the case.

We shall, as already stated, adopt such an axial system in every class. The triad axis will be placed vertically and the parametral plane horizontally; and we shall take the positive directions on the three axes to be measured upwards, the negative directions downwards: the faces and normals of the pyramid which gives the directions of the axes will in all classes be denoted by r, r', r''.

4. In a stereogram the triad axis will be placed perpendicular to the primitive, and the eye will be supposed to be at the lower end of the axis. The pole (111), denoted by C, will be at the centre and will coincide with the extremity of $O\rho$. From the assumption made as to $O\rho$ being a zone-axis, there will be in the primitive a number of possible poles arranged in triads, the faces of which are inclined at 120° to one another. Further, since the angles XOY, YOZ and ZOX are not 90°, the normals r, r', r'' to the axial planes never coincide with the axial points X, Y, Z. The triangles XYZ and $rr'r''$ are polar triangles (see Chap. XI, Art. 6).

We proceed to show the relative positions in a stereogram of the poles r and the axial points X, Y, Z.

Let, in Fig. 310, X, Y, Z be the axial points; and let m, m', m'' be the points in which the planes $XO\rho$, $YO\rho$, $ZO\rho$ of Fig. 309 meet the primitive—the extremity ρ coinciding with the pole C (111). Then $\wedge CX = \wedge CY = \wedge CZ$; and $\wedge XCY = \wedge YCZ = \wedge ZCX = 120°$: these

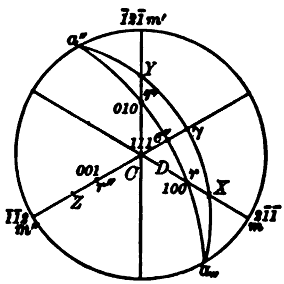


Fig. 310.

latter angles are also the arcs mm', $m'm''$ and $m''m$. Now the pole $r(100)$ is at 90° from Y and Z; for the normal to a face is at 90° to every line in, or parallel to, it: r must therefore be in the plane $XO\rho$, represented in Fig. 310 by the diameter Cm, which bisects the angle between the axes of Y and Z. But the arc $mm' = 120°$, and the arc $Cm = 90°$. Therefore Y, lying between C and m' makes with m an angle $>90°$ and with C an angle $< 90°$. Hence there must be a point r between m and C which is at 90° from Y. The point r is also at 90° from Z; since, for any point r in $[Cm]$, $\wedge rZ = \wedge rY$. The pole $r(100)$ lies therefore between m and C, and on the same side of C as the axial point X. Similarly, r' lies near Y between C and m'; and r'' near Z between C and m''.

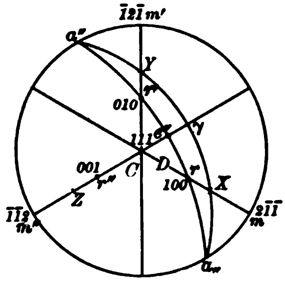


FIG. 310.

Again r, r', r'' are nearer to C, or more remote from it, than the axial poiuts X, Y, Z according as $\wedge XY = \wedge YZ = \wedge ZX$ are each of them greater or less than 90°. For it is clear that $XY > rY$, when r is, as in Fig. 310, nearer than X to C; and rY being 90°, $XY > 90°$.

5. The angle ρOr, or the arc Cr, between the triad axis and the normal to one of the axial planes we shall denote as the *angular element* D of a rhombohedral crystal.

We shall now show how from the angular element D the angle between the poles r, r' of two of the axial planes, and the angle XOY between any pair of the axes can be found; and, vice versâ, how the angle D can be determined when either of the angles rr' or XOY is given.

The spherical triangles rCr', XCY are both isosceles, and have the common angle $rCr' = 120°$. The sides rr' and XY of the triangles are also bisected at right angles by the great circle $ZC\gamma$ at the points e'', γ. Hence,

$$\sin re'' = \sin (rCe'' = 60°) \sin Cr;$$

$$\therefore \sin \tfrac{1}{2} rr' = \sin 60° \sin D \ldots\ldots\ldots\ldots (1).$$

Similarly, $$\sin \tfrac{1}{2} XY = \sin 60° \sin CX \ldots\ldots\ldots\ldots (2).$$

Again, since $Zr = Zr' = 90°$, Z is at 90° from every point in the great circle rr'.

$$\therefore Ze'' = 90°, \text{ and } CZ = 90° - Ce''.$$

Also $CZ = CY = CX$; and from the spherical triangle rCe'',

$$\cos 60^\circ = \tan Ce'' \cot Cr;$$

$$\therefore \tfrac{1}{2} = \cot CZ \cot Cr;$$

$$\therefore \tan CX \tan Cr = \tan CX \tan D = 2 \ldots\ldots\ldots\ldots (3).$$

6. We shall denote by c a length on the triad axis, such as $O\rho$ of Fig. 311, which is the distance of the apex of the axial pyramid (often called the *fundamental pyramid*) from the pedion XYZ; and we shall denote by $3a$ the length of one of the sides of the equilateral triangle XYZ; the factor 3 being introduced to avoid the recurrence of fractions, and to conform to the usage of the crystallographers who take for element the ratio $c : a$. If we take a to be unity, we may call c the *linear element* of a rhombohedral crystal.

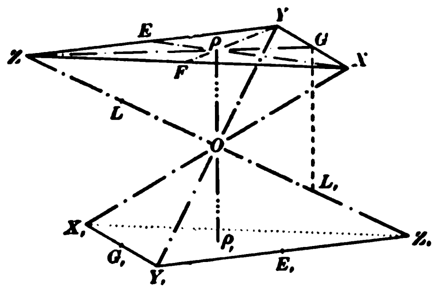


FIG. 311.

We shall now find the relation between $c \div a$ and D from the geometry of Fig. 311.

Since $\wedge\, ZGX = 90^\circ$, and ZG bisects the angle XZY, $\wedge\, GZX = 30^\circ$; and

$$ZG = XZ \cos 30^\circ = 3a \cos 30^\circ \ldots\ldots\ldots\ldots (4).$$

Again, by the well-known properties of the centre of gravity of a triangle, $Z\rho = 2\rho G$, and $ZG = 3\rho G$.

$$\therefore \rho G = a \cos 30^\circ \ldots\ldots\ldots\ldots (4^*).$$

The angular element D is the angle ZGO, the inclination of a face of the fundamental pyramid to the pedion. Hence

$$\tan D = O\rho \div \rho G = c \div a \cos 30^\circ \ldots\ldots\ldots\ldots (5).$$

$$\therefore \text{when } a = 1,\ c = \tan D \cos 30^\circ \ldots\ldots\ldots\ldots (6).$$

The reader must not confuse the $a = XY \div 3$ of this Article with the length OX which is used in Art. 3 as the Millerian parameter. The two are connected by the following relation.

$$OX \sin XO\rho = X\rho = Y\rho = Z\rho = 2a \cos 30^\circ.$$

$$\therefore OX \sin CX = 2a \cos 30^\circ \ldots\ldots\ldots\ldots (7).$$

If D is known, the angle CX is found from (3).

Ogdohedral

I. *Adeistous trigonal class;* $\tau\{hkl\}$.

7. The crystals of this class having a single triad axis and no other element of symmetry, the general form consists of a trigonal pyramid such as was employed in the preceding articles to give the axes of reference of any rhombohedral crystal. Any pyramid possible on a crystal of the class may be selected as the fundamental, or axial, pyramid $\tau\{100\}$. That pyramid which is most frequently met with, and has the largest faces, is usually taken; but, when cleavages are discovered parallel to the faces of a pyramid, it is most convenient to take the intersections of these cleavages for the axes of reference. There is, however, no essential difference between the pyramid $\tau\{100\}$ and that having the general symbol $\tau\{hkl\}$, for the choice has been guided by a desire for simplicity of symbols and facility of identification.

The pedions $\tau\{111\}$ and $\tau\{\bar{1}\bar{1}\bar{1}\}$ are two distinct possible special forms. They give equal parameters on the axes.

8. *The trigonal pyramid,* $\tau\{hkl\}$. From the facts that the axes are interchangeable in order and the parameters are equal, it follows that the form $\tau\{hkl\}$ consists of the three faces:

$$hkl,\ lhk,\ klh \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{a});$$

in which the indices are taken in cyclical order. This cyclical order was explained in Chap. xv, Art. 19, and was there shown to be a consequence of the interchangeability of lengths on axes of reference similarly placed with respect to a triad axis. The fact that the axes are no longer axes of symmetry at 90° to each other in no way affects the cyclical order.

If a plane (hkl), intercepting on the axes the lengths $a \div h$, $a_{,} \div k$, $a_{,,} \div l$ (a being OX), is a possible face, then the parallel plane $(\bar{h}\bar{k}\bar{l})$ is also a possible face; for the indices are rational and only differ in sign from those of the first face. The second face does not belong to the form $\tau\{hkl\}$, but to a complementary form $\tau\{\bar{h}\bar{k}\bar{l}\}$ which includes the faces:

$$\bar{h}\bar{k}\bar{l},\ \bar{l}\bar{h}\bar{k},\ \bar{k}\bar{l}\bar{h}.$$

Since the faces are parallel to those of $\tau\{hkl\}$, the angles over the polar edges (p. 112) of the two forms are equal.

Again, the face (hkl) being possible, then (hlk) is also a possible face. From the equal inclination of the axes to the vertical, and from their lying in vertical planes at 120° to one another, it follows

that the two faces are equally inclined to the horizon and to the vertical plane $XO\rho$; i.e. (hkl) and (hlk) are reciprocal reflexions in the plane $XO\rho$. But (hlk) belongs to a pyramid $\tau\{hlk\}$ consisting of the triad of faces:

$$hlk,\ khl,\ lkh;$$

the symbols of which are in the reverse cyclical order to those of $\tau\{hkl\}$. Since the faces of the two pyramids are equally inclined to the horizon, they are geometrically similar; and the angles over the polar edges are equal. But the two pyramids are distinct and separate forms; and they can only be brought into similar positions by turning one about the principal axis through an angle which varies with h, k and l.

We have also a fourth form $\tau\{\bar{h}\bar{l}\bar{k}\}$, the faces of which are parallel to those last discussed. These four forms being geometrically similar were, according to the principle of merohedrism, formerly regarded as tetartohedral forms derived from the general form of class III.

9. *The trigonal prism*, $\tau\{0\bar{1}1\}$. Since the triad axis $O\rho$ and the edges of the pyramid selected to give the axes of reference are possible zone-axes, a plane containing $O\rho$ and one of the axes, OX (say), is parallel to a possible face. Since the face is parallel to OX the first index is zero. Now the plane containing OX and $O\rho$ passes through the line $X\rho E$ of Fig. 309, which is the trace of the plane on the pedion; and since the triangle formed by the points X, Y, Z, where the axes meet the pedion, is equilateral, it is clear that $YE = EZ$. If the plane $OX\rho E$ is transposed, remaining parallel to its original position, until it passes through Z, it must in its new position meet the plane YOZ in a line $ZY_,$ parallel to OE, as shown in Fig. 312, which gives the lines in the plane YOZ; for parallel planes meet a third plane in parallel straight lines (Euclid XI, 16): therefore the triangles $ZYY_,$, EYO are similar. Hence (Euclid VI. 4),

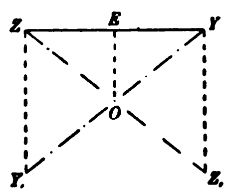


FIG. 312.

$$Y_,Y : YO = ZY : EY = 2:1;\ \therefore\ YY_, = 2OY;$$

and $$OY_, = OY.$$

Hence the plane in its transposed position meets the axes at distances $OX \div 0$, $OY \div \bar{1}$, $OZ \div 1$. But the equal lengths OX, OY and OZ can be taken as the parameters. Therefore the face has the symbol $(0\bar{1}1)$.

If the plane $OX\rho E$ is transposed in the opposite direction so as to pass through Y, it is easy to prove that it will meet OZ at $Z_{,}$ where $OZ_{,} = -OZ$. The symbol of this new face is therefore $(01\bar{1})$. The two parallel faces $(0\bar{1}1)$ and $(01\bar{1})$ are not associated together in crystals of this class, but belong to separate trigonal prisms $\tau\{0\bar{1}1\}$ and $\tau\{01\bar{1}\}$, which are complementary. Again, rotation through 120° about the triad axis $O\rho$ brings the plane $OX\rho E$ into the position $OY\rho F$, and the face $(0\bar{1}1)$ into a position in which it meets OX at X, and OZ at $Z_{,}$ where $OZ_{,} = -OZ$. The new face has therefore the symbol $(10\bar{1})$. A second rotation in the same direction brings the plane through $O\rho$ into the position $OZ\rho G$, and the parallel face to meet OX at $X_{,}$ and OY at Y: its symbol is $(\bar{1}10)$. The trigonal prism $\tau\{0\bar{1}1\}$, Fig. 313, includes the faces:

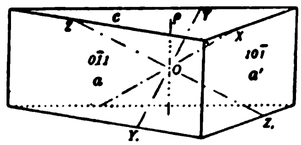


FIG. 313.

$$0\bar{1}1,\ 10\bar{1},\ \bar{1}10 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(b)}.$$

The angles between the faces are 120°; and the poles are those marked a, a' and a'' in Fig. 316.

The complementary trigonal prism, $\tau\{01\bar{1}\}$, having its faces also parallel each to the triad axis and to one of the axes of reference, consists of the faces $01\bar{1}$, $\bar{1}01$, $1\bar{1}0$. The poles are indicated by $\bar{a}$, $a_{,}$ and $a_{,,}$. Besides differing in signs from those of the first triad (b), the symbols of this last triad of faces are also in the reverse cyclical order to those in the first; hence there can be only two complementary forms having these symbols.

10. We shall now establish a relation existing between the indices h, k, l of a face N when it is parallel to the triad axis, *i.e.* is one of the faces of a trigonal prism. From two of the faces of the prism $\tau\{0\bar{1}1\}$ the zone-symbol can be determined, for these faces are not parallel to one another. By the rule (Chap. v, Art. 4) the zone-symbol is $[0\bar{1}1,\ 10\bar{1}] = [111]$; and by Weiss's law,

$$h + k + l = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(8)}.$$

Hence a face is parallel to the triad axis when the sum of its indices is zero: some of the indices must necessarily be negative. The three faces of the prism $\tau\{hkl\}$ have the symbols:

$$hkl,\ lhk,\ klh.$$

11. *The trigonal prism*, $\tau\{11\bar{2}\}$. An important trigonal prism is that of which each of the faces passes through one edge of the equilateral triangle in which the axial pyramid meets the pedion. The symbols of the faces can be found from the geometry of Fig. 314.

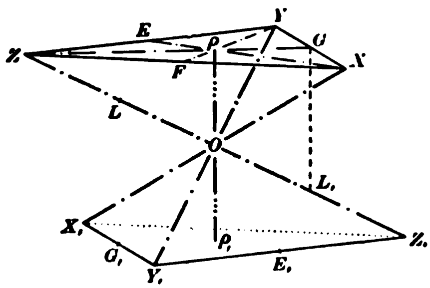


FIG. 314.

Let a plane be drawn through XGY parallel to the triad axis $O\rho$, and meet OZ in $L_,$. Then a plane through OZ and $O\rho$ intersects the prism plane $XYL_,$ in the line $GL_,$, which must be parallel to $O\rho$. Therefore, in the plane $ZGL_,$, the two triangles $ZO\rho$, $ZL_,G$ are similar; for they have the common angle $GZL_,$, and the third sides $O\rho$ and $GL_,$ are parallel. Hence (Euclid VI, 4),

$$OL_, : OZ = \rho G : \rho Z.$$

But (Art. 6), $\rho Z = 2\rho G$; $\therefore\ OZ = 2OL_,$.

Hence, since $L_,$ lies on the negative side of the origin, the prism-face $XYL_,$ intercepts on the axes the lengths OX, OY, $OZ \div \bar{2}$. The symbol is therefore $(11\bar{2})$. It can also be found from (8) and the fact that the face lies in the zone $[111, 001]$.

Rotation through 120° about the triad axis brings the face into positions in which it passes through the other edges of the triangle XYZ. The prism $\tau\{11\bar{2}\}$ consists therefore of the faces:

$$11\bar{2},\ \bar{2}11,\ 1\bar{2}1 \qquad \text{.........................(c).}$$

The complementary trigonal prism $\tau\{2\bar{1}\bar{1}\}$, Fig. 315, has each of its faces parallel to a face of the preceding prism, and its faces can be drawn through points, such as $X_,$, $Y_,$, L of Fig. 315. Its faces and poles will always be denoted by the letters m, m', m''.

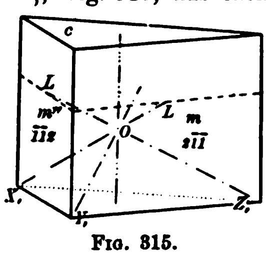


FIG. 315.

The prisms in Figs. 313 and 315 are terminated by the complementary pedions $\tau\{111\}$ and $\tau\{\bar{1}\bar{1}\bar{1}\}$.

12. The position of the normal P to any face (hkl) can be given by the general equations (1) of Chap. IV.; but, since the parameters

are equal, the equations can be simplified by taking the parameter to be unity. Hence, for the normal P to a face (hkl), we have

$$OP = \frac{\cos XP}{h} = \frac{\cos YP}{k} = \frac{\cos ZP}{l} = \frac{\cos XP + \cos YP + \cos ZP}{h+k+l} \ldots(9);$$

the last term being obtained by adding the numerators and denominators of the three preceding ratios to form a new ratio equal to each of the preceding terms.

13. Equation (8) connecting the indices h, k and l of a prism-face N can also be found from equations (9). In Fig. 316, let the triad axis, emerging at the pole C (111), be the diameter through the eye; and let X, Y and Z be the axial points. Then the poles N, N' and N'' of the trigonal prism $\tau\,\{hkl\}$ lie in the primitive at 120° from one another. Describe the great circles XN, YN, ZN; then, from the right-angled triangles XmN, $Ym'N$, $Zm''N$, we have

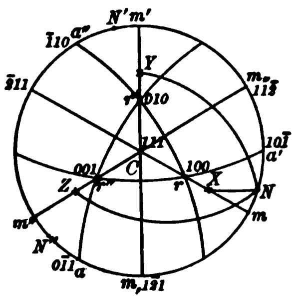


Fig. 316.

$$\left.\begin{aligned}\cos XN &= \cos Xm \cos mN = \sin CX \cos mN,\\ \cos YN &= \cos Ym' \cos m'N = \sin CY \cos m'N,\\ \cos ZN &= \cos Zm'' \cos m''N = \sin CZ \cos m''N\end{aligned}\right\}\ldots\ldots(10).$$

But $CZ = CY = CX$.

Therefore adding equations (10), we have

$$\cos XN + \cos YN + \cos ZN = \sin CX\,(\cos mN + \cos m'N + \cos m''N).$$

But $m'N = 120° - mN$, and $m''N = 120° + mN$;

$$\therefore\ \cos m'N + \cos m''N = \cos(120° - mN) + \cos(120° + mN)$$
$$= 2\cos 120° \cos mN = -\cos mN;$$

since $\qquad 2\cos 120° = -1.$

$$\therefore\ \cos XN + \cos YN + \cos ZN = \sin CX\,(\cos mN - \cos mN) = 0 \ldots(11).$$

The numerator of the last term of (9) therefore vanishes when the pole (hkl) lies in the primitive (*i.e.* when the face is that of a prism): the denominator must consequently vanish; for each of the terms of (9) is equal to a finite length OP, that of the perpendicular on the face from the origin. Hence, the sum of the indices h, k and l of any prism-face is zero, and

$$h + k + l = 0.$$

14. The numerator of the last term of equations (9) can now be expressed in a simple manner whatever may be the position of the pole P (hkl).

Let Fig. 317 be a projection of the poles P of a pyramid $\tau\,\{hkl\}$, and let the diametral zone $[CP]$ meet the primitive at N. Describe the great circles PX, PY and PZ. Then, from the spherical triangles CXP, CYP, CZP, we have (McCl. and P. *Spher. Trig.* I, p. 36)

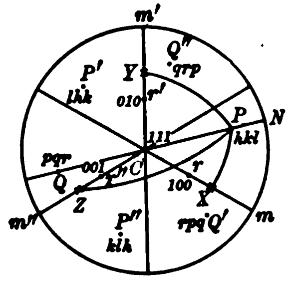


Fig. 317.

$$\left.\begin{aligned}\cos XP &= \cos CX \cos CP + \sin CX \sin CP \cos (XCP = mN),\\ \cos YP &= \cos CY \cos CP + \sin CY \sin CP \cos (YCP = m'N),\\ \cos ZP &= \cos CZ \cos CP + \sin CZ \sin CP \cos (ZCP = m''N)\end{aligned}\right\}\ldots(12).$$

But $CZ = CY = CX$; $\therefore$ by addition,

$$\cos XP + \cos YP + \cos ZP$$
$$= 3 \cos CX \cos CP + \sin CX \sin CP (\cos mN + \cos m'N + \cos m''N).$$

But it was proved in Art. 13 that, for any pole N lying in the primitive,

$$\cos mN + \cos m'N + \cos m''N = 0.$$

$$\therefore\ \cos XP + \cos YP + \cos ZP = 3 \cos CX \cos CP.$$

Hence, equations (9) become

$$OP = \frac{\cos XP}{h} = \frac{\cos YP}{k} = \frac{\cos ZP}{l} = \frac{3 \cos CX \cos CP}{h+k+l} \ldots\ldots(13).$$

15. Since C, the point at which the triad axis meets the sphere, Fig. 317, is the pole (111) of the parametral plane, it follows that N, the point of intersection of the zone-circle $[CP]$ with the primitive, is a possible pole to which we shall give the symbol (efg).

But we have already seen in Art. 10 that the primitive has the zone-symbol $[111]$; and by Chap. V, Art. 4, the symbol of $[CP]$ is $[k-l,\ l-h,\ h-k]$. Hence the indices of N are given in terms of h, k and l by the following equations (Chap. V, table 23):

$$\left.\begin{aligned} e &= h-k-(l-h) = 3h-(h+k+l) = 3h-\theta,\\ f &= k-l-(h-k) = 3k-(h+k+l) = 3k-\theta,\\ g &= l-h-(k-l) = 3l-(h+k+l) = 3l-\theta\end{aligned}\right\}\ldots\ldots(14),$$

where $\theta = h + k + l$—an abbreviation which we shall use in the discussion of rhombohedral and hexagonal crystals to denote the sum of h, k and l.

But, since there are three poles N, P, C in the zone $[CP]$, and two of them, N and C, are at 90° from one another, it follows that Q is also a possible pole, where $\wedge\, CQ = \wedge\, CP$ (Chap. IX, Art. 2); and the H. R. $\{NPCQ\} = 1 \div 2$. Let the symbol of Q be (pqr): it is required to find the relations connecting the indices of P and Q.

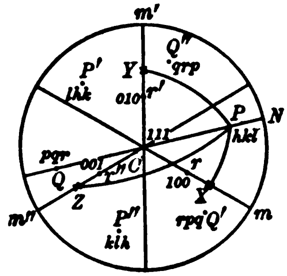


FIG. 317.

Now, H. R. $\{NPCQ\}$ gives

$$\frac{\sin NP}{\sin NC} \div \frac{\sin QP}{\sin QC} = \frac{1}{2} = \left|\begin{matrix} efg \\ hkl \\ \hline efg \\ 111 \end{matrix}\right| \div \left|\begin{matrix} pqr \\ hkl \\ \hline pqr \\ 111 \end{matrix}\right| \quad \ldots\ldots\ldots\ldots (15).$$

Employing the first two columns in the above expression, we have

$$\frac{1}{2} = \frac{ek - fh}{e - f} \div \frac{pk - qh}{p - q};$$

$$\therefore \text{ from (14), } \frac{pk - qh}{p - q} = 2\,\frac{k\,(3h - \theta) - h\,(3k - \theta)}{(3h - \theta) - (3k - \theta)} = \frac{2\theta}{3};$$

$$\therefore\ p\,(2\theta - 3k) = q\,(2\theta - 3h).$$

$$\therefore\ \frac{p}{2\theta - 3h} = \frac{q}{2\theta - 3k} = \text{(by symmetry) } \frac{r}{2\theta - 3l} \ldots.\ (16).$$

The equality of the third ratio involving r and l to the others in (16) is inferred from the symmetrical manner in which the several indices enter into the expression (15) for the harmonic ratio. It can be proved by taking together either the second and third columns of the right side of (15), or the first and third.

We have now to establish that the signs of the equivalents for (pqr) given by (16) are correct. To prove this, we add the numerators and denominators together, when we find each ratio

$$= \frac{p + q + r}{6\theta - 3\,(h + k + l)} = \frac{p + q + r}{3\theta} \ldots\ldots\ldots\ldots\ldots (17).$$

From the equations of the normals to P and Q, Art. 14, we have

$$OP = \frac{\cos XP}{h} = \frac{\cos YP}{k} = \frac{\cos ZP}{l} = \frac{3 \cos CX \cos CP}{\theta};$$

$$OQ = \frac{\cos XQ}{p} = \frac{\cos YQ}{q} = \frac{\cos ZQ}{r} = \frac{3 \cos CX \cos (CQ = CP)}{p + q + r}.$$

The numerators of the last ratios in the equations of the two normals are equal. Hence the denominators, θ and $p+q+r$, must be both positive when P and Q are above the primitive; and both negative when the two poles are below the primitive. Hence, expression (17) is positive. The pole Q given by the ratios (16) is therefore on the same hemisphere as P, as is required by Fig. 317.

The relations between the indices of P and Q can also be given as follows:

$$\left.\begin{aligned} p &= 2\theta - 3h = -h + 2k + 2l, \\ q &= 2\theta - 3k = 2h - k + 2l, \\ r &= 2\theta - 3l = 2h + 2k - l \end{aligned}\right\} \text{.............. (18).}$$

The sum $p+q+r = 3(h+k+l)$; hence the sums of the indices are both positive or both negative at the same time.

16. From equations (16) or (18) it is easy to obtain expressions for h, k and l in terms of p, q and r; so that, Q being the known pole, P (the unknown one) is determined from Q.

For, adding together equations (18), we have

$$p+q+r = 6\theta - 3(h+k+l) = 3(h+k+l) = 3\theta.$$

Hence,
$$\left.\begin{aligned} 9h &= 2(p+q+r) - 3p, \\ 9k &= 2(p+q+r) - 3q, \\ 9l &= 2(p+q+r) - 3r \end{aligned}\right\} \text{.................. (19).}$$

The common factor is cancelled, when the indices are introduced into the symbol.

Poles, such as P and Q, connected together by the relations given in (16) and (19) will be called *dirhombohedral* poles. They belong to separate forms, which will be called *dirhombohedral*; for the forms are geometrically similar in *each class* of the system, and the angles between adjacent faces of the one form are equal to those between corresponding faces of the other. If the forms occur together equably developed, they will compose hexagonal or dihexagonal pyramids according to the class to which the crystal belongs. But the forms are independent, and when both are present it is often easy to distinguish by their physical characters the faces of the one from those of the other.

17. The student should bear in mind that the indices of the faces of a general form need not be all positive, and that there may be several forms on a crystal in which one, two, or all three, of

the indices may be negative. Thus in the prism $\tau\{0\bar{1}1\}$, one of the indices is always negative; in $\tau\{2\bar{1}\bar{1}\}$, two are always negative.

It is therefore often necessary to determine on which hemisphere of the sphere of projection a pole lies; or, what is the same thing, to determine whether a face, having a known symbol (hkl), meets the triad axis at the end which is above or below the plane of the primitive.

For this purpose the last term in equations (13), viz.

$$\frac{3 \cos CX \cos CP}{h+k+l},$$

affords a ready test; for the ratio must be always positive. But, since $\wedge\, CX$ is less than 90°, the sign of the numerator depends on the value of $\wedge\, CP$. But $\cos CP$ is positive when CP is less than 90°; the denominator must then be also positive. Hence, $h+k+l>0$, for any pole lying above the paper in the stereogram, Fig. 317; and for any face which meets the triad axis on the side of the origin directed upwards.

But, if CP is greater than 90°, then $\cos CP$ is negative; and $h+k+l<0$. Hence the sum of the indices of a face, meeting the triad axis at the end directed downwards, is negative.

The positions of the poles with reference to the zone-circles $[r'r'']$, $[r''r]$, $[rr']$, are, similarly, determined from the second, third, and fourth terms of equations (13) respectively. For, if h is positive, $\cos XP$ is positive, and XP is less than 90°. If $h=0$, $\cos XP=0$; and $\wedge XP=90°$: the pole P then lies on $[r'r'']$. Hence, CP being less than 90°, P lies on the same side of $[r'r'']$ as C when h is positive: and, when h is negative, $\wedge XP>90°$, and P lies on the side of $[r'r'']$ remote from C. Similarly, when k is positive, P lies on the same side of $[r''r]$ as C; and when k is negative, P and C lie on opposite sides of $[r''r]$. In the same way l is positive when C and P are on the same side of $[rr']$; and l is negative when C and P are on opposite sides of $[rr']$. These relations are general, and hold for all classes of the system.

18. It was shown in Art. 8 that, for any particular values of h, k and l, two similar pyramids, $\tau\{hkl\}$ and $\tau\{hlk\}$, are possible, the faces (hkl) and (hlk) of which are reciprocal reflexions in the vertical plane $XO\rho$. Similarly, the other faces of the two pyramids are, in pairs, reciprocal reflexions in $XO\rho$; viz. (lhk) and (lkh), (klh) and

(khl). The pyramids are therefore enantiomorphous; and the crystals should rotate the plane of polarization of a beam of plane-polarized light transmitted along the triad axis. This has been established in crystals of sodium periodate. The enantiomorphism can only be recognised geometrically, when pyramids are present having their normals in azimuths differing from 60°; thus, in Fig. 318, the angle between the azimuths containing the normals (010) and ($\bar{1}85$) is 16° 6′, that between the zone-axes [111, 100] and [111, 504] in Fig. 319 is 49° 6′.

The triad axis is uniterminal, and should be a pyro-electric axis: this does not seem to have been proved for crystals of sodium periodate.

19. Crystals of *sodium periodate*, $NaIO_4 . 3H_2O$, belong to this class. Figs. 318 and 319 represent plans on the pedion $\tau\{\bar{1}\bar{1}\bar{1}\}$ of two crystals described by Professor Groth (*Pogg. Ann.* CXXXVII, p. 436, 1869); the

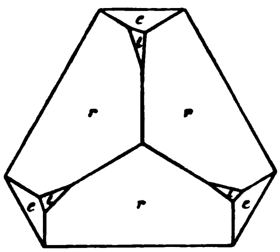


FIG. 318.

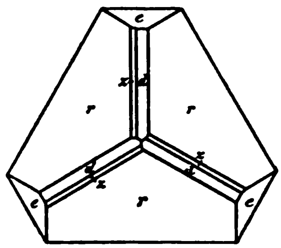


FIG. 319.

forms being: $r=\tau\{100\}$ and $c_{,}=\tau\{\bar{1}\bar{1}\bar{1}\}$ (both largely developed), $d=\tau\{011\}$, $e=\tau\{\bar{1}11\}$, $z=\tau\{504\}$, and $t=\tau\{\bar{1}85\}$. On such crystals the angle $c_{,}r$, or the angle rr', may be measured. In the former case, the angular element $D=180° - c_{,}r = 51° 37{\cdot}6'$ is obtained by direct observation. Measurement of $\wedge rr' = 100 \wedge 010$ gives $85° 31{\cdot}5'$, from which D is calculated by equation (1).

The angle D being known, the height c of the apex of r above the pedion is found from equation (5) in terms of the sides of the triangular base: it is 1·0937.

The crystals rotate the plane of polarization of a beam traversing them in the direction of the triad axis; and sometimes the rotation is to the right, sometimes to the left: those shown in the figures are lævogyral. Composite crystals are sometimes obtained, which show in convergent polarised light Airy's spirals: such crystals are generally regarded as twins of a dextro- and a lævo-gyral crystal united together along a common pedion; the bases $c_{,}$ of both being parallel and directed outwards.

II. *Diplohedral trigonal class;* $\pi\{hkl\}$.

20. When a centre of symmetry is associated with a triad axis, no other element of symmetry is necessarily involved. The class having these two elements of symmetry we shall call the *diplohedral trigonal* class; it has been known as the parallel-faced hemihedral class of the system.

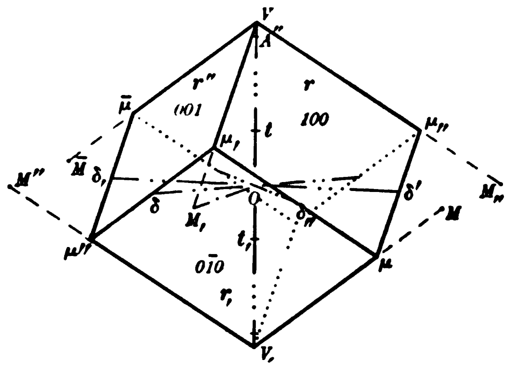


FIG. 320.

Owing to the triad axis, any face r, Fig. 320, inclined at a finite angle (other than 90°) to the triad axis is repeated in two similar faces r' and r'', which have to one another the relations of the three faces forming a pyramid in the last class: but, since the crystals are centro-symmetrical, the above three faces are associated with three similar faces $\bar{r}$, $r_{\prime}$, $r_{\prime\prime}$, each parallel to one of the first set. The first set of faces meeting at an apex V, the second set may be drawn through an apex $V_{\prime}$ at an equal distance from the origin. But, since the face r ($V\mu_{\prime}\mu\mu_{\prime\prime}$) meets the parallel faces r'' and $r_{\prime\prime}$, the edges $V\mu_{\prime}$ and $\mu\mu_{\prime\prime}$ must be parallel (Euclid XI, 16): and, since the face r meets the parallel faces r' and $r_{\prime}$, the edges $V\mu_{\prime\prime}$ and $\mu_{\prime}\mu$ must also be parallel. Hence the face $V\mu_{\prime}\mu\mu_{\prime\prime}$ is a parallelogram. It is also a rhombus, for the two sides $V\mu_{\prime\prime}$ and $V\mu_{\prime}$, meeting at the apex, are interchangeable when the crystal is turned through 120° about the triad axis. Similarly, every other face can be shown to be an equal and similar rhombus. Hence the form is a *rhombohedron*; and it has given the name to the system. We shall call the like and interchangeable edges which meet at the same apex, such as V, *co-polar edges*; those, like $\mu_{\prime}\mu$, $\mu\mu_{\prime\prime}$, &c., which occupy a middle position the *median edges*.

21. Since the three faces meeting at one of the apices are similar to the faces of the pyramid selected in Art. 3 to give the axial planes, we may take the axes of X, Y and Z to be three lines through the middle point O parallel, respectively, to the co-polar edges $V_{,}\mu$, $V_{,}\mu'$, $V_{,}\mu''$ of some conspicuous rhombohedron. This rhombohedron we shall therefore call the *fundamental* rhombohedron; and we shall denote its faces, and their poles, by the letters r, r', r'', &c.

We shall also take the upper face of the pinakoid perpendicular to the triad axis for the parametral plane (111). The parameters are therefore equal; and may be taken to be unity, or any three equal lengths on lines parallel to the axes such as $V_{,}M$, $V_{,}M' = VM_{,}$, and $V_{,}M''$ of Fig. 320. The formulæ of computation and the relations between the several lines of the axial and parametral planes established in preceding Articles hold for crystals of this class; and need not therefore be repeated.

The fundamental rhombohedron $r\{100\}$, Fig. 320, includes the following faces:

$$100\ 010\ 001\ \bar{1}00\ 0\bar{1}0\ 00\bar{1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(d)}.$$

22. Any face (hkl) inclined to the triad axis at a finite angle (other than 90°) gives rise to a rhombohedron $\pi\{hkl\}$, which includes the faces:

$$hkl\ lhk\ klh\ \bar{h}\bar{k}\bar{l}\ \bar{l}\bar{h}\bar{k}\ \bar{k}\bar{l}\bar{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(e)}.$$

The only limits to the relative magnitudes of h, k, and l are: (*1*) that they cannot be all equal, and (*2*) that $h + k + l > 0$.

1. If $h = k = l$, the face belongs to the pinakoid $\{111\}$, which includes the faces (111) and $(\bar{1}\bar{1}\bar{1})$, both perpendicular to the triad axis.

2. When $h + k + l = 0$, the face is parallel to the triad axis and the form $\pi\{hkl\}$ consists of a hexagonal prism, adjacent faces of which are inclined to one another at angles of 60°. The symbols of the faces of the prism $\pi\{hkl\}$ are given in table **e**, the difference between the symbols of the general form and of a prism arising from the particular relation between the indices; thus, $\{10\bar{1}\}$, $\{2\bar{1}\bar{1}\}$, $\pi\{3\bar{2}\bar{1}\}$, $\pi\{4\bar{3}\bar{1}\}$ are hexagonal prisms, $\pi\{4\bar{2}1\}$, $\pi\{421\}$ are rhombohedra.

The rhombohedron $\{100\}$ is not a special form: it only differs from other rhombohedra inasmuch as the axes of reference have

been taken parallel to its three co-polar edges. Similarly, the particular cases of the hexagonal prisms $\{0\bar{1}1\}$, $\{2\bar{1}\bar{1}\}$ differ in no essential respect from the hexagonal prism $\pi\{hkl\}$: they include six faces which are geometrically identical with those of the corresponding pairs of complementary prisms of the last class, and in the case of the two former with the six faces given in tables **f** and **g** of class III.

Crystallographers have not always agreed as to the rhombohedron to be selected to give the axes, *i.e.* as $\{100\}$. By comparing the values of D or c, given at the beginning of the description of the crystals, it is easy to see whether the same fundamental rhombohedron has been selected or not. Thus, in dioptase, CuH_2SiO_4, Miller's $\{100\}$ is Dana's $\{\bar{1}11\}$. The faces are those labelled s in Fig. 321. Dana does not indeed adopt a set of axes such as that described in Art. 3, but employs four axes which will be explained in the next chapter. But in transforming from Dana's representation to Miller's, the fundamental rhombohedron is not the same. Miller took s, the rhombohedron most conspicuously developed on the crystals, to be $\{100\}$; Dana selected as axial rhombohedron that which is parallel to the cleavages and truncates the polar edges of s, which then becomes $\{\bar{1}11\}$. Hence Miller's $D = (111 \wedge 100) = 50° 39'$; whilst Dana's $D = (0001 \wedge 0\bar{1}11) = 31° 40'$, and this latter angle is Miller's $(111 \wedge 011)$. The two angles do not quite accord; for, if Dana's value is accepted as correct, then Miller's angle should be $50° 58'$.

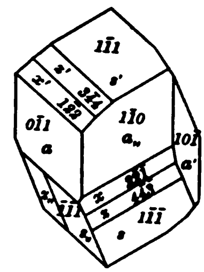


Fig. 321.

The Greek prefix π is omitted before the brackets in the symbol of the form, whenever two of the indices are equal or the forms are one or other of the hexagonal prisms $\{0\bar{1}1\}$ and $\{2\bar{1}\bar{1}\}$; for in these cases the forms are geometrically identical with the corresponding forms of the next class, which was by Miller regarded as the holohedral class of the rhombohedral system. We shall find that one or other of these forms belongs also to other classes of the system.

23. Crystals of dioptase, CuH_2SiO_4, and phenakite, Be_2SiO_4, belong to this class.

Fig. 321 represents a crystal of *dioptase* in which the faces s are $\{11\bar{1}\}$, $z = \pi\{4\bar{4}\bar{3}\}$, $x = \pi\{2\bar{2}\bar{1}\}$ and $a\{0\bar{1}1\}$; the above symbols being obtained from Dana's fundamental rhombohedron. Measurement

of one of the zones $[szxa_{,,}s']$ suffices to determine the symbols of all the faces by means of the A. R. of four tautozonal faces; for the symbols of $s'(111)$, $a_{,,}(1\bar{1}0)$ and $s(11\bar{1})$ are easily determined from the assumption of the fundamental rhombohedron. The angles are: $a_{,,}x = 28° \, 48'$, $a_{,,}z = 39° \, 31'$, $a_{,,}s = a_{,,}s' = 47° \, 43'$.

Two crystals of *phenakite* from Colorado, described by Prof. Penfield, are represented in Figs. 322 and 323. The poles—indicated by the same

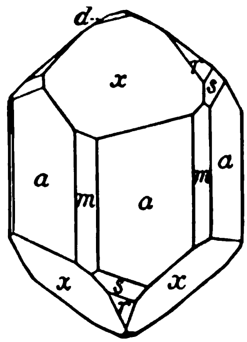


FIG. 322.

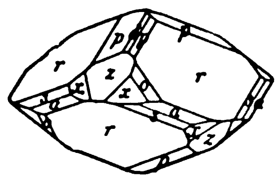


FIG. 323.

letters—are represented in Fig. 324. The forms are: $r\{100\}$, $d\{101\}$, p (on the right of d, Fig. 323) $= \pi\{201\}$, p' (being p on the left of d) $= \pi\{102\}$, $z = \{\bar{1}22\}$, $x = \pi\{2\bar{1}1\}$, $x_{,} = \pi\{1\bar{1}2\}$, $o = \pi\{3\bar{1}1\}$, $s = \pi\{20\bar{1}\}$, $a = \{1\bar{1}0\}$, $m = \{2\bar{1}\bar{1}\}$.

It should be noticed that r and z, p and p', are dirhombohedral forms; and that the indices of the faces of the pairs p and p', x and $x_{,}$, are in reverse cyclical orders. The faces of the latter pairs of forms are associated together in class III; and the members of each pair may therefore be regarded as complementary forms. The symbols of the forms can be determined by observation of, and measurement of the angles in, a few of the more important zones; for in most of them the indices of three or more poles, such as a, m, f', z, are immediately found when r is selected for the fundamental rhombohedron $\{100\}$.

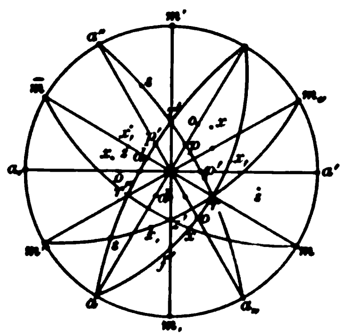


FIG. 324.

Thus, f' lies in $[r'm']$ and $[ar]$, and is $(1\bar{1}1)$; also z' is $(2\bar{1}2)$, and x is in $[ar]$ and $[mr'']$: it is therefore $(2\bar{1}1)$. Similarly, o, p and d are in pairs of zones, the symbols of which are readily determined, and therefore the indices of the faces.

We can determine the element D, and the symbols of p', p and s from measurement of the zone $[a_{,,}rdps]$. For, adopting as measured angles, $a''s = 28° 21'$, $a''r' = 58° 18'$, $a''p = 78° 22'$, $a''d = 90°$, $a''p' = 101° 38'$; then, from the right-angled triangle Cdr (C being the central pole (111): as the pinakoid is not actually present, the pole is not shown)

$$\sin dr = \sin 60° \sin D \text{ (see Art. 5).}$$

But $$dr = 90° - a_{,,}r = 31° 42'; \quad \therefore D = 37° 21 \cdot \dot{3}'.$$

Again, d being the pole between p and p' in which the zones $[a''r]$ and $[r'm_{,,}]$ intersect, then d is (110); and since it is at 90° from a'', the A. R. $\{a''r'sd\}$ gives

$$\frac{\tan a''r'}{\tan a''s} = \left|\begin{matrix} \bar{1}10 \\ 010 \\ \hline \bar{1}10 \\ hk0 \end{matrix}\right| \div \left|\begin{matrix} 110 \\ 010 \\ \hline 110 \\ hk0 \end{matrix}\right| = \frac{k-h}{k+h}.$$

$$L \tan (a''r' = 58° 18') = 10 \cdot 20928$$
$$L \tan (a''s = 28° 21') = \underline{9 \cdot 73205}$$
$$\cdot 47723 = \log 3.$$

$$\therefore \frac{k-h}{k+h} = 3, \quad \therefore h = \bar{1}, \; k = 2;$$

and the symbol of s is $(\bar{1}20)$.

Similarly, for p in the same zone we have from the A. R. $(a''r'pd)$

$$\frac{h+k}{k-h} = \frac{\tan (a''p = 78° 22')}{\tan (a''r' = 58° 18')} = 3, \text{ by computation.}$$

$$\therefore h = 1 \text{ and } k = 2; \text{ and } p \text{ is } (120).$$

Being given $ax = 62° 17'$ and $ao = 70° 42'$, we can, in a similar manner, find the symbols of x and o; for $ar = 90°$, and af' can be easily found from the right-angled spherical triangle $m_{,}af'$ in which $\wedge am_{,}$ and $\wedge ram_{,} = \wedge rm$ are both known. Or, conversely, knowing x to be $(2\bar{1}1)$, we can find the angle ax.

Thus, $\tan af' = \tan (m_{,}a = 30°) \div \cos (maf' = mr) = \tan 30° \div \sin 37° 21 \cdot \dot{3}'$;

$\therefore$ by computation, $af' = 43° 34 \cdot \dot{6}'$.

And, since $ar = 90°$, we have from the A. R. $\{af'xr\}$

$$\frac{\tan ax}{\tan (af' = 43° 34 \cdot \dot{6}')} = \left|\begin{matrix} 0\bar{1}1 \\ 2\bar{1}1 \\ \hline 0\bar{1}1 \\ 111 \end{matrix}\right| \div \left|\begin{matrix} 100 \\ 2\bar{1}1 \\ \hline 100 \\ 1\bar{1}1 \end{matrix}\right| = 2;$$

$\therefore$ by computation, $ax = 62° 16 \cdot 75'$.

The stereogram, Fig. 324, is made as follows. The primitive being described with any convenient radius, arcs of 30° are measured off on it, and diametral zones through these points are then drawn. The alternate points at 60° from one another are the poles $a\{0\bar{1}1\}$, the other alternate points are the poles $m\{2\bar{1}\bar{1}\}$. On the radius through m an arc $rm = 90° - 37° 21 \cdot \dot{3}'$ is marked off (Chap. VII, Prob. 1); and the homologous poles r', r'' as well as the dirhombohedral poles z, z', z'' are then found at the same distance from the centre. The zone-circles $[ar]$, $[a''r]$, $[mz']$, $[m''r]$ are then described, and fix the positions of all the poles.

III. *Scalenohedral class ;* $\{hkl\}$.

24. This class, the most important one of the system, may be derived from class II by the introduction of a dyad axis, or plane of symmetry. Suppose a dyad axis to be added to the elements of symmetry of class II; it must be at right angles to the triad axis, or it will introduce other triad axes, thus contravening the definition of the system, viz. that the crystals have only one triad axis: the cases in which there are several triad axes have been discussed in Chap. xv. But a dyad axis perpendicular to the triad axis must be associated with two other like and interchangeable dyad axes, both of them at 90° to the principal axis and at 120° to the first dyad axis and to one another. Again, a centro-symmetrical crystal must have a plane of symmetry, Σ, perpendicular to each dyad axis. There must therefore be three like and interchangeable planes of symmetry intersecting in the triad axis at angles of 60°. The crystals of this class have therefore the following elements of symmetry: ρ, 3δ, C, 3Σ. The arrangement of the planes and axes of symmetry is shown in Fig. 325.

The triad axis is a possible zone-axis, for it is the line of intersection of three planes Σ, which, by Chap. IX, Prop. 1, are parallel to possible faces. It is also perpendicular to a possible face—that parallel to the three dyad axes. The central plane parallel to this face will, in this and the hexagonal systems, be called the *equatorial plane*: it is not a plane of symmetry in crystals of this class.

FIG. 325.

25. *The rhombohedron,* $\{100\}$. Since a dyad axis is a possible zone-axis, a form is possible having a face parallel to this axis, and perpendicular therefore to the plane of symmetry of which the dyad axis is the normal. When the face, as, for instance, $V\mu_{,}\mu\mu_{,,}$ of Fig. 326, is inclined to the triad axis at a finite angle, other than 90°, there will be three such faces meeting at an apex in the triad axis, each parallel to one of the dyad axes and perpendicular to the corresponding plane of symmetry. Again, since the crystal is centro-symmetrical, there will be three like faces meeting the triad axis at an opposite apex, each of them being parallel to a dyad axis and perpendicular to a plane Σ. But a face parallel to a dyad axis is, by a rotation

of 180° about this axis, brought into the position of the parallel face. Thus the parallel faces $V\mu_{,}\mu\mu_{,,}$, $V_{,}\mu'\bar{\mu}\mu''$ are parallel to $O\delta$ and interchangeable by a semi-revolution about it. Again, the faces $V\mu_{,}\mu''\bar{\mu}$ and $V_{,}\mu\mu'\mu''$ change places when the crystal is turned through 180° about the axis $O\delta$ bisecting the edge $\mu_{,}\mu''$ at right angles. The faces meeting at opposite apices are therefore symmetrical with respect to the dyad axes; and no new faces are introduced by them. Similarly, no new faces are introduced by the planes of symmetry, each of them being perpendicular to a pair of parallel faces, and bisecting the angles between the pairs of other faces meeting at an apex. The figure is geometrically similar to that described in Art. 20, and is a rhombohedron.

In Fig. 326 the dyad axes are the lines $O\delta$, $O\delta'$, $O\delta''$: they pass each through the middle points of opposite median edges, *e.g.* $\mu_{,}\mu''$ and $\mu'\mu_{,,}$, &c., and are perpendicular to the edges which they bisect.

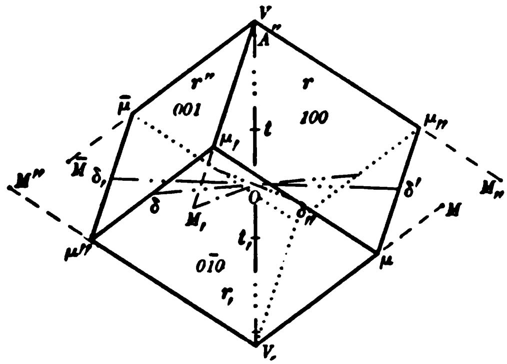


Fig. 326.

The planes Σ pass each through a pair of parallel polar edges, and also through the *polar diagonals* of the pair of faces to which they are respectively perpendicular. Thus Σ passes through the polar edges $V\bar{\mu}$, $V_{,}\mu$, and through the polar diagonals $V\mu$ and $V_{,}\bar{\mu}$.

26. As before, the axes of X, Y and Z are taken parallel to the three co-polar edges of any possible rhombohedron, which is then called the fundamental rhombohedron: its faces are denoted by r, r', &c. The axes lie each in one of the planes of symmetry, and we shall consider $XX_{,}$ of Fig. 309 to lie in Σ, $YY_{,}$ in Σ' and $ZZ_{,}$ in Σ'': they are parallel, respectively, to $V\bar{M}$, $VM_{,}$ and $VM_{,,}$.

The points δ and M of Fig. 326 are projected in the manner described in Chap. VI, Art. 19; $O\delta'$ being placed in the prolongation

of $D'C$ of Fig. 51: the rhombohedral axis $YY_{,}$ which is perpendicular to $O\delta'$ is consequently in the plane $C\gamma A'$, and this plane coincides with Σ'. The positive direction of the rhombohedral axis OX projects to the front and right in a vertical plane through $C\gamma$ inclined to $D'C\gamma$ at an angle of 30°. Similarly, the negative direction of OZ lies to the front and left in a vertical plane through $C\gamma$ inclined to $D'C\gamma$ at an angle of 30°. This arrangement is also adopted in Chap. XVII in the representation of hexagonal crystals by rhombohedral axes.

Or we may, as is the case in most of the drawings of calcite, place $O\delta_{,,}$ in the back-and-fore axis CA' of the cube projected in Fig. 51, when the axis $ZZ_{,}$ lies in the plane $D'C\gamma$ which now coincides with Σ''. The rhombohedral axes OX and OY lie in vertical planes inclined to $D'C\gamma$ at angles of 60° on opposite sides of it. The vertical planes containing the axes of reference are now in azimuths inclined at 30° to those of the first position.

$C\gamma$ is the unit of length on the triad axis, and $OV = c \times C\gamma$; the linear element c being connected with D by equation (6).

The method of finding the rhombohedral axes from Naumann's projection of the cubic axes is the same as that described for the case where Mohs' axes serve as basis.

The parametral plane (111) is taken to be the face perpendicular to the triad axis and parallel therefore to the dyad axes $O\delta$, $O\delta'$, $O\delta''$. The parameters may therefore be taken to be any three equal lengths which may be convenient; and in theoretical expressions, such as, for instance, the equations of a normal, may be taken to be unity. The expressions established in Arts. 5, 6, 12—16, hold therefore for crystals of this class. The angular element D is the inclination to the equatorial plane of each face of the fundamental rhombohedron, and is the angle $Cr = 111 \wedge 100$. It is connected with the angle between the faces r, r' by equation (1), with c by equation (6), and with the inclination of the axes of reference to the principal axis by equation (3).

27. Other special forms are the pinakoid, and hexagonal and dihexagonal prisms.

1. The pinakoid, $\{111\}$, consists of the faces (111) and $(\bar{1}\bar{1}\bar{1})$, both perpendicular to the triad axis.

2. The hexagonal prism, $\{1\bar{1}0\}$. When a face is parallel to one of the planes of symmetry Σ (say), it is parallel to the axis of reference $XX_{,}$ in this plane: the corresponding index is therefore zero.

Thus, in the particular instance of the face being parallel to $XX_{,}$, the first index is zero. The relation between the two other indices can be obtained from the geometry of Fig. 309 in the manner described in Art. 9. Hence, the pair of faces parallel to Σ and to $XX_{,}$ are $(0\bar{1}1)$ and $(01\bar{1})$: they are perpendicular to $O\delta$. By rotation about the triad axis through 120°, the two faces are brought successively into positions in which they are parallel to $YY_{,}$ and $ZZ_{,}$ and are respectively perpendicular to $O\delta'$ and $O\delta''$. The form, Fig. 327, is therefore a hexagonal prism having the faces:

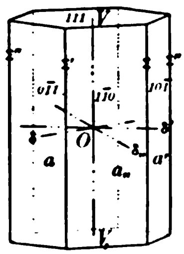


FIG. 327.

$0\bar{1}1$ $10\bar{1}$ $\bar{1}10$ $01\bar{1}$ $\bar{1}01$ $1\bar{1}0$......(f).

The angles between adjacent faces are clearly all 60°. The prism is constructed by drawing lines through the points μ, $\mu_{,}$, μ'', &c., of Fig. 326 parallel to the principal axis.

3. The hexagonal prism, $\{2\bar{1}\bar{1}\}$. When a face is parallel to the triad axis and to a dyad axis, we also get a hexagonal prism, the angles between adjacent faces of which are 60°. A face of the prism may be drawn through one edge of the triangle XYZ of Fig. 309; for the sides of this triangle are parallel each to a dyad axis, and they are perpendicular each to a plane of symmetry. Hence, the symbol of the face $XYL_{,}$ can be obtained from the geometry of the figure in the way employed in Art. 11. The face has therefore the symbol $(11\bar{2})$; and the form $\{2\bar{1}\bar{1}\}$, Fig. 328, includes the six faces:

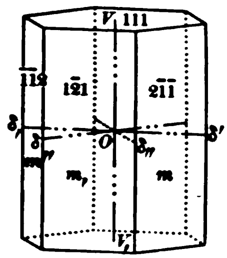


FIG. 328.

$2\bar{1}\bar{1}$ $\bar{1}2\bar{1}$ $\bar{1}\bar{1}2$ $\bar{2}11$ $1\bar{2}1$ $11\bar{2}$......(g).

The symbol of the face $XYL_{,}$ can also be obtained from the relations of the two zones to which it is common: viz. the zones

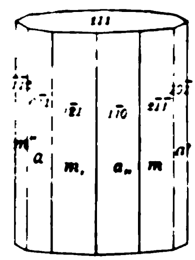


FIG. 329.

$[0\bar{1}1,\ 10\bar{1}] = [111]$ and $[111,\ 001] = [1\bar{1}0]$.

Hence, if the face is taken to be (hkl), we have

from the first zone $\quad h + k + l = 0$,

and from the second, $\quad h - k \quad = 0$.

$\therefore\ h = k = 1$, and $l = -2$; or $h = k = -1$, and $l = 2$. The parallel faces have therefore the symbols $(11\bar{2})$ and $(\bar{1}\bar{1}2)$.

The prism is constructed by drawing lines through the points δ, $\delta_{\prime\prime}$, δ', &c., of Fig. 326 parallel to the triad axis.

The faces of $\{2\bar{1}\bar{1}\}$ will be denoted by the letters m as indicated in Figs. 328 and 329, the faces of $\{0\bar{1}1\}$ by a. The faces of the one prism truncate the edges of the other; and adjacent faces a and m are inclined to one another at angles of 30°. A combination of the two prisms and the pinakoid $\{111\}$ is shown in Fig. 329. The poles of the two prisms are given by the points marked m and a in Fig. 331.

4. The dihexagonal prism, $\{hkl\}$, where $h+k+l=0$, has its faces arranged in pairs which are interchangeable by rotation about the dyad axes δ, and in pairs which are symmetrical about the planes Σ. The faces can also be arranged in triads interchangeable by rotation about the triad axis. Hence the form consists of:—

$$\left.\begin{matrix} hkl & lhk & klh & lkh & hlk & khl \\ \bar{h}\bar{k}\bar{l} & \bar{l}\bar{h}\bar{k} & \bar{k}\bar{l}\bar{h} & \bar{l}\bar{k}\bar{h} & \bar{h}\bar{l}\bar{k} & \bar{k}\bar{h}\bar{l} \end{matrix}\right\}\ldots(\text{h}).$$

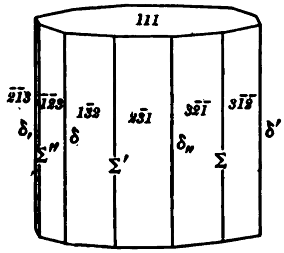


Fig. 330.

The symbols of the faces in each of the four triads making up the form are connected together by cyclical order, and the faces of a triad change places with one another on rotation about the triad axis. Again, the two faces in each column are parallel. The particular case $\{3\bar{1}\bar{2}\}$ is shown in Fig. 330.

It remains to prove that triads in opposite cyclical orders must coexist. This follows from the fact that an axis OX (say) lies in a plane Σ with respect to which the two other axes OY and OZ are reciprocal reflexions, for Σ is the plane $OX\rho E$ of Fig. 309. Hence if two faces are symmetrical with respect to Σ, they must meet OX at the same point, whilst the intercepts on OY and OZ of the two faces are reciprocal reflexions. If therefore (hkl) is one of the faces, the other is (hlk), in which the intercepts on OY and OZ have changed places; the symbols being in opposite cyclical order. Owing to the triad axis, (hkl) is repeated in (klh) and (lhk), and (hlk) in (khl) and (lkh); and pairs of the new faces, one from each of the triads, are symmetrical with respect to Σ. The six faces are also symmetrical in pairs to Σ' and Σ'', for rotations of 120° about the triad axis bring Σ to Σ' and Σ'' in succession.

Again, the dyad axis $O\delta_{\prime\prime}$ is parallel to the side XY of the base in Fig. 309, and is perpendicular to Σ'' and OZ. Hence, it is equally inclined to OX and OY, and bisects the angle $XOY_{\prime}$. A

rotation of 180° about $O\delta_{,,}$ interchanges equal positive and negative lengths on OZ; and changes a positive length on OX with a negative one on OY, and vice versâ. A face (hkl) is therefore by this rotation brought into a position given by the symbol $(\bar{k}\bar{h}\bar{l})$. This face is parallel to one of the six faces already obtained. The form having the faces given in (h) is therefore complete.

The angles between pairs of adjacent faces are constant, and those over two adjacent edges are in all cases unequal, whilst the angles between alternate faces are 60°. For the angle over an edge lying in Σ, such as $hkl \wedge hlk$, is double the angle either face makes with Σ. Similarly, the angle, such as $hkl \wedge \bar{l}\bar{k}\bar{h}$ over the adjacent edge meeting $O\delta'$ is double the angle which (hkl) makes with $O\delta'$, the extremity of which coincides with the pole a' in Fig. 331. But Σ and the adjacent dyad axis $O\delta'$ are at 30° to one another. Hence the angles $hkl \wedge hlk$ and $hkl \wedge \bar{l}\bar{k}\bar{h}$ are together equal to 60°. Further, if the two angles were equal, the zone would be divided isogonally by poles of possible faces into arcs of 15° and 45°; and it was shown in Chap. IX, Art. 12, that angles of 45° cannot occur in a zone containing angles of 30° or 60° repeated in succession between possible faces.

The distribution of the poles of the prism $\{hkl\}$ on the primitive is shown in Fig. 331: from it we can, by the A.R. of four tautozonal poles, find the angles mN and $hkl \wedge hlk$, or the angles $a'N$ and $hkl \wedge \bar{l}\bar{k}\bar{h}$, and prove that they are constant and independent of the crystal-element D which varies with the substance. The angles can indeed be calculated for any particular values of the indices, and apply to all crystals of the system in which faces having the same symbols occur.

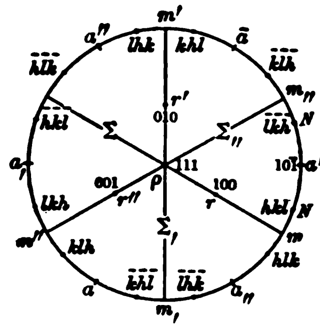


FIG. 331.

Let, in Fig. 331, a', the pole of the plane $\Sigma_{,}$, be $(10\bar{1})$, N be (hkl), m $(2\bar{1}\bar{1})$ and $m_{,}$ $(1\bar{2}1)$, where $ma' = 30°$ and $m_{,}a' = 90°$. Then, from the A.R. $\{a'Nmm_{,}\}$, we have

$$\tan a'N \div \tan a'm = \frac{\begin{vmatrix} 10\bar{1} \\ hkl \end{vmatrix}}{\begin{vmatrix} 10\bar{1} \\ 2\bar{1}\bar{1} \end{vmatrix}} \div \frac{\begin{vmatrix} 1\bar{2}1 \\ hkl \end{vmatrix}}{\begin{vmatrix} 1\bar{2}1 \\ 2\bar{1}\bar{1} \end{vmatrix}} = -\frac{3k}{2h+k} \;\ldots\ldots\; (20).$$

In the figure, and also in the A.R., the pole N is taken to lie somewhere between a' and $m_{,}$; hence k is negative and $2h$ must be numerically greater than k: the expression on the right of (20) is therefore positive, as is required by the way in which the poles have been taken in the A.R. The angle $a'N$ can therefore be computed when the indices h and k are given; and all the angles of any prism can then be found. We have the following angles for the prisms given in the first column:

$\{hkl\}$	$\tan a'N$	$a' \wedge hkl$	$hkl \wedge \bar{l}\bar{k}\bar{h}$	$hkl \wedge hlk$
$\{3\bar{1}\bar{2}\}$	$\frac{2\sqrt{3}}{10}$	19° 6·4′	38° 13′	21° 47′
$\{4\bar{1}\bar{3}\}$	$\frac{\sqrt{3}}{7}$	13 54	27 48	32 12
$\{5\bar{1}\bar{4}\}$	$\frac{\sqrt{3}}{9}$	10 53·6	21 47	38 13
$\{5\bar{2}\bar{3}\}$	$\frac{\sqrt{3}}{4}$	23 24·8	46 49·6	13 10·4.

28. For drawing the dihexagonal prism and other forms of the rhombohedral system, Fig. 332 is useful. In it the paper coincides with the equatorial plane perpendicular to the triad axis; $A\bar{A}$, $A'A_{,}$ and $A''A_{,,}$ are the dyad axes; OM, OM', OM'' are the traces of the planes of symmetry; M, M', M'' are the points in which the lower polar edges $V_{,}\mu$, $V_{,}\mu'$, $V_{,}\mu''$ of the fundamental rhombohedron, Fig. 326, meet the equatorial plane. The apex $V_{,}$ is at a distance c below O. The lengths $V_{,}M$, $V_{,}M'$, $V_{,}M''$, on edges parallel to the axes OX, OY and OZ, may be taken to be the Millerian parameters: they are equal and parallel to OX, OY and OZ of Fig. 309.

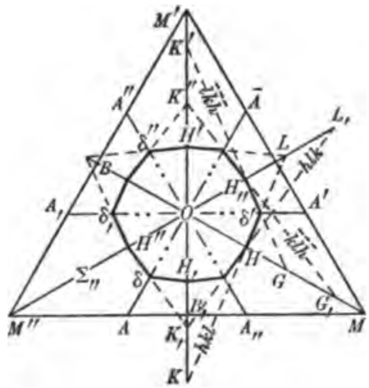


FIG. 332.

We shall first determine the relations between the lengths OM, OA, &c., and the lengths of the lines in the triangle XYZ of Fig. 309.

The triangle $MM'M''$ is equal and similarly placed to XYZ of Fig. 309; for the three rhombohedral faces through $V_{,}$ are parallel to the axial planes through O, and $OV_{,} = O\rho$.

Hence $MM' = M'M'' = M''M = 3a$ of Art. 6.

In Art. 25 the faces of the rhombohedron $\{100\}$ are shown to be parallel each to one of the dyad axes; the positions of which are therefore OA, OA', OA'', parallel respectively to $M'M''$, $M''M$ and MM', and to YZ, ZX and XY of Fig. 309.

Since O is the centre of gravity of the triangle $MM'M''$, we have $OB = OB_{,} = OB_{,,} = OM \div 2$.

Also $$OA : BM'' = OM : BM = 2 : 3,$$

$$\therefore\ OA = 2BM'' \div 3 = M'M'' \div 3 = a.$$

The length $a = XY \div 3$ is therefore an arbitrary length measured on the dyad axis, with which the linear element c is connected by equation (5).

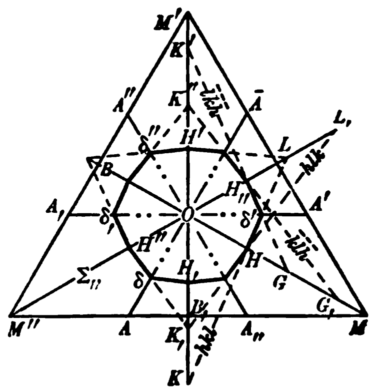


FIG. 332.

We shall, whenever it is necessary, denote a length a measured on OA' by $a_{,}$, and on OA'' by $a_{,,}$; and similarly, we shall denote lengths $= OB$ measured on OM, OM' and OM'' by b, $b_{,}$ and $b_{,,}$, respectively.

Again, $AOA_{,,}$ is an equilateral triangle, for its sides are parallel to those of $MM'M''$; and $OB_{,}$ bisects the angle $AOA_{,,}$, and is perpendicular to $AA_{,,}$. Hence $OB_{,} = OA \cos 30°$.

$$\therefore\ OB = OB_{,} = OB_{,,} = b = a \cos 30° ;$$

and $$OM = OM' = OM'' = 2OB = 2b = 2a \cos 30° \ \ldots\ldots (21).$$

We shall now find the lengths cut off on the lines of Fig. 332 by the prism-face (hkl); and show how to derive the symbols of the other prism-faces, one method having already been given in Art. 27.

The dihexagon $H\delta'H_{,,}\ldots$ is the trace on the equatorial plane of the prism $\{hkl\}$; the face (hkl) meeting the axes and the parallel polar edges at distances $V_{,}M \div h$, $V_{,}M' \div k$, $V_{,}M'' \div l$. But in projections by means of parallel rays, lengths on any line are projected in lengths having to one another the same ratios. Hence, OM being the projection on the equatorial plane of $V_{,}M$, and OH that of $V_{,}M \div h$, it follows that $OH = OM \div h$. Similarly, OM' and $OK = OM' \div k$ are the projections of $V_{,}M'$ and $V_{,}M' \div k$; and OM'' and $OL = OM'' \div l$ are the projections of $V_{,}M''$ and

$V_{,}M'' \div l$. The trace HKL of the prism-face (hkl) is given by $OM \div h$, $OM' \div k$, $OM'' \div l$.

The planes Σ, $\Sigma_{,}$, $\Sigma_{,,}$ pass respectively through OM, OM', OM'', and are perpendicular to the paper. Hence the face (hkl) is repeated by Σ in a face which meets the paper in the trace $K_{,}HL_{,}$, where the angle $OHK_{,} = \wedge\, OHL$. But $\wedge\, HOL = \wedge\, HOK_{,} = 60°$. The triangles HOL and $HOK_{,}$ are therefore equal, and

$$OK_{,} = OL = OM' \div l\text{; and } \wedge\, OLH = \wedge\, OK_{,}H.$$

Again, the triangles OKL and $OK_{,}L_{,}$ are equal, for they have the common angle $KOL = 120°$, and $\wedge\, OLK = \wedge\, OK_{,}L_{,}$. Also the side $OL =$ the side $OK_{,}$. The remaining sides are therefore equal, and $OL_{,} = OK = OM'' \div k$.

Hence the trace $K_{,}HL_{,}$ is given by $OH = OM \div h$, $OK_{,} = OM' \div l$, $OL_{,} = OM'' \div k$; and these lengths are the projections on the equatorial plane of lengths $V_{,}M \div h$, $V_{,}M' \div l$, $V_{,}M'' \div k$ on the axes. The new face is, as before (p. 369), (hlk).

In the figure, OH is the shortest length, and is measured on the *same side as* M: the corresponding index h is positive and numerically the greatest. The points K and L are measured *away from* M' and M'', and the corresponding indices k and l are negative. In no case can the indices be all positive, for $h + k + l = 0$.

By a semi-revolution about the dyad axis $A_{,}A'$ the trace KHL is brought to $G\delta' H_{,,}K'$, where $\wedge\, O\delta' H_{,,} = \wedge\, O\delta' H$. The angle $\delta' OH_{,,} = \wedge\, \delta' OH = 30°$; $\therefore\ OH_{,,} = OH = OM'' \div \bar{h}$; since $OH_{,,}$ is measured on the side of O away from M''.

Similarly, $\qquad OG = OL = OM \div \bar{l}$;

for G is on the same side of O as M, whilst L and M'' are on opposite sides of O: the sign of the index l has therefore to be changed. Again, $OK' = OK$, the two being measured in opposite directions on a line perpendicular to the dyad axis: the signs of the indices are therefore changed. Hence $OK' = -OK = OM' \div \bar{k}$. The trace $GH_{,,}K'$ is given by $OM \div \bar{l}$, $OM' \div \bar{k}$, $OM'' \div \bar{h}$; and the intercepts made by the prism-face through $GH_{,,}K'$ are $V_{,}M \div \bar{l}$, $V_{,}M' \div \bar{k}$, $V_{,}M'' \div \bar{h}$. The symbol of the face is therefore $(\bar{l}\bar{k}\bar{h})$. The rule connecting a pair of prism-faces interchanged by a semi-revolution about a dyad axis is, therefore, (α) that all the signs of the indices are changed, and (β) that the indices referring to the two axes inclined to the axis of rotation change places, whilst the index referring to the perpendicular axis changes sign only: the symbols are therefore in reverse cyclical orders.

The above rules connecting the symbols of consecutive faces symmetrical with respect to a plane Σ and to a dyad axis have been established for the dihexagonal prism only; but they will be shown to hold generally whatever be the position of the face (hkl).

The symbols of the remaining faces of the prism are readily obtained by repeating the above operations. Thus the face through $H_{\prime\prime}K_{\prime\prime}$ is $(\bar{k}\bar{l}\bar{h})$, the next through H' is (khl) and that through $H'\delta''$ is (lhk).

To make an orthographic or clinographic drawing, all that is needed is to describe, in the plane of the horizontal cubic axes of Chap. VI, a figure similar to Fig. 332. The vertical lines through the projected points corresponding to H, δ', $H_{\prime\prime}$, &c. are the edges of the dihexagonal prism.

29. We can now determine the lengths $O\delta'$, $Od_{\prime\prime}$ and OE in which the trace HKL of the prism-face (hkl), Fig. 333, meets the three dyad axes. For, since the triangles $HO\delta'$ and $\delta'OL$ make up the triangle HOL, we have

$$OH\,.\,O\delta' \sin 30^\circ + O\delta'\,.\,OL \sin 30^\circ = OH\,.\,OL \sin 60^\circ.$$

Dividing by $OH\,.\,OL\,.\,O\delta'$ and by $\sin 30^\circ = \cos 60^\circ$, we have

$$\frac{\tan 60^\circ}{O\delta'} = \frac{1}{OH} + \frac{1}{OL}.$$

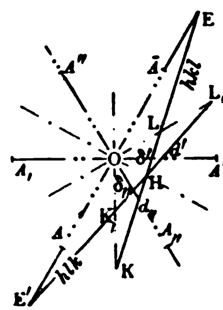


FIG. 333.

In this expression OH and OL are treated as *positive* lengths. Hence, when they are replaced by their equivalents in terms of OM and the indices, attention must be paid to the signs of the latter. Thus for the face taken, h is positive whilst l is negative; and in the above expression $OM \div \bar{l}$ must be taken as the length OL.

Hence, since $\tan 60^\circ = \sqrt{3}$ and, from (21), $OM = a\sqrt{3}$;

$$\frac{\sqrt{3}}{O\delta'} = \frac{h}{OM} - \frac{l}{OM} = \frac{h-l}{a\sqrt{3}}.$$

$$\therefore\ O\delta' = \frac{3a}{h-l} \quad\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

In a similar manner $Od_{\prime\prime}$ ($d_{\prime\prime}$ being the point in which HKL cuts $OA_{\prime\prime}$) can be found from the triangles $HOd_{\prime\prime}$, $d_{\prime\prime}OK$ and HOK.

$$\therefore\ Od_{\prime\prime} = \frac{3a}{h-k} \quad\ldots\ldots\ldots\ldots\ldots\ldots(23).$$

The length OE (E being the point in which HKL meets the axis $A\bar{A}$) is obtained from the triangles $EO\delta'$, $\delta'Od_{\prime\prime}$ and $EOd_{\prime\prime}$, the angles $EO\delta'$, $\delta'Od_{\prime\prime}$ being both 60°. Thus

$$OE \,.\, Od_{\prime\prime} \sin 120° = OE \,.\, O\delta' \sin 60° + O\delta' \,.\, Od_{\prime\prime} \sin 60°.$$

But $\sin 120° = \sin 60°$: dividing therefore by $OE \,.\, O\delta' \,.\, Od_{\prime\prime} \sin 60°$,

we have
$$\frac{1}{O\delta'} = \frac{1}{Od_{\prime\prime}} + \frac{1}{OE},$$

$$\therefore\ OE = \frac{3a}{k-l} \qquad\qquad (24).$$

30. By the aid of Fig. 333, we can give an independent proof of the relation, $h+k+l=0$, holding between the indices of a prism-face. For the triangle KOL contains the two triangles KOH and HOL.

$$\therefore\ OK \,.\, OL \sin 120° = OH \,.\, OK \sin 60° + OH \,.\, OL \sin 60°.$$

Dividing by $OH \,.\, OK \,.\, OL \sin 60°$, we have

$$\frac{1}{OH} = \frac{1}{OL} + \frac{1}{OK}$$

in which the lengths are all positive. We must therefore write $OM \div \bar{k}$ for OK, and $OM \div \bar{l}$ for OL; since the face taken meets the axes of Y and Z on the negative side of the origin.

Hence, $\qquad h = -k-l,\ $ and $\ h+k+l=0.$

The Rhombohedron.

31. When the angle D, or the angle $r'r$ of the fundamental rhombohedron $\{100\}$, is known, we can find the distance c of its apex V from the origin, $O\delta'$ (Fig. 335) being the unit of length a on the dyad axis. Let Fig. 334 represent a section through the plane EOX of Fig. 309 and $V_{\prime}OM$ of Fig. 335, so that the point ρ of Fig. 309 coincides with the apex V, and $XV = 2VK$. Now OX is parallel to $V_{\prime}M$, OE to μV [the polar face-diagonal of (100)]; and the normal Or lies in Σ and meets μV at r. The inclination of the face to the equatorial plane is therefore $D = \wedge VBO = \wedge OEX = \wedge VOr$.

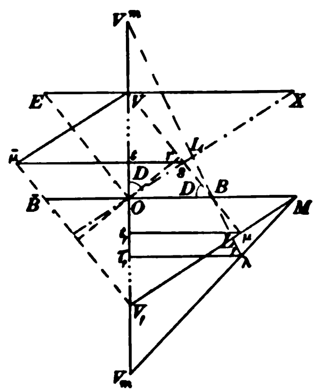


FIG. 334.

Since $OV_{\prime} = OV$, and OX is parallel to $V_{\prime}M$, $\therefore$ from (21), $OM = VX = 2a \cos 30°$.

Also $OB = VE = VX \div 2 = OM \div 2 = a \cos 30°$.........(25).

Hence, $c = OV = OB \tan D = a \cos 30° \tan D$...........(5*),

and $\tan VOX = \tan OV_{,}M = OM \div OV_{,} = 2OB \div c = 2 \cot D$.

Therefore C being the point at which the triad axis meets the sphere aud X being the axial point,

$$\tan CX \tan D = 2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3^*).$$

Fig. 336 represents a section of Fig. 335 in the plane $VV_{,}M''$ exactly similar to, and on the same scale as, Fig. 334.

We can now draw the rhombohedron $\{100\}$.

Taking $O\delta_{,}$ to be the unit of length CD' of Fig. 51, or $OY_{,}$ of Fig. 60, then the apex V is found by marking off on $C\gamma$ or OA'' of the same figures lengths equal respectively to $C\gamma \cos 30° \tan D$, or $OA'' \cos 30° \tan D$: in Fig. 335 OV has been taken to be $OA'' \times 1{\cdot}1$. Again, for the cleavage-rhombohedron r of calcite, $r'r = 74° \, 55'$; and, by computation from (1), $D = 44° \, 36{\cdot}6'$. Therefore

$$OV = OV_{,} = OA'' \cos 30° \tan 44° \, 36{\cdot}6' = OA'' \times {\cdot}8543 :$$

also $XOV = YOV = ZOV = 63° \, 44{\cdot}75'$.

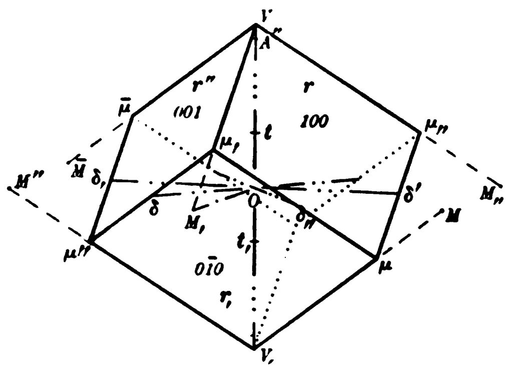


Fig. 335.

The hexagon $\delta_{,}\delta\delta_{,,}$... and the points M'', $M_{,}$, M, &c., of Fig. 57, being projected in the horizontal plane by the method described in Chap. VI, Art. 19, the lower polar edges join $V_{,}$ to the points M, M', M''; and the upper polar edges join V to the

opposite points $\overline{M}$, $M_{,}$, $M_{,,}$. The median edges $\mu\mu_{,}$, $\mu_{,}\mu''$, &c., are then drawn through the points $\delta_{,,}$, δ, &c., parallel respectively to the polar edges $VM_{,,}$, $V_{,}M$, &c.

The median edges are, however, most easily found by means of the following proposition: The median coigns μ, $\mu_{,}$, &c., lie on straight lines parallel to OM, $OM_{,}$, &c., which trisect at $t_{,}$ and t the length $V_{,}V$ intercepted on the principal axis between the apices; μ, $\mu_{,}$, &c., are therefore those points of trisection of $V_{,}M$, $VM_{,}$, &c., nearest to the equatorial plane. For the polar edges meeting at V, being interchangeable, are equal to one another; and so also are those meeting at $V_{,}$. Suppose lines to be drawn from the upper coigns $\bar{\mu}$, $\mu_{,}$, $\mu_{,,}$, and also from the lower coigns μ, μ', μ'' to meet the triad axis at right angles at the points t and $t_{,}$: the pair $t\bar{\mu}$ and $t_{,}\mu$ are shown in Fig. 334, and another pair in Fig. 336. Then

$$Vt = V_{,}t_{,} = V_{,}\mu \cos t_{,}V_{,}M = V_{,}\mu \cos VOX.$$

The lengths Vt and $V_{,}t_{,}$ are the orthogonal projections of the polar edges on $VV_{,}$, and must be the same whichever of the polar edges is taken. Hence, $tt_{,}$ is the orthogonal projection of the median edge $\mu\mu_{,,}$; for $t_{,}$ is the projection of μ, and t of $\mu_{,,}$. But the orthogonal projections on the same line of equal and parallel lines are themselves equal; and $\mu\mu_{,,}$ of Fig. 335 is equal and parallel to $V\mu_{,}$. Hence $tt_{,} = Vt = V_{,}t_{,}$; and the points t and $t_{,}$ are the points of trisection of $VV_{,}$.

Again, the right-angled triangles $Vt\mu_{,,}$, $VOM_{,,}$ of Fig. 336 are similar; hence

$$V\mu_{,,} : VM_{,,} = Vt : VO = 2 : 3.$$

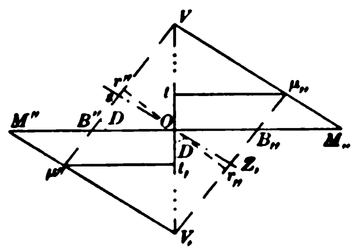


FIG. 336.

Therefore $\mu_{,,}$ is a point of trisection of $VM_{,,}$; and similarly, μ in Fig. 334 is a point of trisection of $V_{,}M$. The same is true of all the other median coigns. The polar edges are therefore found by joining the apices to alternate projected points M, M', &c., and trisecting the lines so obtained by proportional compasses: the points of trisection nearest to the equatorial plane being joined in pairs parallel to the polar edges give the median edges, $\mu_{,,}\mu$, $\mu\mu_{,}$, &c.

32. The geometrical relations of the rhombohedron are so important that we give another method of determining them. Let Fig. 337 represent one of the equal faces of $\{100\}$; $V\mu''$, the face-diagonal in Σ'', being the same line as $VB''\mu''$ of Fig. 336. The face is parallel to $O\delta_{,,}$, and may be supposed to be drawn through the line $M_{,}A_{,}\bar{M}$ of Fig. 57 to meet the principal axis at V. Now the median edges $\bar{\mu}\mu''$, $\mu''\mu_{,}$ are bisected by the dyad axes meeting them at $A_{,}$ and A, coincident with $\delta_{,}$ and δ in Figs. 57 and 335; for by a semi-revolution about $O\delta_{,}$ the edge $V\bar{\mu}$ of Fig. 335 is interchanged with $V_{,}\mu''$, and the points $\bar{\mu}$ and μ'' change places. Hence $AA_{,}$ of Fig. 337 (the same line as $\delta\delta_{,}$ of Fig. 335) is parallel to the horizontal face-diagonal $\bar{\mu}\mu_{,}$. Therefore (Euclid VI, 2), $B''\mu'' = s\mu'' \div 2 = V\mu'' \div 4$; since the diagonals of a rhombus bisect one another.

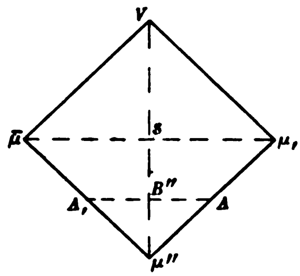


FIG. 337.

Again, from Fig. 336,

$$OV : Ot_{,} = VB'' : B''\mu'' = 3 : 1.$$

$$\therefore \; Ot_{,} = OV \div 3 = OV_{,} \div 3.$$

Hence the median coigns lie on the lines parallel to OM, $OM_{,}$, &c., drawn through the points t and $t_{,}$ which trisect the length $VV_{,}$.

Again, $V\bar{\mu}$ is parallel to OX, and $V\mu_{,}$ to OY; also

$$\mu_{,}s = AA_{,} = OA = a,$$

since $OAA_{,}$, Fig. 332, is an equilateral triangle.

Hence $\tan \tfrac{1}{2}XOY = \tan sV\mu_{,} = \mu_{,}s \div Vs = 3a \div 2\,VB''$.

But in Fig. 336,

$$OB'' = VB'' \cos D, \text{ and } OB'' = a \cos 30°.$$

$$\therefore \; \tan \tfrac{1}{2}XOY = 3a \cos D \div 2OB'' = \sqrt{3} \cos D \text{.........(26).}$$

33. The direct and inverse rhombohedra, mR and $-mR$. The method of drawing the rhombohedron $\{100\}$ applies to all rhombohedra; for any rhombohedron possible on the crystal may be selected to give the directions of the axes: relations equivalent to those given in Arts. 31 and 32 for $\{100\}$ hold between the several

lines, coigns and angles of every rhombohedron. Further, by the law of rational indices, faces belonging to different forms of the same crystal will, if drawn through the same point on any zone-axis, meet other zone-axes at distances measured from the origin, which are to one another in commensurable ratios. Hence three faces of all rhombohedra belonging to the same substance can be drawn through the lines MM', $M'M''$, $M''M$ of Fig. 57 or Fig. 332 to meet the triad axis at apices V^m or V_m, where $OV^m = OV_m = mc$, and m is a commensurable number. Two rhombohedra can be obtained for each value of m, according as the points M, M', M'' of Fig. 57 are joined (i) to the lower apex V_m in the manner adopted in constructing Fig. 335, or (ii) to the upper apex V^m.

i. In the first case the upper polar edges are given by $V^m\overline{M}$, $V^mM_,$, $V^mM_{,,}$; and the lower polar edges by V_mM, &c. The figure is completed in the manner described in Art. 31 by finding the points of trisection λ, Fig. 338, of the polar edges nearest to the equatorial plane. This rhombohedron is called by Naumann the *direct rhombohedron*, and is denoted by the symbol mR. Its poles g lie in the diametral zone-circles Cr, Cr', Cr'' of Fig. 342, p. 382, and on the same side of C as r, r' and r''. When $m=1$, the rhombohedron is R, and is the same as Miller's {100}.

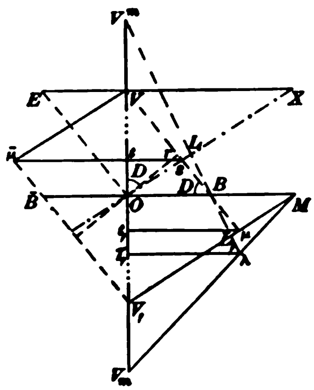


FIG. 338.

Fig. 338 is a vertical section of R and mR showing the lower polar edges $V_,M$ and $V_m M$, and the upper polar face-diagonals $V\mu$ and $V^m\lambda$. The lines $\mu t_,$, $\lambda\tau_,$ are drawn through the median coigns μ and λ parallel to OM; and meet the triad axis at $t_,$ and $\tau_,$.

Hence, $\qquad \mu t_, : OM = V_,\mu : VM = 2:3;$

and since $O\tau_, = OV_m \div 3$,

$$\lambda\tau_, : OM = V_m\lambda : V_m M = 2:3.$$

$$\therefore \; \lambda\tau_, = \mu t_, = 2OM : 3 = (\text{from } (21)) \; 2a \div \sqrt{3} \ldots\ldots (27).$$

ii. The second rhombohedron, called the *inverse rhombohedron*, and denoted by $-mR$, is obtained by joining the points M, M', M'' to V^m, and the opposite points $\overline{M}$, $M_,$, $M_{,,}$ to V_m; *i.e.* the points in

the equatorial plane are joined to the apices in the reverse order to that adopted in the first case, and this is indicated by making m negative. The mode of construction is the same as that given for $\{100\}$ and mR. Fig. 339 represents the inverse rhombohedron $-R$, or $\{122\}$, the faces of which will be denoted by the letter z: it corresponds exactly (except that the scale has been reduced in the ratio of 3 : 4) with the direct rhombohedron R, or $\{100\}$, shown in Fig. 335.

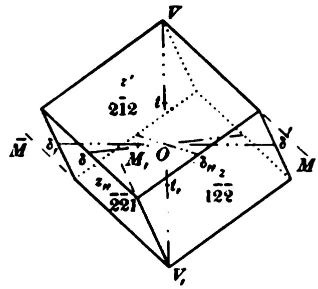


FIG. 339.

The following are a few instances of rhombohedra which are frequently observed in crystals of this class: $\frac{1}{2}R$, R, $2R$, $3R$, $-\frac{1}{2}R$, $-R$, $-2R$, $-3R$. The faces of the direct and inverse rhombohedra, in which m has the same value, are connected by the relation of dirhombohedral faces given in Art. 15; and their poles above the primitive lie in the diametral zones, $[Cr]$, $[Cr']$, $[Cr'']$ at equal distances from C.

34. The special case, where $m = 0$, gives a single plane coinciding with the equatorial plane, $MM'M''$. For the apex coincides with the origin O. Hence $0R$ is Naumann's symbol for the pinakoid $\{111\}$.

Similarly, when $m = \infty$, the vertex is infinitely distant, and the faces are those of the prism drawn through the lines MM', $M'M''$, &c., all parallel to the triad axis. Hence ∞R is Naumann's symbol for the hexagonal prism $\{2\bar{1}\bar{1}\}$ described in Art. 27.

35. To find the Millerian symbols for the faces of mR.

A face through $M_{,}M_{,,}$ of Fig. 335 and an apex V^m meets the axes of Y and Z at the same distance from the origin and on the same side of it, for OY and OZ are parallel to $VM_{,}$ and $VM_{,,}$, respectively. Hence the second and third indices of the face are equal, for the parameters are equal. But the axis OX will be met at a different distance; and on the same side of the origin as OY and OZ, if the face is less steeply inclined to the equatorial plane than the corresponding face of the fundamental rhombohedron, *i.e.* if V^m lies nearer to O than V. The axis of X will be met on the opposite side of the origin, if V^m is further away from O than V. The face is, therefore, (hll) where h and l have the same, or opposite signs

according as the pole of the face is nearer to C, or more remote from it, than r (100).

Let Fig. 340 represent part of a section of the rhombohedra mR and R in the plane Σ containing OX and the polar edges $V_m M$, $V_{,}M$. The coign λ is on the line drawn parallel to OM through $\tau_{,}$, where $O\tau_{,} = OV_m \div 3$. Also, $V^m\lambda$ is the polar face-diagonal and meets OX in L, OM in B and $V_{,}M$ in $L_{,}$. Adopting $V_{,}M$ as parameter, then $OL = V_{,}M \div h$. But, from the similar triangles $V_{,}L_{,}V^m$ and OLV^m, we have

$$\frac{V_{,}L_{,}}{OL} = \frac{V_{,}V^m}{OV^m} = \frac{mc + c}{mc} = \frac{m+1}{m} \qquad\text{...........(28).}$$

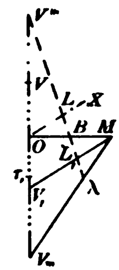


Fig. 340.

From the similar triangles OLB and $MBL_{,}$, we have

$$\frac{L_{,}M}{OL} = \frac{BM}{OB} = 1 \qquad\text{........................(29);}$$

since (Art. 31) $OB = BM$.

Adding (28) and (29), we have

$$h = \frac{V_{,}M}{OL} = \frac{m+1}{m} + 1 = \frac{2m+1}{m} \qquad\text{...............(30).}$$

The face through $V^m\lambda$ of Fig. 340 meets the adjacent plane Σ' in the polar edge $V^m\lambda_{,}$ which, since $OV^m > OV$, must be prolonged to meet OY at K on the negative side of the origin. Fig. 341 represents part of a section of R and mR in Σ'. But, since $VM_{,}$ is parallel to OY, the triangles V^mKO and $V^mM_{,}V$ are similar: we therefore have

$$\frac{VM_{,}}{OK} = \frac{VV^m}{OV^m} = \frac{mc - c}{mc} = \frac{m-1}{m} \qquad\text{.......(31).}$$

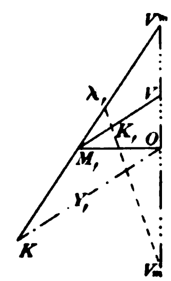


Fig. 341.

In (31) the lengths are all regarded as positive; but l is $V_{,}M' \div -OK$. Hence,

$$l = -\frac{V_{,}M'}{OK} = -\frac{VM_{,}}{OK} = \frac{1-m}{m}.$$

The Millerian symbol (hll) is therefore given by the ratios:

$$\frac{h}{2m+1} = \frac{l}{1-m} = \frac{l}{1-m} \qquad\text{..................(32);}$$

and, if the Millerian symbol is known, the Naumannian symbol mR is found from

$$m = \frac{h-l}{h+2l} \qquad\text{...........................(33).}$$

The same equations are found when OV^m is taken to be less than OV.

36. Equations (32) and (33) can also be obtained from the A. R. of the four poles $\{Cgr\mathfrak{m}\}$[1], where C, Fig. 342, is the pole (111), $g\,(hll)$, $r\,(100)$ and $\mathfrak{m}\,\{2\bar{1}\bar{1}\}$. For, in Fig. 338, OBV^m is the inclination to the pinakoid of the face of mR through $V^m\lambda$, and is therefore the angle Cg. Hence

$$\tan Cg = \tan OBV^m = OV^m \div OB = mc \div b.$$

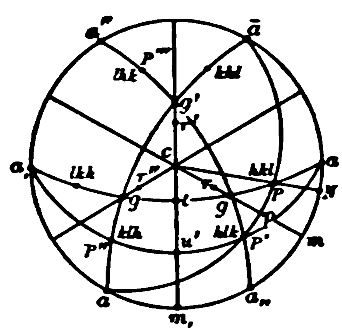


Fig. 342.

Again, $D = \wedge\, Cr = \wedge\, OBV$;

and $\tan D = \tan OBV = c \div b$.

Dividing the former tangent by the latter, we have

$$\tan Cg \div \tan D = m \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

But, from the A. R. $\{Cgr\mathfrak{m}\}$ of the four poles given above, we have

$$\frac{\sin Cg}{\sin Cr} \div \frac{\sin \mathfrak{m}g}{\sin \mathfrak{m}r} = \tan Cg \div \tan Cr = \begin{vmatrix} 111 & \cdot & 2\bar{1}\bar{1} \\ \dfrac{hll}{111} & \div & \dfrac{hll}{2\bar{1}\bar{1}} \\ 100 & . & 100 \end{vmatrix} = \frac{h-l}{h+2l} \ldots (35);$$

since $\mathfrak{m}g = 90° - Cg$, and $\mathfrak{m}r = 90° - Cr$.

The angles on the left sides of (34) and (35) being the same, the terms on the right must be equal.

Hence, $$m = \frac{h-l}{h+2l};$$

and $$\frac{h}{2m+1} = \frac{l}{1-m}.$$

These are the results given in (32) and (33).

The special forms given in Art. 34 are immediately deduced from the above equations; for, if $m = 0$, $h = l$, and the face is (lll) or (111). If $m = \infty$, then $h + 2l = 0$; and the face is $(2\bar{1}\bar{1})$ or $(\bar{2}11)$, and the form is the hexagonal prism $\{2\bar{1}\bar{1}\}$.

[1] A different type has been used to represent the pole $(2\bar{1}\bar{1})$ to avoid confusing it with the Naumannian index m.

37. The Millerian symbol for the inverse rhombohedron $-mR$ is found from equations (32) by changing the sign of m: it can also be obtained from the geometry of the figure as follows.

Let Fig. 343 be part of a section in the plane Σ of the rhombohedra R and $-mR$; and let the polar face-diagonal $V_m\gamma$ meet OX in G and the lower polar edge $V_{,}M$ of R in $G_{,}$. Then from the similar triangles V_mOG and $V_mV_{,}G_{,}$, we have

$$\frac{V_{,}G_{,}}{OG} = \frac{V_{,}V_m}{OV_m} = \frac{mc-c}{mc} = \frac{m-1}{m} \quad \text{..........(36).}$$

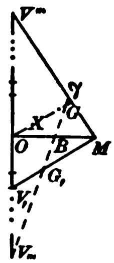


FIG. 343.

Again, from the similar triangles OBG and $MBG_{,}$, we have

$$\frac{G_{,}M}{OG} = \frac{BM}{OB} = 1 \quad \text{...................(37).}$$

$\therefore$ adding (36) and (37),

$$\frac{V_{,}M}{OG} = \frac{m-1}{m} + 1 = \frac{2m-1}{m} \quad \text{.................(38).}$$

Fig. 344 represents a section of the same rhombohedron in the plane Σ', where $V_mM_{,}$ is the polar edge of the face through $V_m\gamma$ of Fig. 343 and meets OY at F on the negative side of the origin.

From the similar triangles V_mFO, $V_mM_{,}V$, we have

$$\frac{VM_{,}}{OF} = \frac{VV_m}{OV_m} = \frac{mc+c}{mc} = \frac{m+1}{m} \quad \text{..........(39).}$$

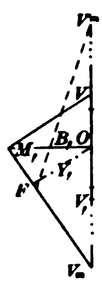


FIG. 344.

Hence, the intercepts on the axes are OG, $-OF$, $-OF$; or

$$\frac{mV_{,}M}{2m-1}, \; -\frac{mV_{,}M}{1+m}, \; -\frac{mV_{,}M}{1+m}.$$

If now the upper face parallel to that taken is denoted by $(h_{,}l_{,}l_{,})$, then the face through $V_m\gamma$ is $(\bar{h}_{,}\bar{l}_{,}\bar{l}_{,})$; and the intercepts it makes on the axes are in the ratios:—

$$\frac{V_{,}M}{-h_{,}} : \frac{V_{,}M}{-l_{,}} : \frac{V_{,}M}{-l_{,}}.$$

Hence,
$$\frac{h_{,}}{1-2m} = \frac{l_{,}}{1+m} \quad \text{........................(40).}$$

Also
$$m = \frac{l_{,}-h_{,}}{h_{,}+2l_{,}} \quad \text{..........................(41).}$$

It is clear that these expressions differ from those for mR given

in (32) and (33) only in the sign of m. Hence, expressions (32) and (33) may be taken to apply to both cases, provided care is taken to give the correct sign to m. Expression (33) serves also as a ready test for determining whether a rhombohedron $\{hll\}$ is a direct or an inverse one. Thus, when the indices of $\{011\}$ are introduced in (33), we have for m the value $-\frac{1}{2}$; similarly, $\{\bar{1}11\}$ gives $m = -2$; and $\{3\bar{1}\bar{1}\}$ gives $m = 4$.

It follows therefore that the simplest way of drawing any rhombohedron, the Millerian symbol of which is given, is to find m from equation (33). The apices V^m and V_m at distances mc from the origin can then be marked off on the vertical axis; and the drawing can be made in the manner described in Art. 31.

38. The symbols of the faces of the rhombohedron $\{100\}$ are given in table **d** of class II; those of $\{hll\}$, which may be either a direct or inverse rhombohedron, are:

$$hll \quad lhl \quad llh \quad \bar{h}\bar{l}\bar{l} \quad \bar{l}\bar{h}\bar{l} \quad \bar{l}\bar{l}\bar{h} \ldots\ldots\ldots\ldots\ldots\ldots\text{(j)}.$$

These rhombohedra are geometrically common to classes II, III and IV.

39. Since the median edges of rhombohedra and scalenohedra (Art. 40) are inclined to the equatorial plane and cross it in zigzag fashion, such forms can be quickly drawn in the following simple manner. The eye is supposed to be situated in the equatorial plane, and the triad axis to be in the paper: any horizontal plane is then reduced to a straight line.

Two straight lines $VOV_{\prime}$ and $FOF_{\prime}$, Fig. 345, are first drawn at

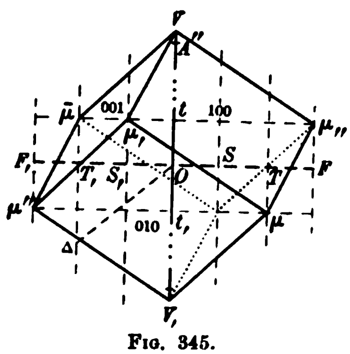


Fig. 345.

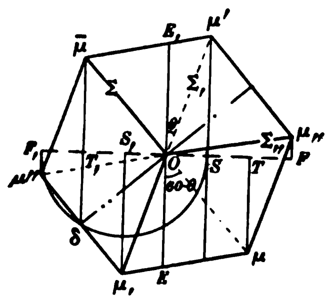


Fig. 346.

right angles to one another. On $FOF_{\prime}$ any six equal lengths $OS = ST = TF = OS_{\prime} = S_{\prime}T_{\prime} = T_{\prime}F_{\prime}$, are marked off. Lengths OV and $OV_{\prime}$,

equal to the distances of the rhombohedral apices from the origin are determined on the vertical line, and the length intercepted between the apices is then trisected in the points t and $t_{\prime}$. Through the points of trisection lines are drawn parallel to $FOF_{\prime}$ to intersect the verticals through T, F, &c., in the coigns μ, $\mu_{\prime\prime}$, &c. If $OV=c$, and $V\mu_{\prime\prime}$, $V\mu_{\prime}$, &c., are drawn as in the figure, we obtain the fundamental rhombohedron $R=\{100\}$. To draw the rhombohedron mR, we take $OV^m=OV_m=mc$, trisect V^mV_m in τ and $\tau_{\prime}$, and join the apices V_m, V^m to the coigns λ, $\lambda_{\prime\prime}$, &c., in which the horizontal lines through $\tau_{\prime}$ and τ meet the verticals through, T, F, &c., the median coigns being taken in the same order as that adopted for R. The inverse rhombohedron $-R=\{\bar{1}22\}$ is given by joining V to the points of intersection of the horizontal line $t\mu_{\prime\prime}$ with the verticals through S, T and $F_{\prime}$, and $V_{\prime}$ to the median coigns on the verticals through $T_{\prime}$, $S_{\prime}$, F: the figure is then easily completed. The rhombohedron $-mR$ is obtained in a similar manner.

All that is now needed to draw any scalenohedron mRn (Art. 40) is to take V^n, V_n on the triad axis at distances mnc from the origin, and join them to the median coigns λ of mR.

To find the unit of length OA'' corresponding to the arbitrary length OS, a circle is described with centre at $S_{\prime}$ and radius $=SS_{\prime}$: the point Δ where the circle cuts the vertical $\bar{\mu}T_{\prime}\Delta$ is at unit distance from O; and $OA''=O\Delta$. The length $OV=c$ for any particular substance is therefore $O\Delta \cos 30° \tan D$.

Since the horizontal plane is reduced to a line the method is not adapted to represent crystals in which lines parallel to the equatorial plane, but inclined to one another, have to be shown.

We proceed to indicate the positions of the planes of symmetry in such a projection as Fig. 345, and to prove that $O\Delta$ is the unit length on OV.

Suppose the rhombohedron represented in Fig. 345 to be turned through 90° about the line $FOF_{\prime}$, and to be projected on the paper: it will then be given by Fig. 346. During the rotation the coigns remain in the vertical planes through the points S, T, F, &c.; and the parallel vertical plane through the eye and centre may be represented by $EE_{\prime}$. The dyad axes (of which $O\delta$ is alone shown) now lie in the paper, and bisect pairs of opposite median edges.

Let $EO\mu_{\prime}=\theta$; then $EO\mu=60°-\theta$, and $F\mu_{\prime\prime}O=\mu_{\prime\prime}OE_{\prime}=60°+\theta$.

From the right-angled triangle $SO\mu'$, we have

$$OS=O\mu'\sin\theta;$$

from the triangle $TO\mu$, $\quad OT=O\mu\sin(60°-\theta)$;

and from the triangle $FO\mu_{\prime\prime}$, $OF=O\mu_{\prime\prime}\sin(60°+\theta)$.

But $$O\mu = O\mu' = O\mu_{,,};$$
and $$ST = OT - OS = O\mu\,\{\sin(60^\circ - \theta) - \sin\theta\};$$
$$TF = OF - OT = O\mu\,\{\sin(60^\circ + \theta) - \sin(60^\circ - \theta)\}.$$
Also $$TF = ST = OS.$$
$$\therefore\ \sin(60^\circ + \theta) - \sin(60^\circ - \theta) = \sin(60^\circ - \theta) - \sin\theta = \sin\theta \ldots (42).$$

The first and last terms of (42) are equal for any value of θ, for
$$\sin(60^\circ + \theta) - \sin(60^\circ - \theta) = 2\cos 60^\circ \sin\theta = \sin\theta;$$
since $$2\cos 60^\circ = 1.$$

The value of θ, corresponding to the orientation of the rhombohedron in the two figures, has therefore to be found by combining the first and second terms, or the second and third. Taking the first pair of sides in (42), we have
$$\sin(60^\circ + \theta) + \sin\theta = 2\sin(60^\circ - \theta) = 2\cos(30^\circ + \theta);$$
since $$60^\circ - \theta = 90^\circ - (30^\circ + \theta).$$

But
$$\sin(60^\circ + \theta) + \sin\theta = \sin(30^\circ + \theta + 30^\circ) + \sin(30^\circ + \theta - 30^\circ)$$
$$= 2\sin(30^\circ + \theta)\cos 30^\circ.$$
$$\therefore\ 2\sin(30^\circ + \theta)\cos 30^\circ = 2\cos(30^\circ + \theta);$$
and $$\cot(30^\circ + \theta) = \cos 30^\circ \ldots\ldots\ldots\ldots (43).$$

Therefore, by computation,

$30^\circ + \theta = 49^\circ\ 6{\cdot}4'$; and $\theta = 19^\circ\ 6{\cdot}4'$.

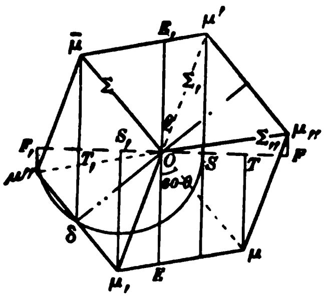


Fig. 346.

Hence, in Fig. 345, the plane Σ, containing $V_{,}\mu$ and the parallel axis OX, is inclined to the paper at an angle of $49^\circ\ 6{\cdot}4'$ to the right front; the planes $\Sigma_{,}$ and $\Sigma_{,,}$ containing OY and OZ respectively, are inclined to the left front at angles of $70^\circ\ 53{\cdot}6'$ and $10^\circ\ 53{\cdot}6'$ respectively.

We have now to determine the unit length OA'' on the vertical axis $VV_{,}$ of Fig. 345 which corresponds to the arbitrary length OS taken on the horizontal line. In Fig. 346 $O\delta$ is a dyad axis, and its length is that denoted by a. But
$$\wedge\ \delta OE = 30^\circ + \theta = \wedge\ T_{,}\delta O.$$
$$\therefore\ T_{,}\delta = T_{,}O\cot T_{,}\delta O = T_{,}O\cot(30^\circ + \theta) = (\text{from } (43))\ T_{,}O\cos 30^\circ = \sqrt{3}\,T_{,}O \div 2.$$
Also, $$S_{,}T_{,} = T_{,}O \div 2;$$
$$\therefore\ S_{,}\delta^2 = T_{,}\delta^2 + T_{,}S_{,}^2 = T_{,}O^2(3+1) \div 4 = T_{,}O^2.$$
$$\therefore\ S_{,}\delta = T_{,}O = S_{,}S = S_{,}F_{,}.$$

Hence δ lies on a circle described with centre at $S_{,}$ and radius $= SS_{,} = 2OS$. To obtain OA'' in Fig. 345, a circle is described with centre at $S_{,}$ and radius $= 2OS$: the point Δ, in which the circle cuts the vertical through $T_{,}$, is at distance a from O. OA'' is then cut off equal to $O\Delta$; and OV is found by multiplying this length by c, or $\cos 30^\circ \tan D$.

The scalenohedron, $\{hkl\}$.

40. By the law of rational indices, faces belonging to a crystal can be drawn through an edge of any of its forms to meet any non-parallel zone-axis at distances which are commensurable multiples of the intercept made on this axis by one of the faces. The form through the edge of which the new faces have to be drawn will be called the *auxiliary form*. Thus a pair of possible faces of a crystal of this class can be drawn through a median edge of any rhombohedron mR to meet the triad axis at points V_n, V^n, where $OV_n = OV^n = nOV_m$. But by the character of the symmetry, similar pairs of faces must be drawn through each of the remaining median edges. Since the points V_n and V^n on the triad axis are equally distant from the centre of the rhombohedron and opposite median edges are parallel, it follows that the form, Fig. 347, has twelve faces arranged in pairs which are parallel. Further, each face is a scalene triangle having its polar edges in two of the planes of symmetry Σ, $\Sigma_{,}$, $\Sigma_{,,}$; and the angles over the three edges are unequal. The form is therefore called the scalenohedron of the rhombohedral system; and as its faces will be shown to meet the Millerian axes of reference at unequal distances, its symbol will be $\{hkl\}$. The symbol mRn was used by Naumann to denote the form. In this symbol the number m *preceding* R indicates the auxiliary rhombohedron, the median coigns λ, $\lambda_{,}$, &c., of which have to be joined to V^n and V_n. The number n *following* R indicates the multiple of OV_m required to give OV_n, which is therefore mnc.

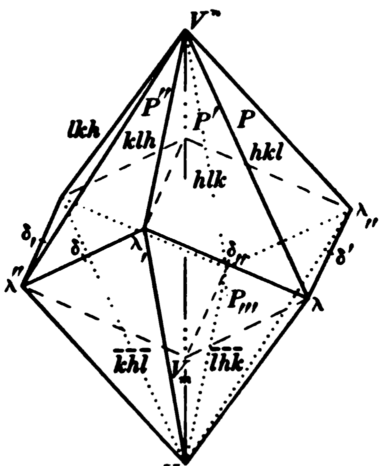


Fig. 347.

The above method of derivation of the general form of the class is similar to that by which the tetrakis-hexahedron $\{hk0\}$ of the cubic system was, in Chap. XIV., Art. 18, derived from the cube. It affords a ready means of drawing the form when the Naumannian indices m and n have been determined.

From the method of construction it follows that the median coigns of the scalenohedron lie in the horizontal planes trisecting at τ and $\tau_,$ the length $V^m V_m$ intercepted between the apices of the auxiliary rhombohedron mR. Since the polar edges of all rhombohedra join the apices to the points M in the equatorial plane, then, (Art. 33), $\tau_,\lambda = \tau_,\lambda'' = \&c. = 2OM \div 3 = 4a \cos 30° \div 3$, a constant distance independent of the values of m and n. Hence the median coigns of all scalenohedra lie in the vertical edges of the prism $\{0\bar{1}1\}$, when its faces are drawn through the points A, A', A'', &c.; OA being the unit of length a on the dyad axis in Fig. 332.

41. The limits of the series of scalenohedra having the same median edges as the auxiliary rhombohedron mR are given by making: (i) $n = 1$, when the scalenohedron becomes identical with mR itself; (ii) $n = \infty$, when all the faces become parallel to the triad axis. The form then becomes the hexagonal prism, the faces of which truncate the median edges of the rhombohedron, and are therefore perpendicular each to one of the dyad axes: this prism was shown in Art. 27 to be $\{0\bar{1}1\}$.

42. We proceed to find the intercepts made by the faces on the axes of reference; and hence to determine the connection between the Millerian symbol $\{hkl\}$ and the Naumannian symbol mRn. Incidentally, expressions will be obtained which will enable us to transform the above symbols to those involving four axes of reference. Rhombohedral crystals are still frequently referred to a set of four axes; but such axes are better suited to represent the forms of hexagonal crystals, and their discussion will be postponed to the next chapter.

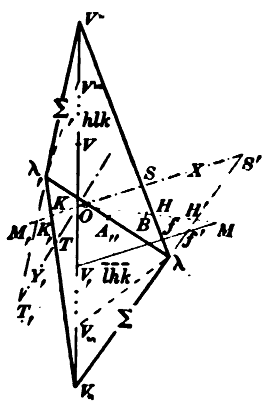


FIG. 348.

Let $V^n\lambda\lambda_,$ be the face (hlk). In Fig. 348 two faces and the axes OX and OY are alone shown. The polar edges $V^n\lambda$, $V_n\lambda$ lie in the plane Σ and meet the axis OX in the points S and S', and the horizontal line OM in the points H and H'. The similar polar edges $V_n\lambda_,$, $V^n\lambda_,$ lie in $\Sigma_,$, and meet OY in the points T and $T_,$, and the horizontal line $OM_,$ in

K and $K_{,}$. The median edge $\lambda\lambda_{,}$ meets the dyad axis at $A_{,,}$, where $OA_{,,} = a$, and $A_{,,}\lambda = A_{,,}\lambda_{,}$.

By a semi-revolution about $OA_{,,}$ the two faces change places, the edges $V^{n}\lambda_{,}$ and $V_{n}\lambda$ changing places, also $V^{n}\lambda$ and $V_{n}\lambda_{,}$. Hence, $OT_{,}$ on $OY_{,}$ is equal to OS' measured on OX, and likewise $OT = OS$. Furthermore, since the same rotation interchanges OM and $OM_{,}$, we have $OK = OH$, and $OK_{,} = OH'$.

Hence, to find the line $HA_{,,}K_{,}$, in which the face $V^{n}\lambda\lambda_{,}$ meets the equatorial plane, it suffices to determine the points H and H' in which the edges $V^{n}\lambda$, $V_{n}\lambda$ meet OM. The points S and S', and the intercepts OS and $OT_{,}$ are then easily found.

Let Fig. 349 represent a section of the scalenohedron and of the rhombohedron mR by the plane Σ, which contains the polar edges $V^{n}\lambda$, $V_{m}\lambda$, and OX parallel to the edge $V_{,}M$ of the fundamental rhombohedron. Now $O\tau_{,} = OV_{m} \div 3 = mc \div 3$, and $\tau_{,}\lambda = 2OM \div 3$.

From the similar triangles OHV^{n} and $\tau_{,}\lambda V^{n}$, we have

$$\frac{\tau_{,}\lambda}{OH} = \frac{\tau_{,}V^{n}}{OV^{n}} = \frac{mnc + mc \div 3}{mnc} = \frac{3n+1}{3n} \quad (44);$$

$$\therefore \frac{OM}{OH} = \frac{3n+1}{2n} \quad \ldots\ldots\ldots (45).$$

From the similar triangles $OH'V_{n}$, $\tau_{,}\lambda V_{n}$, we have

$$\frac{\tau_{,}\lambda}{OH'} = \frac{\tau_{,}V_{n}}{OV_{n}} = \frac{mnc - mc \div 3}{mnc} = \frac{3n-1}{3n} \quad (46);$$

$$\therefore \frac{OM}{OH'} = \frac{3n-1}{2n} \quad \ldots\ldots\ldots (47).$$

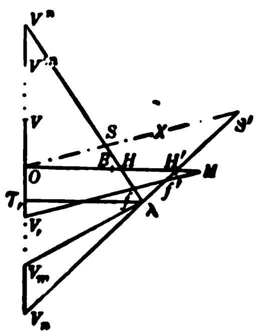


FIG. 349.

The trace of the face in the equatorial plane may be taken to coincide with the line $HK_{,}L_{,}$ of Fig. 332, p. 372. The distance $OL_{,}$ on OM'' is then given by the fact that the triangle $K_{,}OL_{,}$ is made up of $K_{,}OH$ and $HOL_{,}$, of which the angles at O are 120° and 60°. Hence,

$$OK_{,} . OL_{,} \sin 120^\circ = OK_{,} . OH \sin 60^\circ + OH . OL_{,} \sin 60^\circ ;$$

$$\therefore \frac{OM}{OL_{,}} = \frac{OM}{OH} - \frac{OM}{OK_{,}} = \frac{3n+1}{2n} - \frac{3n-1}{2n} = \frac{1}{n} \quad \ldots\ldots (48);$$

since $OK_{,} = OH'$.

43. To find h, k, l in terms of m and n.

The Millerian indices h, l, k, are given by $h = V_{,}M \div OS$,

$l = V_{,}M' \div OT_{,}$, $k = V_{,}M'' \div OU$; if, as before, $V_{,}M$ is the parameter, and OS, $OT_{,}$ and OU are the intercepts on the axes, due attention being paid to the directions in which they are measured. By a semi-revolution about $OA_{,,}$ the polar edge $V^{n}\lambda_{,}T_{,}$, Fig. 348, is brought to the position $V_{n}\lambda H'$ which meets OX in S'. The intercept $OT_{,}$, measured in the negative direction along $OY_{,}$ is equal to OS' measured on OX. Hence

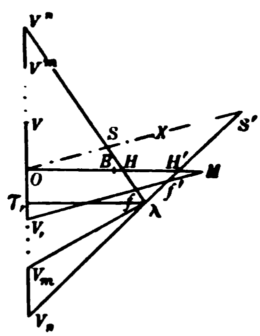


Fig. 349.

$$l = -V_{,}M \div OS'.$$

Since OX is parallel to $V_{,}M$, we can, by pairs of similar triangles, find $V_{,}M \div OS$ and $V_{,}M \div OS'$, and therefore the indices h and l. Let, in Fig. 349, f be the point of intersection of $V_{,}M$ and $V^{n}\lambda$, and f' that of $V_{,}M$ and $V_{n}\lambda$. From the similar triangles OHS, MHf, we have

$$\frac{fM}{OS} = \frac{HM}{OH} = \frac{OM - OH}{OH} = \text{(from (45))}\ \frac{3n+1}{2n} - 1 = \frac{n+1}{2n} \dots(49).$$

And from the similar triangles OSV^{n}, $V_{,}fV^{n}$, we have

$$\frac{V_{,}f}{OS} = \frac{V_{,}V^{n}}{OV^{n}} = \frac{mnc + c}{mnc} = \frac{mn+1}{mn} \dots\dots\dots\dots(50).$$

Adding (49) and (50), we find

$$h = \frac{V_{,}M}{OS} = \frac{n+1}{2n} + \frac{mn+1}{mn} = \frac{3mn + m + 2}{2mn} \dots\dots(51).$$

Again, from the similar triangles $H'f'M$, $H'OS'$, we have

$$\frac{f'M}{OS'} = \frac{H'M}{OH'} = \frac{OM - OH'}{OH'} = \text{(from (47))}\ \frac{3n-1}{2n} - 1 = \frac{n-1}{2n} \dots(52);$$

and from the similar triangles $S'OV_{n}$, $f'V_{,}V_{n}$,

$$\frac{V_{,}f'}{OS'} = \frac{V_{,}V_{n}}{OV_{n}} = \frac{mnc - c}{mnc} = \frac{mn-1}{mn} \dots\dots\dots\dots(53).$$

$\therefore$ adding (52) and (53), we have

$$-l = \frac{V_{,}M}{OS'} = \frac{n-1}{2n} + \frac{mn-1}{mn} = \frac{3mn - m - 2}{2mn} \dots\dots(54).$$

Again, if, in Fig. 350, $V^nL_,$ is the trace of (hlk) in the plane $\Sigma_{,,}$, and if it meets OZ and $V_,M''$ produced in U and G, then from the similar triangles $UOL_,$, $GM''L_,$,

$$\frac{M''G}{OU} = \frac{M''L_,}{OL_,} = \frac{OM''}{OL_,} + 1$$

$$= (\text{from (48)}) \ \frac{1}{n} + 1 \ \ldots\ldots(55).$$

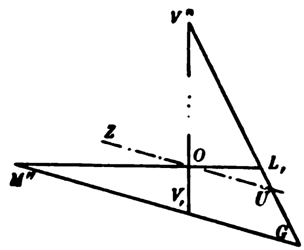


FIG. 350.

And from the similar triangles V^nOU, $V^nV_,G$, we have

$$\frac{V_,G}{OU} = \frac{V_,V^n}{OV^n} = \frac{mn+1}{mn} \ \ldots\ldots\ldots\ldots\ldots\ldots (56).$$

Subtracting (56) from (55), we have

$$\frac{M''V_,}{OU} = \frac{1}{n} + 1 - \frac{mn+1}{mn} = \frac{2(m-1)}{2mn} \ \ldots\ldots\ldots\ldots (57).$$

The above equation gives the numerical value of the ratio $V_,M'' \div OU$. Since, however, OU is measured in the negative direction, the index

$$k = -M''V_, \div OU = 2(1-m) \div 2mn \ldots\ldots\ldots\ldots (58).$$

Hence from (51), (54) and (58),

$$\frac{h}{m+2+3mn} = \frac{l}{m+2-3mn} = \frac{k}{2(1-m)} \ \ldots\ldots\ldots\ldots (59).$$

44. Equations (59) suffice for finding both m and n when h, k and l are given. For, if we add or subtract the numerators, and also the denominators, of any of the ratios, we obtain a ratio equal to any one of them. Thus, taking the first and second terms, we have by addition

$\dfrac{h+l}{2(m+2)} = \dfrac{k}{2(1-m)} =$ (doubling the terms of the latter and subtracting from the former) $\dfrac{h+l-2k}{6m} =$ (by addition) $\dfrac{h+k+l}{6}$.

$$\therefore \ m = \frac{h+l-2k}{h+k+l} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (60);$$

Again, subtracting the numerator and denominator of the second member of (59) from those of the first, we have

$$\frac{h-l}{6mn} = \frac{k}{2(1-m)} = (\text{from above}) \ \frac{h+k+l}{6};$$

$$\therefore \ n = \frac{h-l}{h+k+l}\,\frac{1}{m} = \frac{h-l}{h+l-2k} \ \ldots\ldots\ldots\ldots\ldots (61).$$

Knowing then the indices h, k, l of one face of the scalenohedron, we can find m and n; and can then draw the form by the method given in Art. 40.

45. To find the symbols of all the faces of $\{hkl\}$.

The pair of faces passing through $\lambda\lambda_{,}$ of Fig. 351 change places after a semi-revolution about $O\delta_{,,}$. This axis, being perpendicular to OZ, interchanges positive with equal negative lengths on it; and positive lengths on OX with equal negative ones on OY, and vice versâ. Hence, the face $V^n\lambda\lambda_{,}$ being (hlk), the face $V_n\lambda\lambda_{,}$ is $(\bar{l}\bar{h}\bar{k})$. This is also obvious from the discussion in the two last articles and Fig. 348.

The above two faces are associated with two parallel faces drawn through the median edge parallel to $\lambda\lambda_{,}$. The symbols of the latter faces are, therefore, $(\bar{h}\bar{l}\bar{k})$ and (lhk). The four faces are necessarily tautozonal with the four rhombohedral faces meeting in the same median edges, and with the two prism-faces $(1\bar{1}0)$ and $(\bar{1}10)$ truncating these edges.

Again, by rotations of 120° about the triad axis the above four faces are brought into the positions of two other sets of four similar faces. We have already seen that the triads of interchangeable faces have their symbols in the same cyclical order; hence the faces of the scalenohedron $\{hkl\}$, Fig. 351, taken in order from $V^n\lambda\lambda_{,}$ and $V_n\lambda\lambda_{,}$, have the following symbols:

$$\left.\begin{matrix} hlk & hkl & khl & lhk & lkh & klh \\ \bar{l}\bar{h}\bar{k} & \bar{l}\bar{k}\bar{h} & \bar{k}\bar{l}\bar{h} & \bar{h}\bar{l}\bar{k} & \bar{h}\bar{k}\bar{l} & \bar{k}\bar{h}\bar{l} \end{matrix}\right\} \quad \ldots\ldots(\mathrm{k}).$$

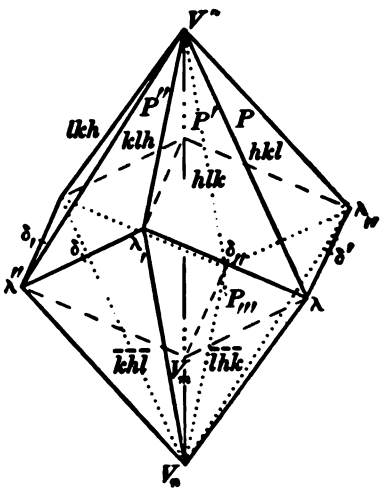


FIG. 351.

The pairs of faces in the columns are interchangeable by semi-revolutions about the dyad axes $\delta_{,,}$, $\delta_{,}$ and δ, respectively.

It is easy to see that the above faces are symmetrically placed with respect to the planes of symmetry. Thus the pairs of faces meeting in the edges $V^n\lambda$, $V_n\lambda$ meet OX in S and S' respectively, where $OS = V_{,}M \div h$,

$OS_{,} = V_{,}M \div \bar{l}$, for l is a negative number (Art. 43). But the plane Σ bisects the angle between OY and OZ, and the intercepts on these axes are reciprocal reflexions. The face $V^{n}\lambda\lambda_{,}$ being (hlk), the symbol of $V^{n}\lambda\lambda_{,,}$ is (hkl); and the face $V_{n}\lambda\lambda_{,}$ being $(\bar{l}\bar{h}\bar{k})$ the symbol of $V_{n}\lambda\lambda_{,,}$ is $(\bar{l}\bar{k}\bar{h})$. Similar proofs can be applied to the pairs which meet in polar edges $V^{n}\lambda_{,}$, &c.; and also to pairs of faces which do not meet in edges, such as $V^{n}\lambda_{,}\lambda''$ (klh) and $V^{n}\lambda'\lambda_{,,}$ (khl).

46. The relations of the poles P of the scalenohedron $\{hkl\} = mRn$ and of g those of the inscribed auxiliary rhombohedron mR, are shown in Fig. 352. The two faces of each form which meet in the edge $\lambda\lambda_{,}$ must be in a zone with $a_{,,}(1\bar{1}0)$, the prism-face perpendicular to the dyad axis $O\delta_{,,}$; and this face would, if developed, truncate the edge. It is clear, however, that the two faces of the scalenohedron make a smaller angle with $(1\bar{1}0)$ than the two faces of the rhombohedron g. Hence the pole P' lies between $a_{,,}$ and g. If P' is (hlk) and g in $[a_{,,}P']$ is $(h_{,}l_{,}l_{,})$; then, by Weiss's zone-law,

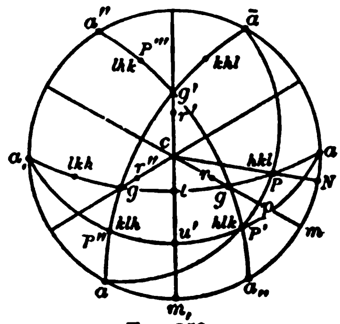


FIG. 352.

$$h_{,}k + l_{,}k - l_{,}(h + l) = 0.$$

This equation is satisfied by making $h_{,} = h - k + l$, and $l_{,} = k$. The inscribed rhombohedron mR has therefore the symbols $\{h - k + l, k, k\}$. But, in Art. 36, it was shown that, if mR is identical with $\{h_{,}l_{,}l_{,}\}$, then $m = (h_{,} - l_{,}) \div (h_{,} + 2l_{,})$. Introducing into this expression the values of $h_{,}$ and $l_{,}$ just found, we have

$$m = \frac{h - k + l - k}{h - k + l + 2k} = \frac{h - 2k + l}{h + k + l} = \frac{\theta - 3k}{\theta};$$

the same result as is given in (60).

The number n can now be found from equations (59).

47. Although, in the method of derivation employed in Art. 40 and in all the succeeding Articles, the number m has been supposed to be positive, the process is perfectly general; and all the relations hold true if m is negative and the inscribed rhombohedron is the inverse form, $-mR$. All that is necessary is to make m negative in equations (59) and in all equations into which m enters.

Such a scalenohedron may, when it is desired to indicate its precise position with respect to the axes of reference, be called an *inverse scalenohedron*, $-mRn$.

It is clear therefore that, for the same numerical values of m and n, we have two tautomorphous scalenohedra, the positions of which in space are determined by the sign of m; and the poles of which are shown in Fig. 353.

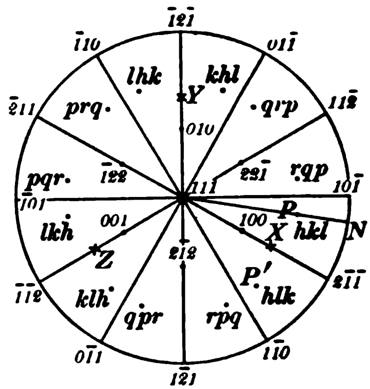


FIG. 353.

It is required to find the relation between the Millerian indices of the direct scalenohedron $\{hkl\} = mRn$ and the correlative inverse scalenohedron $\{pqr\} = -mRn$.

In Arts. 43 and 44 we have seen that for the face (hlk),

$$\frac{h}{m+2+3mn} = \frac{l}{m+2-3mn} = \frac{k}{2(1-m)} = \frac{h+k+l}{6} \ldots\ldots(62).$$

Changing the sign of m, we have, for the *opposite* face (prq)

$$\frac{p}{-m+2-3mn} = \frac{r}{-m+2+3mn} = \frac{q}{2(1+m)} \ldots\ldots\ldots(63).$$

If from double the numerator and denominator of the last term of equations (62) we subtract three times the numerator and denominator of each of the preceding terms in turn, we have

$$\frac{h+k+l}{6} = \frac{2(h+k+l)-3h}{3(2-m-3mn)} = \frac{2(h+k+l)-3l}{3(2-m+3mn)}$$
$$= \frac{2(h+k+l)-3k}{6(1+m)} \ldots\ldots\ldots\ldots (64).$$

The denominators in the last three terms of (64) are in the same ratio as the denominators of equation (63): hence the numerators must be in the same ratios.

$$\therefore \frac{p}{2(h+k+l)-3h} = \frac{q}{2(h+k+l)-3k} = \frac{r}{2(h+k+l)-3l} \ldots(65).$$

These are the same relations as were found in Art. 15 to connect the symbols of two dirhombohedral faces. As in that Article, it is easy to show that

$$\frac{h}{2(p+q+r)-3p} = \frac{k}{2(p+q+r)-3q} = \frac{l}{2(p+q+r)-3r} \ldots\ldots(66).$$

48. As a general rule the forms will fall into zones in which the angles can be measured, and in which the symbols of several faces are known or can be easily found. Hence, the anharmonic ratio of four tautozonal faces will usually give the Millerian symbols of the unknown faces.

When, however, the symbols cannot be thus obtained, we can use the elegant relations between the angles of the scalenohedron and the face-indices which we proceed to determine in this and the two next articles.

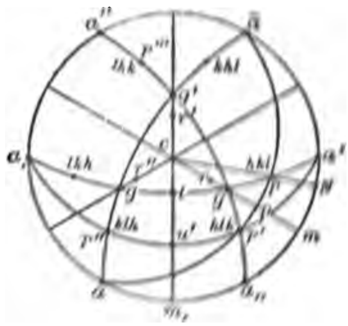


Fig. 354.

Let the angles PP', $P'P''$, $P'P_{\prime\prime\prime}$, over the dissimilar edges $V''\lambda$, $V''\lambda_{\prime}$ and $\lambda\lambda_{\prime}$ of Fig. 351 be denoted by 2ξ, 2η and 2ζ, respectively. Then, in Fig. 354,

$$\xi = Pp = 90^\circ - \bar{a}P\,;$$

$$\eta = u'P' = 90^\circ - a'P' = 90^\circ - a_{\prime\prime}P\,;$$

and $$\zeta = a_{\prime\prime}P' = a'P.$$

Hence, from the right-angled spherical triangles $\bar{a}PN$, $a'PN$, $a_{\prime\prime}PN$ we have

$$\sin\xi = \cos\bar{a}P = [illegible]$$
$$\sin\eta = \cos a_{\prime\prime}P = [illegible] \qquad [illegible]$$
$$\cos\zeta = \cos a'P = [illegible]$$

Adding the first and second equations we have

$$\sin\xi + \sin\eta = [illegible]$$
$$= [illegible]$$
$$= [illegible],$$

since [illegible]

[illegible]

This equation can be [illegible] logarithmic computation [illegible] found, the third can be [illegible]

Thus [illegible]

[illegible]

[illegible]

49. In [illegible] those of [illegible]

Art. 27 an expression (20) for $\tan a'N$ was found in terms of the indices of N. Hence, introducing the values of e and f in terms of h, k and l, into equation (20), we have

$$\sqrt{3}\tan a'N = \frac{-3f}{2e+f} = \frac{3(\theta-3k)}{2(3h-\theta)+3k-\theta}$$

$$= \frac{\theta-3k}{h-l} \quad\text{.........(70).}$$

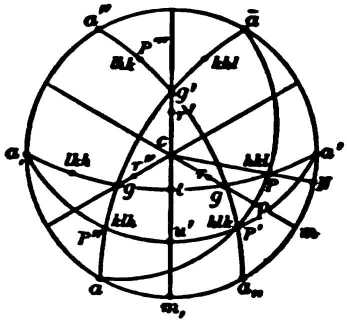


Fig. 354.

But from (67),

$$\frac{\sin\xi}{\cos\zeta} = \frac{\cos \bar{a}N}{\cos a'N} = \frac{\cos(60^\circ + a'N)}{\cos a'N}$$

$$= \cos 60^\circ - \sin 60^\circ \tan a'N$$

$$= \tfrac{1}{2}(1 - \sqrt{3}\tan a'N).$$

Hence, introducing the value of $\sqrt{3}\tan a'N$, we have

$$\frac{\sin\xi}{\cos\zeta} = \tfrac{1}{2}\left(1 - \frac{\theta-3k}{h-l}\right) = \frac{k-l}{h-l} \quad\text{...............(71).}$$

Similarly,

$$\frac{\sin\eta}{\cos\zeta} = \frac{\cos a_{,,}N}{\cos a'N} = \frac{\cos(60^\circ - a'N)}{\cos a'N}$$

$$= \tfrac{1}{2}(1 + \sqrt{3}\tan a'N) = \frac{h-l+\theta-3k}{2(h-l)} = \frac{h-k}{h-l} \quad\text{...... (72).}$$

And, since $\wedge CP = 90^\circ - PN$, we have from the last equation in (67)

$$\frac{\sin CP}{\cos\zeta} = \frac{1}{\cos a'N} = \sqrt{(1+\tan^2 a'N)} = \text{(by transformation from (70))}$$

$$\frac{\sqrt{2\{(h-k)^2+(k-l)^2+(l-h)^2\}}}{(h-l)\sqrt{3}} \quad\text{... (73).}$$

Hence from (71)—(73),

$$\frac{\sin\xi}{k-l} = \frac{\sin\eta}{h-k} = \frac{\cos\zeta}{h-l} = \frac{\sqrt{3}\sin CP}{\sqrt{2\{(h-k)^2+(k-l)^2+(l-h)^2\}}} \quad\text{......(74).}$$

When therefore the indices and one of the above angles are known, equations (74) enable us to compute each of the remaining angles.

50. To connect the angles involved in (74) with the angular element of the crystal we shall employ equations (12) of Art. 14. Thus,

$$\cos YP = \cos CY \cos CP\,(1 + \tan CY \tan CP \cos m'N)$$

$$= \cos CX \cos CP\,(1 - \tan CX \tan CP \sin a'N);$$

since $\qquad CY = CX$, and $m'N = 90^\circ + a'N$.

Introducing this value of $\cos YP$ into equations (13), we have

$$1 - \tan CX \tan CP \sin a'N = \frac{3k}{\theta}.$$

But from (3), $\tan CX = 2 \cot D$,

$$\therefore\ 2 \tan CP \cot D \sin a'N = \frac{\theta - 3k}{\theta} \ldots\ldots\ldots\ldots (75).$$

And from (70), it can be shown that

$$\sin a'N = \frac{\theta - 3k}{\sqrt{2\{(h-k)^2 + (k-l)^2 + (l-h)^2\}}}.$$

Introducing into equation (75), we have

$$\tan CP = \frac{\sqrt{(h-k)^2 + (k-l)^2 + (l-h)^2}}{(h+k+l)\sqrt{2}} \tan D \ldots\ldots (76).$$

If now the value of $\sin CP$, deduced from (76), is introduced into equations (74), the latter can be given in the following form:

$$\frac{\sin \xi}{k-l} = \frac{\sin \eta}{h-k} = \frac{\cos \zeta}{h-l} = \frac{\sin 60^\circ \sin D}{\sqrt{(h+k+l)^2 - 3(hk+kl+lh)\sin^2 D}} \ldots (77).$$

The last term of (77) is not however in a form suitable for logarithmic computation: it is better therefore to compute CP from (76), and then to substitute the value in equations (74).

Another easy way of finding CP is first to calculate Cp, p being the pole of the face truncating the obtuse edge $V^n\lambda$ of Fig. 351. Hence, the arc Cp of Fig. 354 is the angle V^nHO of Fig. 349;

$$\therefore\ \tan Cp = \tan V^nHO = OV^n \div OH;$$

and

$$\frac{OV^n}{OH} = \frac{OV^n}{OM} \times \frac{OM}{OH} = \frac{mnc}{2a \cos 30^\circ} \times \frac{3n+1}{2n} = \frac{3mn+m}{4} \tan D,$$

since $$c \div a \cos 30^\circ = \tan D.$$

But from (62),

$$3mn + m + 2 = 2h(1-m) - k - 6h - \theta;$$

$$\therefore\ \tan Cp = \frac{3mn+m}{4} \tan D = \frac{3h-\theta}{2\theta} \tan D = \frac{2h-k-l}{2(h+k+l)} \tan D$$
$$\ldots\ldots\ldots\ldots\ldots (78)$$

Expression (76) for $\tan CP$ is easily deduced from (78) by the aid of the right-angled spherical triangle pCP, and the known value of $a'N$; for $$\tan CP = \tan Cp \div \cos(mN - 30^\circ - a'N).$$

51. If the scalenohedron $P\{hkl\}$ is the only form present in a crystal of which the element D is known, the equations given in Arts. 48—50 enable us to determine the indices, when two of the

angles 2ξ, 2η and 2ζ have been measured; or, assuming the indices to be any numbers consistent with (74), to find the angular element D which corresponds to the measured angles and the assumed indices. Thus two of the angles being measured, the third is found from equations (69). The angle $P\bar{a}N$ can then be computed by the formula for finding the angle of a triangle of which three sides are known: thus in the triangle $a'P\bar{a}$, Fig. 354, the sides are: $\bar{a}P=90°-\xi$, $a'P=\zeta$ and $a'\bar{a}=60°$. Hence, $Cp=90°-pm=90°-P\bar{a}N$ is known; and equation (78) gives a simple equation between the indices. A second simple equation is given by (74). The solution of these two equations gives the ratios $h \div l$, $k \div l$, and the symbol of the face.

Example. Miller gives the element of calcite as 44° 36·6′, and for the scalenohedron v, Fig. 355, common on Derbyshire crystals he gives $2\xi=35°\,36'$, $2\eta=75°\,22'$. Hence, from (69)

$$\cos\zeta=2\sin 27°\,44{\cdot}5'\cos 9°\,56{\cdot}5';$$

and $$\zeta=23°\,30{\cdot}5'.$$

The sides of the triangle $a'P\bar{a}$ are:

$$\bar{a}P=90°-\xi=72°\,12',\quad a'P=23°\,31',\quad \bar{a}a'=60°.$$

$$\therefore\ \tan\tfrac{1}{2}a'\bar{a}P=\tan\tfrac{1}{2}mp=\sqrt{\frac{\sin 5°\,39{\cdot}5'\sin 17°\,51{\cdot}5'}{\sin 77°\,51{\cdot}5'\sin 54°\,20{\cdot}5'}};$$

and, by computation,

$$a'\bar{a}P=mp=22°\,4{\cdot}8'.$$

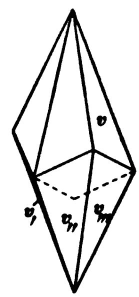


FIG. 355.

Hence, by equation (78)

$$\frac{2h-k-l}{2(h+k+l)}=\cot 22°\,4{\cdot}8'\cot 44°\,36{\cdot}6'=\text{(by computation) } 5\div 2;$$

$$\therefore\ h+2k+2l=0 \qquad\qquad (79).$$

Again, by equations (74)

$$\frac{h-k}{k-l}=\frac{\sin 37°\,41'}{\sin 17°\,48'}=\text{(by computation) } 2;$$

$$\therefore\ h-3k+2l=0 \qquad\qquad (80).$$

Subtracting (80) from (79), we have $5k=0$, $\therefore k=0$: and therefore $h+2l=0$. The face is therefore $(20\bar{1})$ and the form $\{20\bar{1}\}$.

If it be desired to determine the element from the measured angles, it is necessary to assume definite indices, $(20\bar{1})$ say, for the face P. Then after computation of mp, we have $\cot D=5\tan 22°\,4{\cdot}8'\div 2$. But it is clear that equations (74) will not be satisfied if any three numbers are assumed for indices: an arbitrary choice can only be made of two of them, and the third is then deduced from (74).

52. Each scalenohedron $\{hkl\}=mRn$ has five associated rhombohedra, the symbols of which can be readily obtained by the aid of the stereogram, Fig. 356, and Weiss's zone-law, or from the geometrical relations given in preceding Articles. A knowledge of

these rhombohedra frequently [illegible] a combination such as that in Fig. [illegible]

i. The inscribed rhombohedron [illegible] edges as the scalenohedron.

ii. The rhombohedron [illegible] polar edges [illegible] to these edges [illegible] Fig. 349 [illegible]

$$\frac{OT_a}{OT_a} = \frac{OI}{OE} = [illegible]$$

The figure is then [illegible] h, k and l can [illegible]

iii. The rhombohedron [illegible] edges $T'A$. The [illegible] parallel to $T'A$ [illegible] Fig. [illegible]

[illegible]

iv and v. The [illegible] have for polar edges [illegible] scalenohedron [illegible] tance of those [illegible] M parallel to the [illegible]

From the [illegible] obtained [illegible] the [illegible]

Fig. [illegible]

Hence [illegible]

[illegible]

The poles having the above symbols are on the upper hemisphere, for the sum of the indices is in all cases either θ or 2θ, a known positive number.

By the aid of equations (32) and (59) the student can prove that OV^p has the value already given. Thus from (32)

$$OV^p = \frac{2h - k - l}{2(h + k + l)} c = [\text{from (59)}] \; \frac{3m + 9mn}{12} c = \frac{3n + 1}{4} mc.$$

The distances from the origin of the apices V_x, V_y, V_q can be obtained in a similar manner.

As illustrations of the above we may take the scalenohedra $\{20\bar{1}\}$ and $\{410\}$ of calcite. For the former, the five rhombohedra are: (i) $\{100\} = R$; (ii) $\{1\bar{1}1\} = -2R$; (iii) $\{4\bar{1}\bar{1}\} = \frac{5}{2}R$; (iv) $\{3\bar{1}\bar{1}\} = 4R$; (v) $\{2\bar{3}2\} = -5R$. For the latter, the rhombohedra are: (i) $\{311\} = \frac{2}{5}R$; (ii) $\{101\} = -\frac{1}{2}R$; (iii) $\{811\} = \frac{7}{10}R$; (iv) $\{100\} = R$; (v) $\{4\bar{3}4\} = -\frac{7}{5}R$.

The hexagonal bipyramid.

53. When the pole P of a face (hkl) lies in the zone $[Ca_{,}] = [1\bar{2}1]$, Fig. 354, the form $\{hkl\}$ undergoes an important modification, although the indices h, k, l are unequal and their sum differs from zero. By Weiss's zone-law we then have

$$h - 2k + l = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(81);$$

i.e. one of the indices is the arithmetic mean of the other two. But the zone $[a'P]$ contains the pole g of the auxiliary rhombohedron mR. The three upper poles g must, when $[a'P]$ coincides with $[a'C]$, coalesce in C (111), and the rhombohedron becomes the pinakoid $0R$. The form $\{hkl\}$ is then a bipyramid having its median edges all horizontal and its polar edges all equal; and the angles 2ξ and 2η over these latter edges are therefore also equal. The particular instance of a bipyramid $\{31\bar{1}\}$, not unfrequent on crystals of sapphire, is shown in Fig. 357.

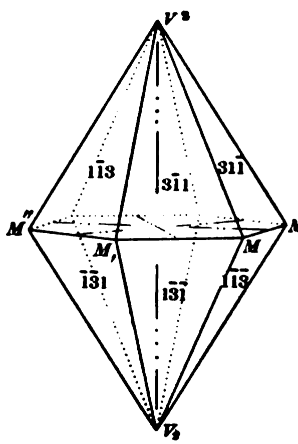


FIG. 357.

The relation (81) between the face-indices of a bipyramid can also be obtained from (60), which gives the value of m for the inscribed rhombohedron mR. When the median edges of this rhombohedron

become horizontal $m = 0$, and $h - 2k + l = 0$. It is clear that the distance of the apex cannot be obtained in the manner employed in the case of a scalenohedron. The form may, however, be constructed as follows. Lines are drawn from the six points M, $M_{,}$, M'', &c., to meet the triad axis at apices V^n, V_n, where $OV^n = OV_n = nc$.

We proceed to determine the relation between the *pyramid-index* n and the indices h, k and l of one of the faces.

Let Fig. 358 be part of a section of the bipyramid by the plane of symmetry Σ containing the polar edges V^nM, V_nM, the axis OX and the parallel edge $V_{,}M$ of the fundamental rhombohedron; and let the polar edges meet OX in S and S'. The faces meeting in V^nM are hkl and hlk; those meeting in V_nM are $\bar{l}\bar{k}\bar{h}$, $\bar{l}\bar{h}\bar{k}$.

Then, from the similar triangles $V_{,}MV^n$, OSV^n, we have

$$\frac{V_{,}M}{OS} = \frac{V_{,}V^n}{OV^n} = \frac{nc + c}{nc} = \frac{n+1}{n} \quad \text{........ (82)};$$

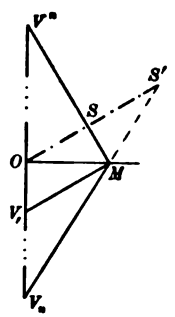


Fig. 358.

from the similar triangles $OS'V_n$, $V_{,}MV_n$, we have

$$\frac{V_{,}M}{OS'} = \frac{V_{,}V_n}{OV_n} = \frac{nc - c}{nc} = \frac{n-1}{n} \quad \text{...... (83)}.$$

As was the case with the scalenohedral face $V^n\lambda\lambda_{,}$ having the symbol (hlk) (Art. 43), the intercept $OT_{,}$ on $OY_{,}$ of the face through $MM_{,}$ is equal to OS' but measured in the negative direction. Since the indices h, k, l, are whole numbers given in their lowest terms, and $V_{,}M$ is a definite length selected arbitrarily as parameter, the plane given by $V_{,}M \div h$, $V_{,}M \div l$, $V_{,}M \div k$ does not necessarily pass through $MM_{,}$: it may be only parallel to the line. An arbitrary factor f is therefore introduced when the ratios in (82) and (83) are expressed in terms of h and l.

$$\left.\begin{aligned} \text{Hence,} \qquad & fh = \frac{V_{,}M}{OS} = \frac{n+1}{n} \\ & fl = -\frac{V_{,}M}{OS'} = \frac{1-n}{n} \\ \text{and from (81),} \qquad & 2fk = f(h+l) = 2 \div n \end{aligned}\right\} \quad \text{....................(84)}.$$

Hence, n is inversely as that index which is the arithmetic mean of the other two.

When the symbol (hlk) is given, equations (84) suffice to determine n and f. Thus, for the face (201) of the pyramid $\{210\}$ we have

from the second equation $n = 1$; and then from the first equation f is also 1: this result is also consistent with the third equation. Similarly, for $\{3\bar{1}1\}$, Fig. 357, we have from the third equation $nf = 1$; and from the first we then have $3 = n + 1$. $\therefore n = 2$.

Equations (84) can be put in the following form,

$$\frac{n+1}{h} = \frac{1-n}{l} = \frac{1}{k} = \text{(by addition) } \frac{2}{h+l}$$
$$= \text{(by subtraction) } \frac{2n}{h-l} \quad \ldots\ldots\ldots (85).$$

Hence, from the third and fourth terms, $h - 2k + l = 0$, the result given in (81); and from the third and fifth terms,

$$n = \frac{h-l}{2k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (86).$$

The index n, giving the length nc on the principal axis, is therefore known in terms of the Millerian indices; and the pyramid is drawn by joining the apices V^n and V_n ($OV^n = OV_n = nc$) to the points M, $M_{,}$, &c., projected in the horizontal plane $D'CA'$ of Fig. 51, or $Y_{,}OX_{,}$ of Fig. 60.

The edge $MM_{,}$ is bisected at right angles by $O\delta_{,,}$ at a distance

$$O\delta_{,,} \text{ (say)} = OM \cos 30° = 2a \cos^2 30° = 3a \div 2.$$

It also meets the adjacent dyad axes $O\delta$ and $O\delta'$ at distances

$$O\Delta = O\Delta' = O\delta_{,,} \div \cos 60° = 2O\delta'' = 2OM \cos 30° = 3a.$$

Naumann therefore denoted the form by the symbol $nP2$; indicating that the apex is at distance nc from the origin, and that each median edge meets the dyad axes inclined to it at 30° at double the distance at which the edge meets the axis perpendicular to it. His value for n is however two-thirds that adopted in this work; for he drew the faces through the points A, A', &c., on the dyad axes each at distance a from the origin.

54. We can now find the relation of the angle 2ζ over a median edge $MM_{,,}$ to 2ξ over a polar edge V^nM, and that of these to the pyramid-index n and the indices h,k,l. For, δ' being the point in which the edge $MM_{,,}$ meets the dyad axis, then $O\delta'V^n$ is one-half the face-angle over $MM_{,,}$ = arc CP, if P is the pole (hkl).

Hence, $\cot \zeta = \tan CP = \tan O\delta'V^n = OV^n \div O\delta' = 2nc \div 3a$.

But, from (5), $\qquad c = a \cos 30° \tan D$;

$\therefore \tan CP = 2n \cos 30° \tan D \div 3 = n \tan D \div \sqrt{3}$

$$= \text{(from (86)) } \frac{h-l}{2k\sqrt{3}} \tan D \quad \ldots . (87).$$

If the angle measured is $PP' = 2\xi$ of Art. 48, then, since $\xi = \eta$, we have from equation (68)

$$\cos\zeta = 2\sin\xi \quad\ldots\ldots\ldots\ldots (88).$$

Since $\zeta = 90° - CP$, a relation can from (87) and (88) be found between ξ and the indices. It is, however, simpler to compute ζ from (88), and afterwards to introduce the value of CP in (87).

55. The hexagonal bipyramid is the special form in which the direct scalenohedron coalesces with the inverse form. For, if $h + l = 2k$, then $\theta = 3k$, and $2k - h = l$, &c.

Hence
$$\left.\begin{aligned} p &= 2\theta - 3h = 3(2k-h) = 3l \\ q &= 2\theta - 3k = \qquad\qquad\quad 3k \\ r &= 2\theta - 3l = 3(2k-l) = 3h \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (89).$$

The opposite face (pqr) is therefore (lkh), which is one of the faces of the pyramid $\{hkl\}$.

56. Crystals of the following substances are placed in this class:

Substance.	*Composition.*	D.	*Common forms.*
Arsenic	As	58° 17′	*r*.
Antimony	Sb	56 48	*rcu* {211}.
Bismuth	Bi	56 24	*r*, *cf* $\{\bar{1}11\}$, *rcf*.
Tellurium	Te	56 55	*mr*.
Selenium	Se	56 53 (nearly)	*mr*.
Graphite	C	58 (?)	
Ice	H_2O	58 18 ? (Nordenskiöld)	
		35 10 ? (Kenngott)[1]	
Corundum	Al_2O_3	57 34	*ca*, *cr*, *cra*, *n* $\{31\bar{1}\}$, &c.
Hematite	Fe_2O_3	57 37	*cr*, *cra*, *rnu* {211}, &c.
Sodium nitrate	$NaNO_3$	43 42	*r*.
Calcite	$CaCO_3$	44 36·6	*e*, *cm*, *v* $\{2\bar{1}0\}$, and numerous other forms and combinations.
Siderite (Chalybite)	$FeCO_3$	43 23	*r*, *e* {011}, *crvs* $\{\bar{3}22\}$ (Fig. 369).
Rhodochrosite	$MnCO_3$	43 23	*r*.
Calamine	$ZnCO_3$	42 57·3	*r*.

The letters denoting forms in the above table will also be used to denote the same forms in other crystals of the system.

Corundum includes the gems, ruby and sapphire, and the semi-trans-

[1] Kenngott's $(\bar{1}11)$ may possibly be Nordenskiöld's (100); but on this assumption the latter's element D should be 54° 38′.

lucent dull-coloured variety of alumina more especially known as corundum. Ruby consists of the red crystals: in them the basal pinakoid $c\{111\}$ is largely developed; and the common habit is shown in Figs. 359—62. In the blue and the colourless crystals of sapphire, pyramids $n\ \{31\bar{1}\}$, $z\ \{71\bar{5}\}$,

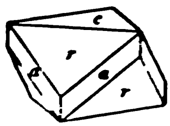


Fig. 359.

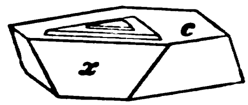


Fig. 360.

$\omega\ \{15, 11, \bar{3}\}$, &c., are the predominant forms. Simple pyramids, like Fig. 357, are sometimes found; but more commonly several pyramids occur together in association with the rhombohedron $\{100\}$, the basal

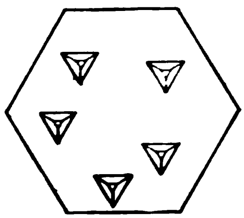


Fig. 361.

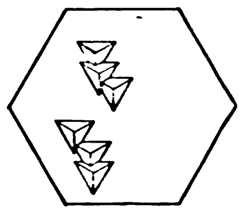


Fig. 362.

pinakoid and sometimes the prism $\{10\bar{1}\}$. Fig. 363 shows a combination of c, r, n, z and ω. Fig. 364 represents a crystal of sapphire from Emerald

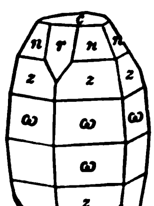


Fig. 363.

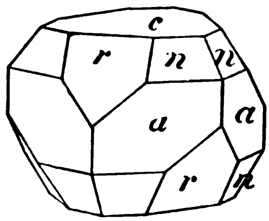


Fig. 364.

Bar, Montana, described by Dr. Pratt (*Am. Jour. of Sci.* [iv], IV, p. 417, 1897): the habit closely resembles that of hematite crystals from Elba. The rhombohedron x shown in Fig. 360 is $\{8\bar{1}\bar{1}\}$, with a computed angle $cx = 67° 2{\cdot}5'$. Figs. 361 and 362 are plans of the base showing the natural corrosion-figures of the face c: these are symmetrical to a triad axis and three planes of symmetry intersecting in it.

Hematite. Fig. 365 represents the habit of crystals from Elba. The faces *u* are generally much rounded, and accurate measurements of the angles between them and other faces are seldom possible: for $u\{211\}$, the angle $uu' = 37° 2'$, and $\wedge\, ur = 36° 6'$. The crystals from St Gotthard are often in rosettes consisting of plates in nearly parallel positions: the forms being $c\{111\}$, $a\{10\bar{1}\}$ and $m\{2\bar{1}\bar{1}\}$.

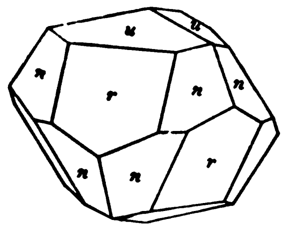


FIG. 365.

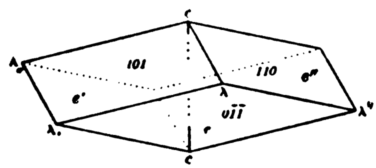


FIG. 366.

57. *Calcite* affords numerous instances of simple forms of different kinds, and an almost endless number of combinations in which a greater or less number of forms of different types enter. Thus we have $e\{110\} = -\frac{1}{2}R$, Fig. 366, in obtuse rhombohedra; $f\{\bar{1}11\} = -2R$, Fig. 370, in acute rhombohedra. The fundamental rhombohedron $r\{100\} = R$ is rare as a separate form, but is common in the allied minerals siderite, rhodochrosite, &c.; and it is fairly common in combinations: it is conspicuous by the perfect cleavage parallel to its faces.

The scalenohedron $v\{2\bar{1}0\} = R3$ is abundant in Derbyshire either as simple, or as twinned, crystals. It is one of the commonest forms in combinations: instances of its association with $e\{110\}$, and with e and $m\{2\bar{1}\bar{1}\}$, are shown in Figs. 367 and 368.

Pyramids are rare as simple forms; and they are far from common in combinations. The prism $m\{2\bar{1}\bar{1}\}$ and pinakoid enter fairly frequently into combinations. The prism $a\{10\bar{1}\}$ and dihexagonal prisms are infrequent. A combination of m and c frequently found in the Harz Mts. is shown in Fig. 328: it is geometrically the same whatever may be the substance. A combination of m and e is shown in Figs. 371 and 372.

The above-mentioned figures have been drawn as follows. The clinographic cubic axes having been projected with the numbers $n = s = 3$ (Chap. VI, Art. 22); the points M, $M_{,}$, M'', &c., are determined in the way described in Chap. VI, Art. 19, placing $O\delta_{,,}$ in the axis OX of Fig. 60. Lengths $OV = OV_{,} = \cdot 8543 \times OA''$ (see Art. 31) are then cut off on the vertical axis; the points of bisection and trisection of OV and $OV_{,}$, and also the points at distances $2OV$ and $3OV$ are then marked. The fundamental rhombohedron is drawn in the manner described in Art. 31, and the median coigns μ are found by trisecting $V_{,}M$, &c. The rhombohedra e and f can be drawn in a similar manner, using for apices points at distances $OV \div 2$, and $2OV$, respectively.

The scalenohedron $\{20\bar{1}\}$ is now drawn by joining the apices V^3 and V_3 at distances $3OV$ to the coigns μ in the way already described in Art. 40. To introduce the faces of $\{110\}$ in Fig. 367, points c and $c_{,}$ are taken on the triad axis equally distant from the origin, and lines (not shown in the figure) are drawn through them parallel to the polar edges $V\mu''$, $V\mu'$, &c. of $\{100\}$: they meet the acute polar edges $V^3\mu''$, $V^3\mu'$, &c., in the points a'', a' &c. These lines of construction are parallel to the diagonals of the faces (110), (101), &c.; for each face e truncates an edge of the rhombohedron r. From the same points c and $c_{,}$ edges are now drawn parallel to the lines joining the points of bisection of OV and $OV_{,}$ to the points M, M', &c.: they give the edges cn, $c_{,}n_{,}$, &c., which meet the obtuse polar edges of the scalenohedron in n, $n_{,}$, &c. The adjacent points a and n are then joined and the figure is completed.

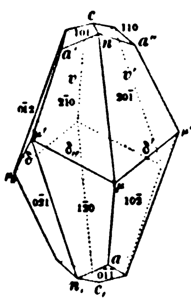


FIG. 367.

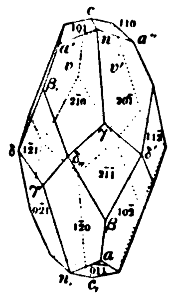


FIG. 368.

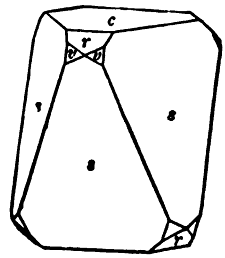


FIG. 369.

The faces of $m\,\{2\bar{1}\bar{1}\}$ shown in Fig. 368 can be inserted on the median coigns of the preceding figure as follows. The lines $\delta\delta_{,,}$, $\delta_{,,}\delta'$, &c., joining the points of bisection of the median edges of $\{20\bar{1}\}$ are bisected in points B at distances $OM \div 2$ from the origin. Through each point B a vertical line is drawn to meet the polar edges in the same plane Σ at points β and γ, respectively. The

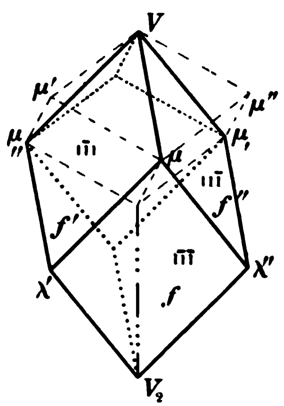


FIG. 370.

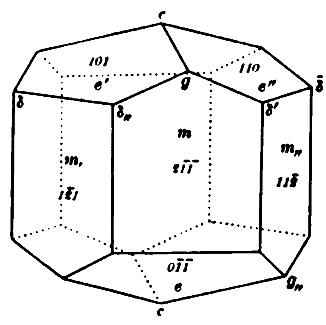


FIG. 371.

lines joining β and γ to the adjacent points δ give the edges in which the faces m and v intersect. The crystals are not often developed with the edges $[mv]$ meeting at the points δ. Sometimes the faces m are much smaller, and often they are much larger. Having obtained the directions of the edges $[mv]$, there is no difficulty in either diminishing or increasing the relative size of the faces m.

Fig. 366 can be drawn, in the way described in Art. 31, by joining apices at the distance $OV \div 2$ to the points M, M', &c.; or by the following method. The polar edges of $R = \{100\}$ are all prolonged to double their length, and each of the points so obtained is joined to the apex opposite to that through which the respective lengthened line passes. That this construction is correct follows from the fact that each face of $\{110\}$ truncates the edge of the fundamental rhombohedron; the polar edges of the latter must therefore coincide in direction with the polar face-diagonals of the former, and the coigns of the two forms on the same face-diagonal lie on the horizontal lines which pass through t and $t_{,}$, the points of trisection of $VV_{,}$.

A similar relation connects the rhombohedron $f\{\bar{1}11\}$ and $\{100\}$, and is illustrated by Fig. 370. Each face of $\{100\}$ truncates a polar edge of $\{\bar{1}11\}$. Hence each polar edge of $\{\bar{1}11\}$ is one-half the corresponding polar face-diagonal of $\{100\}$. It is simpler however to remove the lower apex of $\{\bar{1}11\}$ to V_2, doubling the size of the rhombohedron. The face-diagonals $V\mu$, $V\mu_{,}$, $V\mu_{,,}$ of $\{100\}$ give the upper polar edges of $\{\bar{1}11\}$; the figure is then easily completed. All rhombohedra connected together by the relation that the one truncates the polar edges of the other, or has its polar edges truncated, can be drawn in a manner similar to that described for e and f, when one of the series is known. Thus the diagonals $V\lambda'$, $V\lambda''$, &c. of $\{\bar{1}11\}$, Fig. 370, are the polar edges of $\{3\bar{1}\bar{1}\}$; and so on.

In Fig. 371, $e\{110\}$ is first drawn; the faces m are then introduced in the way described for obtaining the points β and γ in Fig. 368. Thus in Fig. 371 the point g is on the vertical line bisecting the line $\delta'\delta_{,,}$, and the lines $g\delta'$, $g\delta_{,,}$ are the edges $[me'']$, $[me']$. The edge $[me]$ is horizontal and parallel to $\delta'\delta_{,,}$.

Fig. 372 gives the same combination of m and e as Fig. 371, but the triad axis has been displaced from the vertical to show the upper faces e more distinctly; the figure representing one of the crystals on a specimen in the Cambridge Museum. Three grooves (represented by the short, black lines) filled with a dark earthy matter lie in the face-diagonals of $\{110\}$ of each crystal. The triad axis in Fig. 372 was taken to coincide in direction with the cube-diagonal $O\rho$ of Fig. 226, and the dyad axes of calcite coincide with $O\delta'''$, $O\delta^4$ and $O\delta^5$ of that figure. But the lines $O\rho$ and $O\delta'''$ of Fig. 226 are to one another as $\sqrt{3} : \sqrt{2}$. If then $O\delta'''$ is taken as the unit length, that on $O\rho$ is $O\rho\sqrt{2} \div \sqrt{3}$; and this length must be multiplied by $\cos 30^\circ \tan D$, or $\cdot 8543$, to give c for calcite. A length $= O\rho \times \cdot 8543 \div \sqrt{6}$ has therefore been taken to give the apex of $e\{110\}$ at distance $c \div 2$ from the origin. The points δ''', δ^4, δ^5 are then the same points as those marked δ, &c., in Fig. 371. The figure is now completed in a manner similar to that employed for drawing Fig. 371.

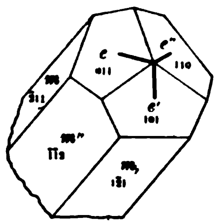


FIG. 372.

The combination shown in Fig. 374 is common in crystals from Iceland. The forms are: $r\,\{100\}$, $v\,\{20\bar{1}\}$, $w\,\{410\}$. From the stereogram, Fig. 373, it is seen that $\{311\}$ is the auxiliary rhombohedron of $\{410\}$, and that $\{100\}$ is the auxiliary rhombohedron of $\{20\bar{1}\}$. We therefore first construct $\{100\}$, its

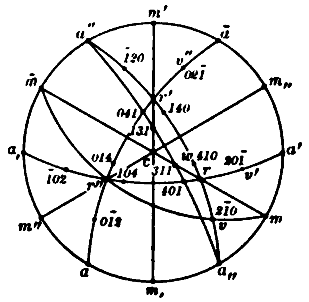


Fig. 373.

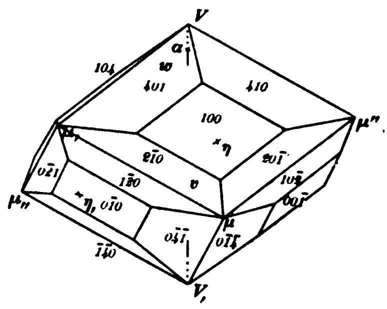


Fig. 374.

median edges are also those of $\{20\bar{1}\}$, and the polar edges of this latter form are got by joining the coigns μ to V^3 and V_3 at distances $3OV$ from the origin. Again, it was shown in Art. 52 that $\{100\}$ is the rhombohedron iv associated with $\{410\}$; *i.e.* the polar edges of $\{100\}$ are parallel to the acute polar edges of $\{410\}$, and we may draw the faces of the latter through the edges $V\mu_,$, $V\mu''$, &c. By (60) and (61), the apex of $\{410\}$ is at $\mathfrak{a}$, where $O\mathfrak{a}=4OV\div 5$; and the coign λ is in the vertical through μ at a distance $c\div 5$ nearer to the equatorial plane. For, by (33), the apex of $\{311\}$ is at distance $2c\div 5$, whilst that of $\{100\}$ is at distance c: also $\lambda\mu=\tau t=Ot-O\tau$. But (Arts. 31 and 33) $Ot=c\div 3$, and $O\tau=2c\div 15$. $\therefore\ \lambda\mu=c\div 3-2c\div 15=c\div 5$. We therefore draw through V an edge [401, 410], parallel to $\mathfrak{a}\lambda$, to meet the obtuse polar edge $[2\bar{1}0, 20\bar{1}]$ in the point η, the positions of which and of its homologue $\eta_,$ are indicated by crosses. The lines joining $\mu_,$, μ'' to η are the edges $[401, 2\bar{1}0]$, $[410, 20\bar{1}]$. The remaining edges of the two forms are obtained in a similar manner. The rhombic faces of $\{100\}$ are now introduced; the coigns nearest to the apices V and $V_,$ being found by cutting off equal lengths on the polar edges $V\eta$, $V_,\eta_,$, &c.

IV. *Trapezohedral class;* $\alpha\,\{hkl\}$.

58. In this class the triad axis $O\rho$ is associated with three like and interchangeable dyad axes making 90° with it and 120° with one another; and there is no other element of symmetry. The above axes resemble those associated together in the preceding class, but the opposite ends of each dyad axis are no longer similar; for the crystals of this class cannot be centro-symmetrical without introducing a plane of symmetry perpendicular to each dyad axis. These

axes are therefore uniterminal; and they are pyro-electric axes: the opposite ends are in Figs. 377—380 indicated by letters δ and d.

59. Any face inclined to the triad axis at a finite angle, other than 90°, must be repeated in two other similar faces which meet at an apex on the axis. If, moreover, the face is parallel to one of the dyad axes, a parallel face must also be present; for a semi-revolution about a dyad axis brings the face into a position parallel to its original one. The same must be true of each face of the triad meeting at an apex on the principal axis. The form must therefore be a rhombohedron similar geometrically to those described in the two preceding classes. The method of construction has been fully given in Arts. 31 and 39.

Again, the plane containing the three dyad axes is parallel to a pair of possible faces; for the dyad axes are possible zone-axes. Hence we obtain a second special form—a pinakoid—similar geometrically to that of the two preceding classes.

We can now select as axes of reference lines parallel to the three polar edges of any rhombohedron possible on the crystal, and for parametral plane (111) a face of the pinakoid. The parameters are therefore equal; for the axes are three similar lines equally inclined to the triad axis and interchangeable by rotations of 120° about this axis, whilst the parametral plane retains the same direction after each rotation. Hence all the analytical relations established in the preceding sections hold also for crystals of this class. The linear element c and the angular element $D = 111 \wedge 100$ are determined from the rhombohedron selected to give the axes of reference.

60. Crystals of this class have the following special forms:

1. The pinakoid $\{111\}$, having the faces 111, $\bar{1}\bar{1}\bar{1}$.

2. Rhombohedra $mR = \{hll\}$, and $-mR = \{h_{,}l_{,}l_{,}\}$: these are geometrically identical with those discussed in Arts. 31—39, and their geometrical relations are given in those Articles.

3. A trigonal prism $a\,\{0\bar{1}1\}$, Fig. 375, each face of which is perpendicular to one of the dyad axes, and parallel therefore to one of the axes of reference. It is geometrically the same as $\tau\,\{0\bar{1}1\}$ of class I, Art. 9; and consists of the faces: $a\,(0\bar{1}1)$, $a'\,(10\bar{1})$, $a''\,(\bar{1}10)$.

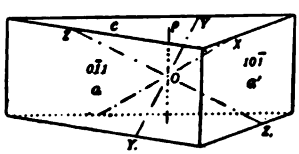


Fig. 375.

The complementary prism $a\,\{01\bar{1}\}$ has its faces parallel to those of $a\,\{0\bar{1}1\}$.

4. A hexagonal prism $\{2\bar{1}\bar{1}\}$, the faces of which are parallel in pairs to one or other of the dyad axes: it includes the faces given in (g) of Art. 27. The face-symbols can be found in any of the ways given in Arts. 11 and 27.

5. The *ditrigonal prism*, $a\{hkl\}$, where $h+k+l=0$. The prism $a\{hkl\}$ consists of the faces:

$$hkl,\ \bar{l}\bar{k}\bar{h},\ lhk,\ \bar{h}\bar{l}\bar{k},\ klh,\ \bar{k}\bar{h}\bar{l}\ \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{m}).$$

The alternate faces have their symbols in the same cyclical order, and are at 120° to one another. There is therefore but one independent angle, which may be taken to be $hkl \wedge \bar{l}\bar{k}\bar{h}$: it is independent of the crystal-element, and can be computed from expression (20) of Art. 27, for it is double $a'N$ of that equation. Fig. 376 represents the particular instance $a\{3\bar{2}\bar{1}\}$, in which $3\bar{2}\bar{1} \wedge 2\bar{3}1 = 38°\ 13'$.

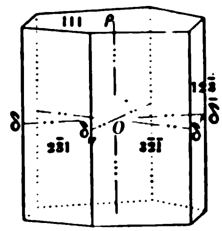


FIG. 376.

The complementary ditrigonal prism $a\{hlk\}$ has its faces parallel to those of $a\{hkl\}$; and the two are geometrically tautomorphous, for the faces of the former can be brought into the position of those of the latter by a rotation of 180° about the triad axis: dissimilar ends of the dyad axes are however interchanged by such a rotation.

6. The *trigonal bipyramid*, $a\{hkl\}$, where $h-2k+l=0$, consists of six equal and interchangeable faces, each of which is an isosceles triangle. The faces of this form, of the ditrigonal prism and of the general form (Art. 61) have the general symbols given in tables **m** and **n**: in the special forms the indices of each of the faces are subject to the conditions given with the form-symbol. Geometrically the trigonal bipyramid may be supposed to consist of the alternate pairs of faces of the hexagonal bipyramid (Arts. 53—55) which are interchanged by rotations of 120° about the triad axis. The faces of the latter form were drawn through apices V^n, V_n and horizontal lines $MM_{,}$, &c., where M, $M_{,}$, &c., are the auxiliary points in the equatorial plane at distances $2a \cos 30°$ from the origin. In the trigonal bipyramid we shall suppose three of these same horizontal edges to be retained: hence each of them is bisected by a dyad axis at a point d, where $Od = OM \cos 30° = 3a \div 2$; and has its extremities on the adjacent dyad axes at points δ, where $O\delta = Od \div \cos 60° = 3a$. The distance nc of the apex V^n is given by equation (86); viz. $n = (h-l) \div 2k$.

Particular instances, having the same symbols as trigonal bipyramids observed on crystals of quartz in association with a hexagonal prism and the rhombohedra $\{100\}$, $\{\bar{1}22\}$, are represented by Figs. 377 and 378—the parameter c in these figures is not however that of quartz. The first is $\alpha\{41\bar{2}\}$ and consists of the

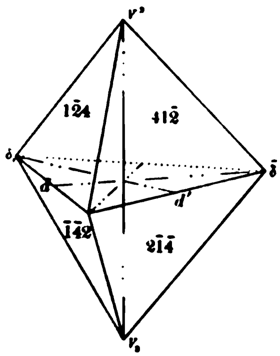


FIG. 377.

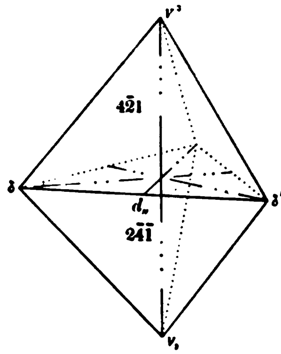


FIG. 378.

faces: $41\bar{2}$, $\bar{2}41$, $1\bar{2}4$, $2\bar{1}\bar{4}$, $\bar{4}2\bar{1}$, $\bar{1}\bar{4}2$. By (86) the value of n is 3. The figure is therefore constructed by joining apices V^3, V_3 at distances $3c$ to points δ on the similar extremities of the dyad axes at distances $3a$ from the origin. Fig. 378 represents the complementary form $\alpha\{4\bar{2}1\}$, in which the median coigns are on the opposite side of the origin from those of the first. Geometrically the two forms are tautomorphous, but by a semi-revolution about the principal axis, opposite and dissimilar ends of the dyad axes are interchanged.

61. *The trapezohedron*, $\alpha\{hkl\}$. When each of the faces meets the axes of reference at finite distances, and all the dyad axes at unequal distances from the origin, we have a six-faced form known as the trapezohedron $\alpha\{hkl\}$, Fig. 379; for each face is bounded by four edges, which do not form a parallelogram. The face (hkl) must, owing to the triad axis, be accompanied by the faces (lhk), (klh) having their symbols in cyclical order, and all meeting at an apex V^n on the triad axis. The face (hkl) meets the adjacent dyad axes at points δ' and $d_{\prime\prime}$ unequally distant from the origin and lying on parts of these axes which are not interchangeable. Since $O\delta'$ is perpendicular to the axis of reference $YY_{\prime}$ and is equally inclined to $XX_{\prime}$ and $ZZ_{\prime}$, a semi-revolution about

$O\delta'$ interchanges positive with equal negative lengths on OY; and exchanges a positive length on OX with a negative one on OZ, and vice versâ: hence the face is brought into a position in which it meets the triad axis at V_n and has the symbol $(\bar{l}\bar{k}\bar{h})$. Similarly, $Od_{\prime\prime}$ is perpendicular to $ZZ_{\prime}$; and a semi-revolution about $Od_{\prime\prime}$ brings (hkl) into a position given by $(\bar{k}\bar{h}\bar{l})$. The third face at V_n is $(\bar{h}\bar{l}\bar{k})$; the three faces meeting at V_n being necessarily in cyclical order. No other faces are introduced by a semi-revolution about $\bar{d}\delta$; for this axis is perpendicular to $XX_{\prime}$, and interchanges equal positive

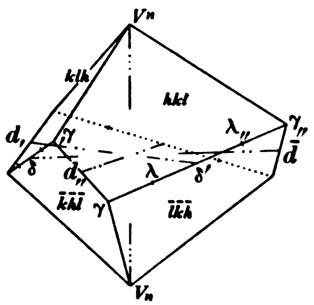


FIG. 379.

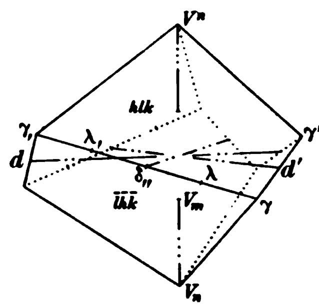


FIG. 380.

and negative lengths on it: hence the face $(\bar{k}\bar{h}\bar{l})$ is brought into a position given by (klh), and $(\bar{l}\bar{k}\bar{h})$ into that given by (lhk). But these two faces are two of those meeting at V^n. Hence, the trapezohedron $\alpha\{hkl\}$, Fig. 379, consists of the six faces:

$$hkl,\ lhk,\ klh,\ \bar{l}\bar{k}\bar{h},\ \bar{h}\bar{l}\bar{k},\ \bar{k}\bar{h}\bar{l} \ldots\ldots\ldots\ldots\ldots(\text{n}).$$

The rule for the deduction of the symbols of the several faces from that of any one of them is—that the three faces meeting at one of the apices have their indices in cyclical order; the sign of each index, whether it be positive or negative, remaining the same to whichever axis of reference it may be transferred: at the opposite apex the cyclical order and also the signs of all the indices are changed. The positions of the poles of particular trapezohedra $\alpha\{4\bar{2}\bar{1}\}$ and $\alpha\{4\bar{1}\bar{2}\}$ are shown in Figs. 384 and 385.

The geometry of this form is most easily understood by constructing it from one-half of the faces of the scalenohedron described in Art. 40; pairs of faces being drawn through each of the alternate median edges of the inscribed rhombohedron mR. But since there are no planes of symmetry in crystals of this class, the similar faces through adjacent median edges do not occur together.

We can therefore obtain two complementary trapezohedra $\alpha\{hkl\}$, Fig. 379, and $\alpha\{hlk\}$, Fig. 380, from the same scalenohedron. The two complementary forms are enantiomorphous, for the one is the reflexion of the other in any of the vertical planes containing an axis of reference.

No crystal showing only a general form has yet been observed, but forms of the kind are common on crystals of quartz in combination with the rhombohedra $\{100\}$, $\{\bar{1}22\}$ and the prism $\{2\bar{1}\bar{1}\}$. The method of drawing the form has therefore no great interest; but when it is needed, the rhombohedron mR, where $m=(h-2k+l)\div(h+k+l)$ (Art. 44) should be first constructed. The median edges of this rhombohedron are also those of the scalenohedron $\{hkl\}$. To obtain $\alpha\{hkl\}$, the apices V^n and V_n and the median edges through δ', δ'' and δ are retained: these edges have now to be extended to meet new polar edges in which the alternate scalenohedral faces at each apex meet. Thus the edge $\lambda\lambda_{\prime\prime}$, in which the faces (hkl) and $(\bar{l}\bar{k}\bar{h})$ of the scalenohedron meet, is extended to meet the faces (lhk) and $(\bar{k}\bar{h}\bar{l})$ at the points $\gamma_{\prime\prime}$ and γ, respectively. The extensions in the two directions are clearly equal; for the edge is perpendicular to the dyad axis $O\delta'$, and the two extremities are interchanged by a semi-revolution about the axis. The coign γ is found by taking (see Chap. XIX)

$$\delta'\gamma=\frac{h-2l+k}{h-k}\delta'\lambda=\left(1+2\frac{k-l}{h-k}\right)\delta'\lambda=(\text{from } (74))\left(1+2\frac{\sin\xi}{\sin\eta}\right)\delta'\lambda\ldots(90).$$

The first expression for the factor of $\delta'\lambda$ is similar to that given in (61) for $1\div n$, but the indices k and l have changed places; the others give for the extension

$$\lambda\gamma=\lambda_{\prime\prime}\gamma_{\prime\prime}=2\frac{k-l}{h-k}\delta'\lambda=2\frac{\sin\xi}{\sin\eta}\delta'\lambda\ldots\ldots\ldots\ldots\ldots\ldots(91).$$

In these expressions the indices h, k and l are taken in descending order of magnitude, 2ξ is the angle over the obtuse polar edge and 2η that over the acute polar edge of the scalenohedron $\{hkl\}$. Thus for $\alpha\{71\bar{2}\}$, to which Fig. 379 corresponds, $m=1\div2$, $n=3$ and $\delta'\gamma=2\delta'\lambda$.

The points γ having been marked by proportional compasses on the alternate median edges, each of them is joined to the nearest apex V_n or V^n; and the adjacent pairs of coigns γ are joined to form the second set of median edges intersecting the dyad axes at the points d.

The polar edges can also be found by the rule for finding a zone-axis (Chap. V, Art. 4). Thus $V^n\gamma_{\prime\prime}$ is parallel to

$$[hkl,\ lhk]=[k^2-lh,\ l^2-hk,\ h^2-kl];$$

and $\gamma_{\prime\prime}$ is the intersection of a line through V^n, parallel to this zone-axis, with the median edge $\lambda\lambda_{\prime\prime}$.

62. *Quartz*, SiO_2. Occasionally the crystals appear as fairly regular bipyramids, which may be represented by Fig. 381, consisting of the

two forms $r\{100\}$ and $z\{\bar{1}22\}$. The faces (100) and ($\bar{1}22$) are connected by the relations given in Art. 15 and belong to dirhombohedral forms. Very rarely the faces r predominate to such an extent that those of z appear only as slight modifications on six coigns of the rhombohedron. Since $rr'' = 85° 46'$, such a crystal looks much like a cube. Knowing the angle rr'' we find D by equation (1) to be 51° 47′, when $c = 1{\cdot}0999$.

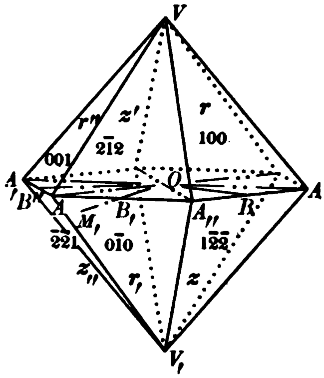


Fig. 381.

More commonly the crystals are hexagonal prisms $\{2\bar{1}\bar{1}\}$ terminated by the dirhombohedral forms $r\{100\}$ and $z\{\bar{1}22\}$: the faces of the prism are usually striated in a direction perpendicular to the triad axis; and those of the rhombohedron r are generally more largely developed than those of z. The faces of the two rhombohedra and of the prism are, however, often very unequally developed; and it is occasionally hard to make out the symmetry. Faces of a trigonal bi-

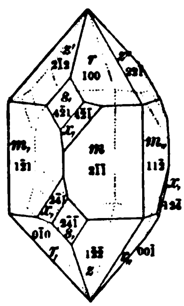


Fig. 382.

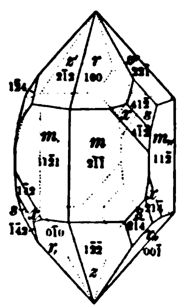


Fig. 383.

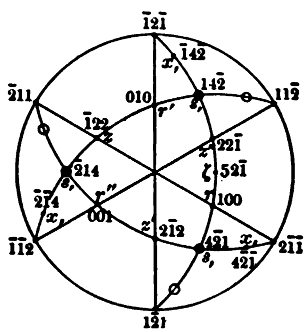


Fig. 384.

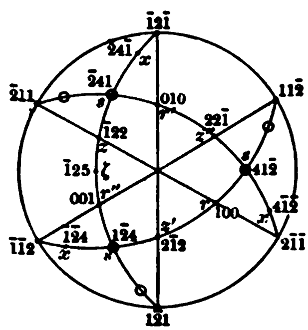


Fig. 385.

pyramid $s = a\{41\bar{2}\}$ or $s_{,} = a\{4\bar{2}1\}$, and of a trapezohedron $x = a\{4\bar{1}\bar{2}\}$ or $x_{,} = a\{4\bar{2}\bar{1}\}$ often occur as small modifications of the coigns of crystals of the bipyramidal habit; but, owing to the remarkable tendency to twinning shown by quartz-crystals (see Chap. XVIII), the homologous coigns are rarely all similarly modified. By the examination of a considerable number of crystals, it has been established that they are of two kinds. In the one, represented by Fig. 382 and the corresponding stereogram Fig. 384, the tautozonal faces z', $s_{,}$, $x_{,}$ and m, and their poles, follow one another in this order along the spiral of a left-handed screw: such crystals will be called lævogyral or left-handed, and rotate the plane of polarization of light transmitted along the triad axis to the left; *i.e.* to an observer receiving the light, the rotation is counter-clockwise. In Figs. 383 and 385 a similar combination of forms is represented, in which the faces z'', s, x, m follow one another along the spiral of a right-handed screw: these crystals are dextrogyral or right-handed, and rotate the plane of polarization to the right; *i.e.* to an observer receiving the light, the rotation is clockwise. The rule can also be given as follows. The prism being vertical, and a face r of the direct rhombohedron $\{100\}$ facing the observer, then the faces s and x in the zone $[z''sxm]$ are to the right of the observer in a dextrogyral crystal; and the faces $s_{,}$, $x_{,}$ in the similar zone are to the left in a lævogyral crystal.

The faces s of the trigonal bipyramid are parallelograms, and in a dextrogyral crystal lie each in two slanting zones; for instance, the face s $(41\bar{2})$ lies in the zones $[m_{,,}r] = [021]$ and $[z''m] = [102]$, and the form is $a\{41\bar{2}\}$: the faces are sometimes striated parallel to the edge $[sr]$. In a lævogyral crystal the faces $s_{,}$ lie each in two zones, such as $[rm_{,}] = [012]$ and $[z'm] = [120]$: the face $s_{,}$ is $(4\bar{2}1)$, and the form is $a\{4\bar{2}1\}$. The faces are occasionally striated parallel to $[rs_{,}]$.

Measurement of the angles in the zone $[mxsz'']$ gives: $mx = 12°\ 1'$, $xs = 25°\ 57'$, $sz'' = 28°\ 54'$, $z''r' = 46°\ 16'$ and $r's\ (\bar{2}41) = 28°\ 54'$. The angles in the corresponding zone $[mx_{,}s_{,}z']$ of Fig. 382 are the same.

Knowing the symbols of m, z'' and r', those of x and s can be found by the A. R. of four poles. Thus, taking x to be (hkl), the A. R. $\{r'z''xm\}$ gives

$$\frac{\sin r'x}{\sin r'z''} \div \frac{\sin mx}{\sin mz''} = \begin{vmatrix} 010 \\ hkl \\ \overline{010} \\ 22\bar{1} \end{vmatrix} \div \begin{vmatrix} 2\bar{1}\bar{1} \\ hkl \\ \overline{2\bar{1}\bar{1}} \\ 22\bar{1} \end{vmatrix} = \frac{3h}{h+2k} = \frac{3l}{l-k};$$

according as the first and second columns are combined, or the second and third.

$$\begin{array}{ll} L \sin (r'x = 101°\ 7') = 9{\cdot}99177 & L \sin (r'z'' = 46°\ 16') = 9\ 85888 \\ L \sin (mz'' = 66°\ 52') = 9{\cdot}96360 & L \sin (mx = 12°\ \ 1') = 9{\cdot}31847 \\ \qquad 19{\cdot}95537 & \qquad \overline{19{\cdot}17735} \\ \qquad \underline{19{\cdot}17735} & \\ \qquad {\cdot}77802 = \log 5{\cdot}998. & \end{array}$$

$$\therefore \frac{3h}{h+2k} = \frac{3l}{l-k} = 6. \quad \text{Hence, } h = -4k, \text{ and } l = 2k.$$

The face x is therefore $(4\bar{1}\bar{2})$, and the form is $\alpha\{4\bar{1}\bar{2}\}$.

The symbols of the faces and poles to the front of the paper are inscribed in the diagrams.

The dyad axes, being uniterminal, are pyro-electric axes; and in crystals, such as those represented in Figs. 382 and 383, the analogous poles are on the alternate prism-edges where the faces s and x appear, the antilogous poles on those prism-edges where no such faces occur.

Optical characters. With a few exceptions mentioned in Chap. XVIII, plates of all quartz crystals, cut perpendicularly to the principal axis, rotate the plane of polarization of a beam of plane-polarised light transmitted along the axis. The amount of rotation of the plane of polarization depends on the thickness of the plate and on the colour of the light, and is the same for light of one colour in plates of equal thickness but opposite directions of rotation. Biot found that the angle of rotation ϕ is given by the approximate expression, $\phi = \pm kt \div \lambda^2$; where t is the thickness of the plate, λ the wave-length of the light, and ϕ is taken to be positive or negative according as the plate is dextro- or lævo-gyral. This expression for ϕ is the first term of a series in which higher powers of $1 \div \lambda^2$ enter.

For plates 1 mm. thick the principal rotations are given in the following table (Stephan, *Pogg. Ann.* CXXII, p. 631, 1864).

Fraunhofer's line.	λ.	ϕ.
B	0·000687 mm.	15·55°
C	,, 656	17·22
D	,, 589	21·67
E	,, 527	27·46
F	,, 486	32·69
G	,, 431	42·37
H	,, 397	50·98

By interposing a plate of quartz of no great thickness in a beam of plane-polarised white light, the direction of rotation can be readily determined by observing the order in which the colours succeed one another on turning the analyser round. Thus, a plate of a dextrogyral crystal gives in succession—red, yellow, green, blue—as the analysing Nicol is turned to the right; a lævogyral plate gives the colours in the same order when the Nicol is turned to the left. The direction of rotation can also be determined by Airy's spirals described below.

When a single plate cut perpendicularly to the optic axis is inserted in convergent white light between crossed Nicols the coloured rings characteristic of uniaxal crystals are seen. These rings are intersected by the arms of a dark rectangular cross similar to those observed in calcite, corundum, &c.; but the arms do not for plates of appreciable thickness penetrate within the first circular ring in which the difference of phase is λ. The centre is uniformly coloured; the colour depends on

the thickness of the plate and is the same as that given when the plate is observed in parallel light between crossed Nicols.

Airy's spirals. Superposition of two plates of like rotation only causes a contraction of the rings and a change in the central colour. But, if equally thick plates of opposite rotations are superposed in the polariscope, four dark spirals start from the centre and bend round with, or against, the clock according as the dextrogyral or lævogyral plate is that *first* traversed by the light; and the direction of the curvature is opposite to the rotation of the plate *nearest* to the observer. The phenomena were first described and explained by Airy (*Camb. Phil. Trans.* IV, pp. 79 and 199, 1833); and the spirals are known as Airy's spirals.

The spirals can also be seen by causing the light to traverse a single plate twice, viz. *away from* and *towards* the observer. A piece of silvered glass is placed on the table, the plate is placed on the mirror and a lens of short focus is put over the plate so that its focus is approximately in the plate. Light from a window or lamp is reflected down on the lens by means of two or three pieces of ordinary window-glass held parallel to one another. The light—partially polarised by reflexion—is rendered convergent by the lens, and traverses the plate *away* from the observer: it is then reflected by the mirror and re-traverses the plate *towards* the observer, and its rays are rendered approximately parallel by the lens. The observer looking down on the lens and crystal through the glass-plates sees Airy's spirals; the glass-plates serving the double function of polariser and analyser, for the planes of polarization of the light reflected by, and transmitted through, them are at right angles. The curvature of the spirals is in the direction opposite to the character of the plate; for, after reflexion from the mirror, the light traverses the plate and analyser in exactly similar conditions to those under which, in the ordinary polariscope, it traverses the upper plate and Nicol after passing through a plate of opposite rotation. Thus, if the plate is dextrogyral, the spirals bend towards the left; and if lævogyral, they bend towards the right. For success in this observation the plate must be accurately cut.

Fig. 386 represents a crystal of *cinnabar*, HgS, which also belongs to this class. The crystals are usually combinations of several rhombohedra associated with the pinakoid: bipyramids and trapezohedra are rare. The forms shown in the figure are: $c\,\{111\}$, $m\,\{2\bar{1}\bar{1}\}$, $r\,\{100\}$, $g_1\,\{110\}$, $n_1\,\{\bar{1}11\}$ and $F = a\,\{6\bar{2}1\}$.

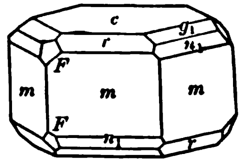


FIG. 386.

The crystals rotate the plane of polarization; and, if the rule which has been shown in crystals of quartz to connect the position of trapezohedral faces and the direction of rotation holds also for cinnabar, the crystal in Fig. 386 is lævogyral. But the connection between the trapezohedral faces and the direction of rotation has not yet been fully established.

V. *Acleistous ditrigonal class;* $\mu\{hkl\}$.

63. The crystals of this class have a triad axis associated with three like planes of symmetry intersecting one another in the axis at angles of 60°, but have no other element of symmetry. The triad axis is uniterminal, and is a pyro-electric and piezo-electric axis.

With the exception of the prisms, the forms can most readily be derived from the corresponding forms of class III by regarding them as consisting geometrically of those faces which meet the principal axis at one of the apices. The forms are all open forms, and the class may be called the *acleistous ditrigonal*, or the *polar ditrigonal* class: they were regarded as hemihedral with inclined faces.

A face perpendicular to one of the planes of symmetry and inclined to the horizon at a finite angle, will be repeated in two others equally inclined to the horizon and each perpendicular to one of the two other planes of symmetry. The three faces form a trigonal pyramid similar to that characteristic of class I; but here the faces of every trigonal pyramid are limited to zones, such that their poles lie in the zone-circles in which the planes of symmetry meet the sphere. The axes of reference are taken parallel to the three polar edges of one of the possible trigonal pyramids, and lie each in one of the planes Σ.

Again, the plane perpendicular to the triad axis is a possible face, for it is parallel to the three normals of the planes of symmetry; and these normals are, by Chap. IX, Prop. 1, possible zone-axes. This face is taken as the parametral plane (111). But by rotations of 120° about the triad axis, the direction of the parametral plane is unchanged, whilst the axes of reference are interchanged, for they are parallel to the interchangeable polar edges of a trigonal pyramid. The parameters are therefore equal, and may, as in the preceding sections, be taken to be any three equal lengths, such as OX or $V_{,}M$ of Figs. 309 and 335. The equations established in preceding sections between the face-symbols, the angles of the forms and axes with one another and with the element D or c, hold also for crystals of this class.

64. The special forms are:

1. Pedions $\mu\{111\}$ and $\mu\{\bar{1}\bar{1}\bar{1}\}$, which consist of the single face given by the indices in the symbol.

2. **Trigonal prisms** $\mu\{2\bar{1}\bar{1}\}$ **and** $\mu\{\bar{2}11\}$. The prism $\mu\{2\bar{1}\bar{1}\}$, Fig. 387, has the three faces $2\bar{1}\bar{1}$, $\bar{1}2\bar{1}$, $\bar{1}\bar{1}2$, each of which is perpendicular to one of the planes of symmetry: it is geometrically similar to that described in Art. 11, and can be drawn in the same manner. The complementary prism $\mu\{\bar{2}11\}$ consists of the faces $\bar{2}11$, $1\bar{2}1$, $11\bar{2}$ parallel to those of $\mu\{2\bar{1}\bar{1}\}$.

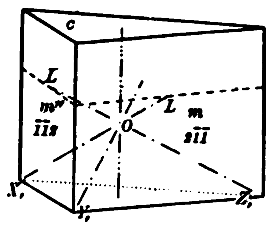


FIG. 387.

3. A hexagonal prism $\{0\bar{1}1\}$, identical geometrically with the similar prism of classes II and III. The faces are parallel in pairs to one of the planes of symmetry in which the axes of reference lie. The form can be drawn in the manner given in Art. 27. The symbols of the faces are given in (f), p. 368.

4. ***The ditrigonal prism***, $\mu\{hkl\}$, where $h+k+l=0$. This prism may be regarded as consisting of those alternate pairs of faces of the dihexagonal prism of Art. 27, the traces of which in Fig. 332 intersect in the points H, L, H', &c., lying on the lines OM, OM' and OM'': and the prism can easily be drawn by the aid of a projection of Fig. 332 in the horizontal planes of Figs. 51 or 60. The form consists of the faces:

$$hkl,\ hlk,\ lhk,\ khl,\ klh,\ lkh \ldots\ldots\ldots\ldots\ldots\ldots\text{(o)}.$$

It has the same general appearance as $\alpha\{hkl\}$ of the preceding class; but, when the indices are transferred from one face to its homologues, they do not change sign: they are the same triad taken in the two possible cyclical orders. Alternate faces are inclined to one another at angles of 120°, and there is only one independent angle $hkl \wedge hlk$. But $hkl \wedge hlk = 2mN = 60° - 2a'N$ (see Fig. 331): the angle can therefore, for any given values of the indices, be calculated by (20). Thus, for $\mu\{3\bar{2}\bar{1}\}$,

$$3\bar{2}\bar{1} \wedge 3\bar{1}\bar{2} = 60° - 38°\ 13' = 21°\ 47'.$$

5. The acleistous trigonal pyramid $\mu\{hll\}$, may be regarded as consisting of the three faces of mR which meet at V^m. The polar edges can therefore be obtained by finding m from equation (33), and taking $OV^m = mc$. The point V^m is then joined to $\bar{M}$, $M_{,}$, $M_{,,}$ of Fig. 335, or to M, M', M'', according as m is positive or negative. The pyramid is geometrically the same as that of class I; but now the poles all lie in the zone-circles $[Cr]$, $[Cr']$, $[Cr'']$.

6. *The adelstous hexagonal pyramid*, $\mu\{hkl\}$, where $h-2k+l=0$. Each face is equally inclined to two adjacent planes of symmetry, and, if produced, meets the equatorial plane and the third plane of symmetry in parallel straight lines: it is therefore tautozonal with the pedion and one of the faces of $\{0\bar{1}1\}$. The pyramid can be constructed in the manner given in Art. 53; the number n giving the distance nc of the apex from the origin is found from (86).

65. *The ditrigonal pyramid*, $\mu\{hkl\}$, Fig. 388, consists of the six faces:

$$hkl,\ khl,\ lhk,\ lkh,\ klh,\ hlk \ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{p}).$$

The polar edges are identical with those of the scalenohedron of class III which meet at V^n; and the form is easily drawn by finding the indices m and n from equations (60) and (61) and the points H, $K_{,}$, &c., from equations (45) and (47).

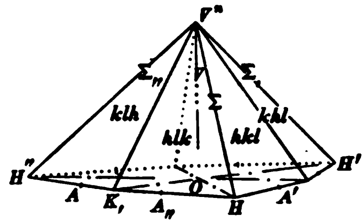


FIG. 388.

$$OV^n = mnc;\ OH = \frac{2n}{3n+1}OM;\ OK_{,} = \frac{2n}{3n-1}OM_{,}.$$

When $m=\infty$, the apex V^n is infinitely distant; and the pyramid becomes the ditrigonal prism $\mu\{hkl\}$, where, by equation (60),

$$h+k+l=0.$$

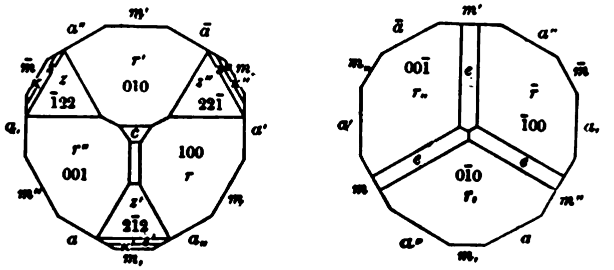
When $OH=OK_{,}$, $n=\infty$: if, however, OV^n is to remain finite, then $m=0$. The two equations are satisfied by making in (60) and (61) $h-2k+l=0$—the expression already obtained for the hexagonal pyramids.

66. Crystals of sodium lithium sulphate, $NaLiSO_4$; of spangolite, $(AlCl)SO_4 . 6Cu(OH)_2+3H_2O$; of greenockite, CdS; of proustite, Ag_3AsS_3; of pyrargyrite, Ag_3SbS_3; and of tourmaline belong to this class.

Tourmaline. Jannasch and Calb (*Ber. Ch. Ges.* XXII, p. 216, 1889) gave the formula $(Mg,Fe,Li_2,H_2\ldots)_3AlO . BO(SiO_4)_2$ or $R_9{}^IBO_2(SiO_4)_2$, R^I including the elements given in the first formula. Messrs. Penfield and Foote (*Am. Jour. of Sci.* [iv], VII, p. 97, 1899) have shown that all tourmalines may be derived from $H_{20}B_2Si_4O_{21}$, and have proposed as a special formula $H_9Al_3(B . OH)_2Si_4O_{19}$, in which the nine hydrogen-atoms are replaced by Fe, Ca, Mg, Li, &c.

The crystals often manifest the development of forms of this class in a striking manner. The trigonal prism $\mu\{2\bar{1}\bar{1}\}$ is sometimes the predominant form, but it is generally associated with the hexagonal

prism $\{0\bar{1}1\}$. A section perpendicular to the triad axis of such a combination will either be a triangle having its angles bevelled, or else a hexagon having alternate angles truncated, according as the faces of $\mu\{2\bar{1}\bar{1}\}$ are large or small. Occasionally as shown in Figs. 389 (*a*) and (*b*) both the complementary prisms $\mu\{2\bar{1}\bar{1}\}$ and $\mu\{\bar{2}11\}$ represented by the letters *m* are present. The crystals are usually terminated by acleistous trigonal pyramids, which may or may not be complementary. Occasionally one end shows a pedion. Figs. 389 (*a*) and (*b*) represent the opposite ends of a small, brown crystal from Ceylon in the Cambridge Museum. At one



(*a*) FIGS. 389. (*b*)

end the trigonal pyramids $r=\mu\{100\}$, $z=\mu\{\bar{1}22\}$ (a rare form), $s=\mu\{\bar{1}11\}$, $\kappa=\mu\{\bar{3}22\}$ are associated with the pedion $c=\mu\{111\}$, one face of $e=\mu\{110\}$ and a single face of a ditrigonal pyramid $u(30\bar{2})$ not shown in the figure: this end is the antilogous pole. At the other end, the faces of the complementary pyramid $\bar{r}=\mu\{\bar{1}00\}$ are very large, and those of $e=\mu\{\bar{1}\bar{1}0\}$ are very narrow: this end is the analogous pole. The prisms are very short, $a\{0\bar{1}1\}$ being predominant and the two complementary prisms $\mu\{2\bar{1}\bar{1}\}$ and $\mu\{\bar{2}11\}$ being unequal. The crystal is markedly dichroic; being more translucent across the crystal from $a_,$ to a' than from one end to the opposite; although the thickness in the former direction is much the greater.

Fig. 390 represents a crystal of common habit; the trigonal prism $m=\mu\{2\bar{1}\bar{1}\}$ is largely developed, and its edges are bevelled by the hexagonal prism $a\{0\bar{1}1\}$. At one end the pyramid $\mu\{100\}$ is shown; and at the other end the complementary pyramid $\mu\{\bar{1}00\}$. But though these forms, if they occurred alone, would compose a rhombohedron similar to that of classes II—IV, the appearance of the two ends is different. At the upper end the faces $r(100)$ and $m(2\bar{1}\bar{1})$ meet in a horizontal edge, and the same is true of the two other pairs of faces *r* and *m*: at the other end each face, such as $r_,$, meets two faces *m* and m'' at equal angles and the edges are not horizontal.

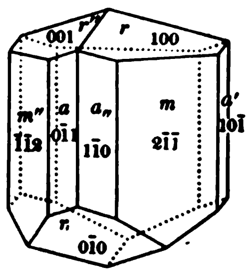


FIG. 390.

In the former case the face r may be said to stand on the prism-face m; in the latter the face r, may be said to stand on the prism-edge $[mm'']$. The rule connecting the electrification may generally be given thus: the analogous pole is at that end at which the faces r of $\mu\{100\}$ stand on the faces of the trigonal prism; the antilogous pole is at that end where the faces of $\mu\{\bar{1}00\}$ stand on the edges of the trigonal prism.

The crystals are optically negative; the principal indices of refraction for sodium-light vary with the colour of the crystal, and sometimes even in its successive layers. Dark coloured varieties are strongly pleochroic, absorbing the ordinary beam to a far greater extent than the extraordinary. Hence the ordinary beam is practically extinguished by a thin plate of a dark coloured crystal cut parallel to the optic axis: and such plates of one or two mm. thickness are often employed for producing beams of plane-polarised light.

In the zone $[rcz s\kappa]$ of Fig. 389 the following angles were measured: $rc=27° 30'$, $cz=27° 35'$, $cs=46° 6'$, $c\kappa=68° 45'$, $c\bar{m}=89° 58'$, $\bar{m}\bar{r}=52° 30'$. Adopting r as (100) and c as (111), then $\bar{m}$ is $(\bar{2}11)$ and $m\,(2\bar{1}\bar{1})$; and z, the dirhombohedral face to (100), is $(\bar{1}22)$. Taking P to be any pole (hll) in $[cm]$, we have from the A.R. $\{cPz\bar{m}\}$

$$\tan cP \div \tan cz = \frac{l-h}{2l+h}.$$

Hence we have, for s, $l-h=2\,(2l+h)$; $\therefore\ l+h=0$, and s is $(\bar{1}11)$.

For κ, $l-h=5\,(2l+h)$, and $3l+2h=0$; $\therefore\ \kappa$ is $(\bar{3}22)$, and the form is $\mu\,\{\bar{3}22\}$.

Again u was found to lie in $[r''ra']$, and the angle ru to be $41° 58'$: hence by the A.R. $\{c'rua'\}$, we have $h-l=5\,(h+l)$. $\therefore\ h=3$, and $l=-2$; and u is $(30\bar{2})$.

VI. *Trigonal bipyramidal class*; $\tau\,\{hkl,\ \bar{p}\bar{q}\bar{r}\}$.

67. The triad axis may be associated with a plane of symmetry, Π, perpendicular to it, provided the crystal has no centre of symmetry. Were a centre of symmetry present, the principal axis would become one of even degree of symmetry, and would then be a hexad axis (Chap. IX, Prop. 4). No crystal has yet been discovered showing the symmetry of the trigonal bipyramidal class.

The general form is a trigonal bipyramid similar geometrically to those represented in Figs. 377 and 378; but, whereas in class IV such forms have their faces limited to zones each containing the pinakoid and one of the faces of the trigonal prism $a\,\{0\bar{1}1\}$, in this class every form is a bipyramid, save when the faces are parallel or perpendicular to the triad axis.

The faces parallel to the plane of symmetry will form a pinakoid similar to that found in classes II—IV; and the triad axis, being the normal to a plane of symmetry, is a possible zone-axis (Chap. IX, Prop. 1).

68. We may take as axes of reference three lines parallel to the co-polar edges of any bipyramid, and a face of the pinakoid as parametral plane (111). Since the axes of reference are interchangeable by rotations of 120° about the triad axis, whilst the parametral face retains the same direction, the parameters are equal. We therefore take as angular element D the inclination of the face of the axial pyramid to the plane of symmetry, and as linear element c a length on the triad axis connected with unit length in the equatorial plane by the equation $c = \cos 30^\circ \tan D$ (see Art. 6). The analytical expressions given in preceding sections apply therefore to crystals of this class.

69. The co-polar triad of faces meeting at an apex are represented by symbols in which the same indices are taken in cyclical order; for the three faces are geometrically similar to the triad forming the trigonal pyramid of class I. The pair of faces situated on opposite sides of the plane of symmetry generally need for their representation symbols having different indices: but a simple relation between their symbols can be found from the fact that the angle between them is bisected by the plane of symmetry parallel to (111), and their edge is truncated by a prism-face, so that the four faces form a harmonic ratio.

Hence the face P above the plane of symmetry being denoted by (hkl), the homologous face below Π is parallel to that which in Art. 15 was called the dirhombohedral face $Q\,(pqr)$. The homologous face has therefore the symbol $(\bar{p}\bar{q}\bar{r})$, where

$$\left.\begin{aligned} \bar{p} &= 3h - 2\theta, \\ \bar{q} &= 3k - 2\theta, \\ \bar{r} &= 3l - 2\theta; \end{aligned}\right\} \quad \text{..........................(92)};$$

θ being $h + k + l$.

We shall therefore denote the form by the symbol $\tau\,\{hkl,\ \bar{p}\bar{q}\bar{r}\}$, in which the symbols of the two faces are connected by equations (92). It is immaterial which of the symbols is derived from the other; for by addition,

$$\bar{p} + \bar{q} + \bar{r} = -3\,(h + k + l) = -3\theta.$$

Hence,

$$9h = 3\bar{p} + 6\theta = 3\bar{p} - 2(\bar{p} + \bar{q} + \bar{r}),$$
$$9k = 3\bar{q} + 6\theta = 3\bar{q} - 2(\bar{p} + \bar{q} + \bar{r}),$$
$$9l = 3\bar{r} + 6\theta = 3\bar{r} - 2(\bar{p} + \bar{q} + \bar{r}).$$

The form $\tau\{hkl, \bar{p}\bar{q}\bar{r}\}$ consists of the six faces:

$$hkl,\ lhk,\ klh,\ \bar{p}\bar{q}\bar{r},\ \bar{r}\bar{p}\bar{q},\ \bar{q}\bar{r}\bar{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(q)}.$$

70. The special forms are:

1. The pinakoid $\{111\}$.

2. Trigonal prisms $\tau\{hkl\}$, geometrically identical with those of class I. The prisms $\tau\{0\bar{1}1\}$ and $\tau\{2\bar{1}\bar{1}\}$ are particular instances, the faces of which have simple relations to the particular edges selected to give the axes: but by a change of the bipyramid selected to give the axial planes general symbols, such as $\tau\{3\bar{2}\bar{1}\}$, may be assigned to either.

The bipyramids represented by Figs. 377 and 378 need only one set of indices and may be represented by $\tau_\sigma\{41\bar{2}\}$, $\tau_\sigma\{4\bar{2}1\}$; but they are not special forms. The faces are tautozonal, each with the pinakoid and a face of the prisms $\tau\{10\bar{1}\}$ and $\tau\{1\bar{1}0\}$. They cannot, however, be represented by the symbols $\tau\{41\bar{2}\}$, $\tau\{4\bar{2}1\}$; for these are the symbols of trigonal pyramids of class I. The σ subscript may be used to indicate that the faces are symmetrical to the equatorial plane; or the forms can be given by the general symbol $\tau\{41\bar{2}, 2\bar{1}4\}$, $\tau\{4\bar{2}1, 2\bar{4}\bar{1}\}$. The same is true of any other bipyramids $\tau_\sigma\{h, (h+l) \div 2, l\}$ having their faces in the same vertical zones.

VII. *Ditrigonal bipyramidal class;* $\kappa\{hkl, \bar{p}\bar{q}\bar{r}\}$.

71. The only other arrangement of elements of symmetry consistent with a single triad axis is one in which the elements of symmetry of class V are associated with a plane of symmetry Π perpendicular to the triad axis and to each of the like planes of symmetry Σ. The lines of intersection of Π with each of the Σ planes are dyad axes (Chap. IX, Prop. 8). The elements of symmetry are therefore:

$$\rho,\ \Pi,\ 3\Sigma,\ 3\Delta.$$

The dyad axes are uniterminal, and should be pyro-electric axes. No crystal has however been discovered showing the symmetry of this class.

72. The forms possible in this class can be readily derived from those of class V in a manner similar to that by which forms of class VI can be obtained from class I. For to obtain any form of class VII, all that is needed is to unite with the corresponding form of class V a similar one, such that the two are reciprocal reflections in the plane of symmetry Π. But retaining the axial arrangement adopted for crystals of class V, the pair of homologous faces symmetrical with respect to Π are connected by the relations (92) given in Art. 69.

Hence the general form $\kappa\{hkl, \bar{p}\bar{q}\bar{r}\}$, Fig. 391, consists of the faces:

$$\left.\begin{matrix} hkl, & khl, & lhk, & lkh, & klh, & hlk, \\ \bar{p}\bar{q}\bar{r}, & \bar{q}\bar{p}\bar{r}, & \bar{r}\bar{p}\bar{q}, & \bar{r}\bar{q}\bar{p}, & \bar{q}\bar{r}\bar{p}, & \bar{p}\bar{r}\bar{q}. \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots \text{(r)}.$$

The two apices are interchangeable by rotations of 180° about one or other of the dyad axes. The drawing can be made in the manner described for $\mu\{hkl\}$.

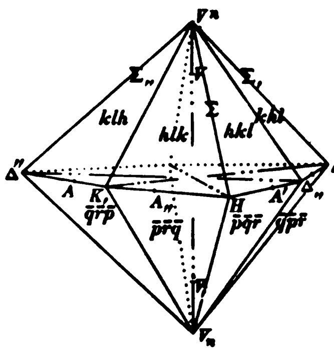


Fig. 391.

73. The special forms are:

1. The pinakoid {111}.

2. A trigonal prism $\kappa\{2\bar{1}\bar{1}\}$ geometrically the same as that of class V.

3. A hexagonal prism $\{0\bar{1}1\}$, having its faces parallel in pairs to the three like planes of symmetry Σ. This prism is common to classes II, III, V and VII.

4. A ditrigonal prism $\kappa\{hkl\}$, geometrically the same as $\mu\{hkl\}$ of class V, and in which $h+k+l=0$.

5. A hexagonal bipyramid $\{hkl\}$, where $h-2k+l=0$. This is geometrically similar to the corresponding form of class III, and can be constructed in the way described in Art. 53.

6. A trigonal bipyramid $\kappa\{hll, \bar{p}\bar{r}\bar{r}\}$, geometrically the same as the corresponding bipyramid of class VI. In this class the trigonal bipyramids are limited to the zones in which the faces are perpendicular to one or other of the like planes Σ; whilst in class VI every form having faces inclined to the axis and equatorial plane at finite angles is a trigonal bipyramid. The particular bipyramid $\kappa\{100, 1\bar{2}2\}$ has its upper polar edges parallel to the axes of reference.

CHAPTER XVII.

THE HEXAGONAL SYSTEM.

1. This system is characterised by having a single hexad axis. It was shown in Chap. IX, Art. 11, that no crystal can have an axis of symmetry of degree higher than six; and in Chap. IX, Art. 21, i. that no crystal can have more than one such axis.

The system comprises five classes, which differ in the elements of symmetry associated with the hexad axis. Since there can be only one hexad axis, it follows that a centre of symmetry, planes of symmetry, parallel and perpendicular to the axis, and dyad axes perpendicular to it are alone admissible. The five classes are:

Pyramidal Hemimorphie

I. The *acleistous hexagonal* (*hexagonal-pyramidal, hemimorphic-hemihedral*) class, in which the hexad axis is the only element of symmetry.

II. The *diplohedral hexagonal* (*hexagonal-bipyramidal, pyramidal-hemihedral*) class, in which the hexad axis is associated with a centre of symmetry and a plane of symmetry, Π, perpendicular to it.

III. The *acleistous dihexagonal* (*dihexagonal-pyramidal, hexagonal-hemimorphic*) class, in which the hexad axis is associated with six planes of symmetry intersecting in it at angles of 30°. Owing to the hexad axis, these planes consist of two triads of like and interchangeable planes S and Σ, arranged alternately so that the S planes bisect, respectively, the angles between the Σ planes; and vice versâ.

IV. The *diplohedral dihexagonal* (*dihexagonal-bipyramidal, holohedral-hexagonal*) class, in which the elements of symmetry of class III are associated with a centre of symmetry. The crystals, consequently, have a plane of symmetry, Π, and six dyad axes

perpendicular to the hexad axis. The lines of intersection of the planes S and Π are three like dyad axes δ; and the lines of intersection of the planes Σ and Π are three like dyad axes Δ. The dyad axes δ are interchangeable by rotations of 60° about the principal axis; and so are the Δ axes; but an axis of one kind is not interchangeable with one of the other; nor is a Σ plane interchangeable with an S plane. The elements may be given by:

$$H,\ 3S,\ 3\Sigma,\ C,\ \Pi,\ 3\delta,\ 3\Delta.$$

V. The *trapezohedral* (*hexagonal-trapezohedral*, *trapezohedral-hemihedral*) class, in which the hexad axis is associated only with six dyad axes perpendicular to it. The dyad axes consist of a triad of like and interchangeable axes δ inclined to one another at angles of 60°, and of three like and interchangeable axes Δ also inclined to one another at angles of 60°. Adjacent δ and Δ axes are inclined to one another at angles of 30°.

2. The hexad axis is perpendicular to a possible face (Chap. IX, Prop. 3), which belongs to a pedion in classes I and III, and to a pinakoid in the other classes; the plane through the origin parallel to this face will be called the *equatorial plane*.

When the faces of a form are inclined to the hexad axis at finite angles, other than 90°, the homologous faces and edges meeting at the same apex will be six or twelve in number, and will be said to be *co-polar*. In classes I, II and V the number of co-polar faces and edges is in all cases six; in classes III and IV the number is six only when the faces are perpendicular each to one of the six tautozonal planes of symmetry S and Σ. When treating of the several classes we shall see that certain of these forms, or all of them, are either acleistous or diplohedral hexagonal pyramids, similar in shape to those represented in Figs. 392 and 393.

For purposes of analysis, axes of reference may be selected in two different ways, which we may denote as (i) *rhombohedral* (or *Millerian*) axes, and (ii) *hexagonal* axes.

i. *Rhombohedral axes.*

3. The axes are taken parallel to the three possible edges in which alternate co-polar faces of a hexagonal pyramid would, if produced, intersect; and the possible face perpendicular to the hexad axis is taken as parametral plane (111). The pyramid selected to give the directions of the axes we shall call the

fundamental pyramid. Thus, in Figs. 392 and 393, the axes of X, Y and Z are parallel to the lines joining the apex V to the points $\bar{M}$, $M_{,}$, $M_{,,}$ which lie on the lines OB, $OB_{,}$, OB'' at distances $2 \times OB$ from the origin and are interchanged by rotations of 120° about the hexad axis.

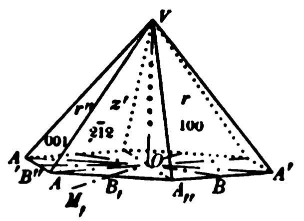


FIG. 392.

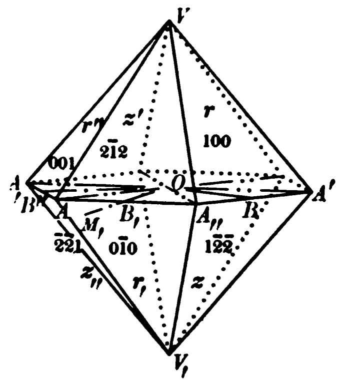


FIG. 393.

We shall, for the sake of easy comparison with the relations of rhombohedral crystals, represent the three co-polar faces meeting in $V\bar{M}$, $VM_{,}$, $VM_{,,}$ by the letters r, r', r''; and we shall take OX to be parallel to $V\bar{M}$ in which r' and r'' intersect, OY to $VM_{,}$ in which r'' and r intersect, and OZ to $VM_{,,}$ the line of intersection of r and r'. The three faces have therefore the symbols: r (100), r' (010), r'' (001). We shall take for unit length the equal semi-diagonals OA, OA', OA'' of the hexagon formed by the fundamental pyramid in the equatorial plane, and $OV = c$ for the linear parameter of the crystal. But from Figs. 392 and 393 it is clear that $OB = OA' \cos 30°$, and $\tan VBO = OV \div OB = OV \div OA' \cos 30°$. But $\wedge VBO$ is equal to the angle between the principal axis and the normal to the face r (100): this angle we shall denote as the angular element D of a hexagonal crystal. Hence, when $OA' = 1$,

$$c = \cos 30° \tan VBO = \cos 30° \tan D \quad \ldots\ldots\ldots\ldots(1).$$

The parameter c is the only crystal-element which in the hexagonal system varies with the substance.

From the fact that the three axes of reference are interchanged by rotations of $120° = 2 \times 60°$ about the hexad axis, whilst the parametral plane retains the same direction, it follows that the parameters on the Millerian axes are all equal, and may be taken to be any three equal lengths parallel to the axes, such as $V\bar{M}$, $VM_{,}$, $VM_{,,}$.

4. To find the Millerian symbols of the faces of a hexagonal pyramid.

Rotations of 120°, double the least rotation characteristic of a hexad axis, being always possible, it is clear that any face (*hkl*) must be associated with two others (*lhk*) and (*klh*) which meet the principal axis at an apex V^n, these faces having their symbols in cyclical order. Further, if the crystal is centro-symmetrical, the parallel faces will also be present; viz. $(\bar{h}\bar{k}\bar{l})$, $(\bar{l}\bar{h}\bar{k})$, $(\bar{k}\bar{l}\bar{h})$. And, again, if planes of symmetry pass through the principal axis and each of the axes of reference, the above triads must be associated with like triads having the same indices taken in the reverse cyclical order. For purposes of analysis a selection of one half of the faces of a hexagonal form may be made, which can all be represented in exactly the same manner as if the crystals belonged to one or other of classes I—V of the rhombohedral system. But the faces homologous to (*hkl*) obtained by rotations of 60° or 180° about the hexad axis cannot be represented by the same indices. The face connected with (*hkl*) by a rotation of 180° is that denoted by (*pqr*) in Chap. XVI, Art. 15. Hence, the second half of the faces of a hexagonal form may be regarded as consisting of the faces of the dirhombohedral form. The face of this form opposite to (*hkl*) being (*pqr*), the symbols are connected by the equations (see Chap. XVI, Art. 15),

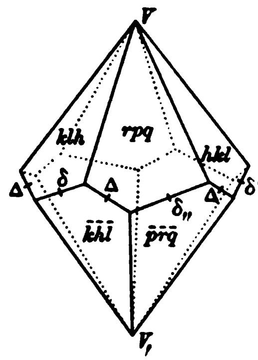


FIG. 394.

$$\left.\begin{aligned} p &= 2(h+k+l) - 3h = 2\theta - 3h, \\ q &= 2(h+k+l) - 3k = 2\theta - 3k, \\ r &= 2(h+k+l) - 3l = 2\theta - 3l \end{aligned}\right\} \ldots (2).$$

Hence the six co-polar faces meeting at V in Fig. 394 which are connected by rotations of 60° about the hexad axis are:

hkl, *qrp*, *lhk*, *pqr*, *klh*, *rpq* ... (a).

The poles of six homologous faces may also be represented by the stereogram, Fig. 395, in which the poles of one triad are denoted by letters P and those of the homologous co-polar triad by Q. The axes meet the sphere at the points X, Y, Z indicated by crosses.

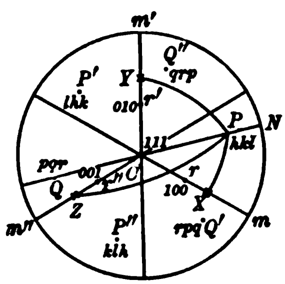


FIG. 395.

OA', and the positive direction is taken to the right: the parameter OA' is equal to OA, and the axis is perpendicular to Miller's OY. We shall denote by OU the third equatorial axis having the direction $OA_{\prime\prime}$, and we shall take $OA_{\prime\prime} = OA$ to be the parameter on it measured on the negative side of the origin: this axis is perpendicular to Miller's OZ. The three first indices will be written in the order in which they refer to the axes OX, OY and OU, respectively.

Since by rotations of 120° about the hexad axis, the positive directions of OX, OY and OU are interchanged, and also equal positive lengths or equal negative lengths on them, it is convenient to take equal parameters on the three equatorial axes; and we shall denote by $a_{\prime}$ a length a when measured on OY, and by $a_{\prime\prime}$ when measured on OU.

We can now take OV of Figs. 392 and 393 to be the parameter c on OZ, when the face (100) of the fundamental pyramid becomes $(01\bar{1}1)$; for $VA'A_{\prime\prime}$, passing through $A'A_{\prime\prime}$, is parallel to OX, and $OA_{\prime\prime} = -OA'$. The face therefore meets the axes at distances given by the ratios $a \div 0 : a_{\prime} : a_{\prime\prime} \div \bar{1} : c$; and its symbol is $(01\bar{1}1)$. Similarly, (010) is $(\bar{1}011)$, and (001) is $(1\bar{1}01)$.

Again, the face $VAA_{\prime\prime}$—Miller's $(2\bar{1}2)$—meets the axes at distances given by the ratios $a : a_{\prime} \div 0 : a_{\prime\prime} \div \bar{1} : c$; and has the symbol $(10\bar{1}1)$. The two remaining co-polar faces of the pyramids have the equivalent symbols $(\bar{1}22) = (0\bar{1}11)$ and $(22\bar{1}) = (\bar{1}101)$.

There are cases however in which Miller has not adopted the same fundamental pyramid as other crystallographers; and OX, OY, OZ may accordingly coincide with OA, OA', OA'', or with OB, OB' and OB''. In the former case, Miller's (100) becomes $(0h\bar{h}l)$; and the two values of c given by equation (1) are in the ratio of h : l, and depend on the change in D only. In the second case, Miller's (100) becomes $(2h, \bar{h}, \bar{h}, l)$; and the two values of c are not commensurable when the faces of the fundamental pyramid pass through points on OB, &c., at unit distance from the origin; for $OB = a \cos 30°$.

We shall however assume that the same fundamental pyramid is adopted in both notations, unless the contrary is stated. The student will discover whether this is the case in any particular substance by comparing the symbols and the corresponding angles given by the authors consulted. Hence, for the same value of D or c, we have the following equivalent symbols

$$\left.\begin{array}{cccccc} 100 & 22\bar{1} & 010 & \bar{1}22 & 001 & 2\bar{1}2; \\ 01\bar{1}1 & \bar{1}101 & \bar{1}011 & 0\bar{1}11 & 1\bar{1}01 & 10\bar{1}1. \end{array}\right\} \quad \text{............(b).}$$

6. The faces perpendicular to the hexad axis have the symbols $000 1$ and $000\bar{1}$, according as they meet it on the positive or negative side of the origin: they correspond, respectively, to Miller's (111) and $(\bar{1}\bar{1}\bar{1})$; and they form pedions or a pinakoid according to the elements of symmetry associated with the hexad axis.

Prism-faces parallel to OZ and to one of the equatorial axes of reference OX (say) meet the other pair of axes at equal distances OA', $OA_{\prime\prime}$, &c., from the origin; but the one intercept is positive whilst the other is negative. The faces have therefore the symbols $(01\bar{1}0)$ and $(0\bar{1}10)$: they must occur together, for they are both parallel to a hexad axis. They are repeated in two other pairs; and the three pairs form a hexagonal prism, Fig. 396, represented by the following equivalent symbols:

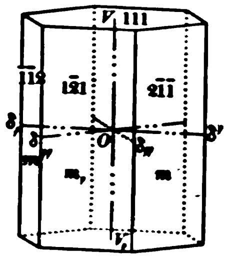


Fig. 396.

$$\left.\begin{matrix} \{2\bar{1}\bar{1}\}, & 2\bar{1}\bar{1} & 11\bar{2} & \bar{1}2\bar{1} & \bar{2}11 & \bar{1}\bar{1}2 & 1\bar{2}1; \\ \{01\bar{1}0\}, & 01\bar{1}0 & \bar{1}100 & \bar{1}010 & 0\bar{1}10 & 1\bar{1}00 & 10\bar{1}0. \end{matrix}\right\} \ldots(\mathrm{c}).$$

The reader will, on reference to Chap. XVI, Art. 27, see that the prism-face through $A'\ A_{\prime\prime}$, coinciding with $\delta'\delta_{\prime\prime}$, is $(2\bar{1}\bar{1})$.

The other hexagonal prism $\{1\bar{1}0\}$ described in Chap. XVI, Art. 27, belongs also to this system: its faces are perpendicular each to one of the axes OA, OA', &c., of Fig. 392, and parallel therefore to the corresponding Millerian axis. We may now suppose one of the faces to pass through A and A', for the line AA' is perpendicular to and bisects $OA_{\prime\prime}$. The intercepts on the axis are $a : a_{\prime} : a_{\prime\prime} \div \bar{2} : c \div 0$; and the symbol is $(11\bar{2}0)$. But this face is Miller's $(1\bar{1}0)$. Hence this hexagonal prism, Fig. 397, has the equivalent symbols:

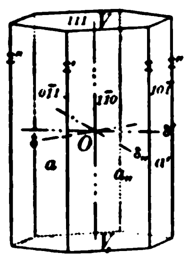


Fig. 397.

$$\left.\begin{matrix} \{1\bar{1}0\}, & 1\bar{1}0 & 10\bar{1} & 01\bar{1} & \bar{1}10 & \bar{1}01 & 0\bar{1}1; \\ \{11\bar{2}0\}, & 11\bar{2}0 & \bar{1}2\bar{1}0 & \bar{2}110 & \bar{1}\bar{1}20 & 1\bar{2}10 & 2\bar{1}\bar{1}0. \end{matrix}\right\} \ldots(\mathrm{d}).$$

The above prisms are common to all classes of the hexagonal system, and are also geometrically the same in classes II and III of the rhombohedral system.

7. When the intercepts made by a face (hkil) on two of the equatorial axes are known, the points in which it meets any other

known lines in the equatorial plane can be found; the intercept on the third axis can therefore be determined.

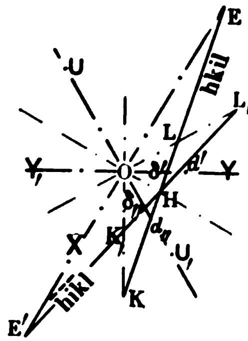


FIG. 398.

Let, in Fig. 398, ELK be the trace of (hkil) in the equatorial plane. Then $OE = a \div \mathrm{h}$, $O\delta' = a_{,} \div \mathrm{k}$, $Od_{,,} = a_{,,} \div \mathrm{i}$. Now the whole triangle $EOd_{,,}$ is made up of the two triangles $EO\delta'$, $\delta' Od_{,,}$.

$$\therefore\ OE \, . \, Od_{,,} \sin 120^\circ$$

$$= OE \, . \, O\delta' \sin 60^\circ + O\delta' \, . \, Od_{,,} \sin 60^\circ;$$

and $\sin 120^\circ = \sin 60^\circ$.

Dividing by $OE \, . \, O\delta' \, . \, Od_{,,} \sin 60^\circ$, we have

$$\frac{1}{O\delta'} = \frac{1}{Od_{,,}} + \frac{1}{OE}.$$

In obtaining this equation all the sides of the triangles have been treated as positive lengths. When, however, their values in terms of the indices are introduced, attention must be paid to the signs of the latter. But in the particular case represented by the figure, h and i are both negative, for the trace meets OX and OU on the negative sides of the origin. Consequently OE must be replaced by $a \div \bar{\mathrm{h}}$, $Od_{,,}$ by $a \div \bar{\mathrm{i}}$ and $O\delta'$ by $a \div \mathrm{k}$.

Hence $$\frac{\mathrm{k}}{a} = -\frac{\mathrm{i}}{a} - \frac{\mathrm{h}}{a};$$

$$\therefore\ \mathrm{h} + \mathrm{k} + \mathrm{i} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

A similar relation can be readily established for any other position of the face, when the order or signs of h, k and i have to be changed to correspond with the intercepts on the axis. Hence *the algebraic sum of the three indices referring to the equatorial axes is always zero.*

It may be noticed that in the face actually taken the positive index is the greatest, and the corresponding intercept $O\delta'$ is the least; also that $O\delta'$ divides the larger triangle into the two smaller ones. Thus the face may be $(\bar{1}3\bar{2}1)$ or $(\bar{2}5\bar{3}1)$. For a face like that through the trace $E'd'L_{,}$, the shortest intercept is now negative and lies on $OU_{,}$. The signs of the three first indices must all be changed; and the symbol is given in the general case by $(\bar{\mathrm{h}}\mathrm{i}\bar{\mathrm{k}}\mathrm{l})$, and in the particular cases by $(12\bar{3}1)$ and $(23\bar{5}1)$.

8. To find the symbols of the six co-polar faces connected by rotations of 60° about the hexad axis.

Let, in Fig. 399, a face (hkil) meet the equatorial plane in the trace $Kd_{,,}H\delta E$ (E on OX not being shown) and let the intercepts

on the axes be $OE = a \div h$, $O\delta' = a_{\prime} \div k$, $Od_{\prime\prime} = a_{\prime\prime} \div i$. A rotation clockwise of 120° about OZ (perpendicular to the paper) brings OX to OU, OU to OY, and OY to OX; and also the trace $KH\delta'$ into the position $H''\delta K_{\prime}$. The intercept OE is now measured on OU and must be written $a_{\prime\prime} \div h$, $O\delta'$ has become $O\delta$ on OX and must be written $a \div k$, and $Od_{\prime\prime}$ is measured on $OY_{\prime}$ and must be written $a_{\prime} \div i$. The face through the trace $H''\delta K_{\prime}$ has therefore the symbol (kihl). A second rotation of 120° in the same direction again

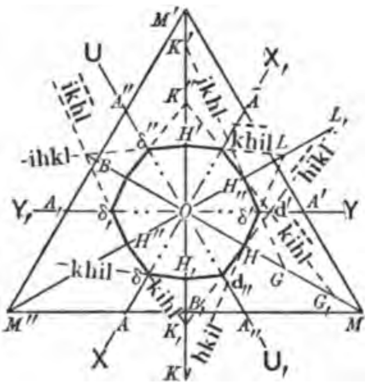


Fig. 399.

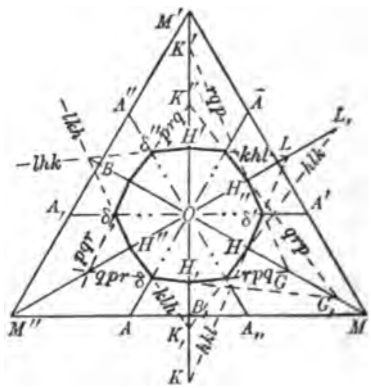


Fig. 400.

changes the same axes and brings the trace to $H'\delta''$, the face through which has the symbol (ihkl). The first three indices are therefore changed in cyclical order; and the three faces have the symbols (hkil), (kihl), (ihkl). If the first face has the Millerian symbol (*hkl*), the three correspond to (*hkl*), (*klh*), (*lhk*) respectively.

When however the crystal is turned about OZ through 180°—a rotation interchanging homologous faces which may be said to be *opposite*—equal positive and negative lengths are interchanged on each equatorial axis. Thus, the face (kihl) through $H''\delta K_{\prime}$ is brought into a position in which it passes through $H_{\prime\prime}K''$: the symbol of the opposite face is therefore ($\bar{k}\bar{i}\bar{h}l$). The triad of faces opposite to the first triad are ($\bar{h}\bar{k}\bar{i}l$,) ($\bar{k}\bar{i}\bar{h}l$), ($\bar{i}\bar{h}\bar{k}l$). But a semi-revolution about the hexad axis interchanges the Millerian triad (*hkl*), (*klh*), (*lhk*) with the dirhombohedral triad (*pqr*), (*qrp*), (*rpq*).

Hence the six faces connected by the hexad axis are, when taken in order counter-clockwise, given by the following equivalent symbols:—

$$\left.\begin{array}{cccccc} hkl & qrp & lhk & pqr & klh & rpq \\ \mathrm{hkil} & \bar{\mathrm{k}}\bar{\mathrm{i}}\bar{\mathrm{h}}\mathrm{l} & \mathrm{ihkl} & \bar{\mathrm{h}}\bar{\mathrm{k}}\bar{\mathrm{i}}\mathrm{l} & \mathrm{kihl} & \bar{\mathrm{i}}\bar{\mathrm{h}}\bar{\mathrm{k}}\mathrm{l} \end{array}\right\} \ldots\ldots (\mathrm{e}).$$

The traces of these faces in the equatorial plane are shown in Figs. 399 and 400; in the former some of the symbols have been omitted for the sake of greater clearness.

9. To find the symbols of the six co-polar faces symmetrical to those in table **e**, when planes of symmetry intersecting in the hexad axis are present.

If planes of symmetry parallel to the hexad axis are present, there must be six. Three of the planes, denoted by letters S, pass through opposite co-polar edges of the fundamental pyramid and meet the equatorial plane in the three equatorial axes: the three others, denoted by letters Σ, are perpendicular each to a pair of opposite faces of the pyramid and meet the equatorial plane in the lines OB, OB', OB''. The planes forming each triad are like planes of symmetry, for they are interchangeable by rotations of 60° about the hexad axis.

Now the Millerian axes OX, OY, OZ lie in the planes Σ; and it was shown in Chap. XVI, Art. 45, that (hkl) being one face of the scalenohedron, the adjacent face meeting the first in the polar edge $V^*\lambda$ has the symbol (hlk); and a similar relation holds for each pair of faces symmetrically placed with respect to each of the planes Σ. Similar relations hold for pairs of faces of the inverse scalenohedron $\{pqr\}$, and also for ditrigonal pyramids $\mu\{hkl\}$ and $\mu\{pqr\}$ of class V of the rhombohedral system.

But each of the planes Σ is perpendicular to one of the equatorial axes of reference and bisects the angle between the other two. Thus Σ passing through BOM of Fig. 399 is perpendicular to OX and bisects the angle between the axes OY and $OU_,$. Hence the face (hkil) through the trace $H\delta'L$ is repeated in a similar face through $K_,HL_,$. The intercepts on OX are equal but opposite in sign, whilst those on OY and OU are interchanged. But $a_, \div \mathrm{k}$ measured on OY is positive, and is therefore changed to $a_{,,} \div \bar{\mathrm{k}}$ when measured along $OU_,$. Again, $a_{,,} \div \mathrm{i}$ on OU is negative, and must be changed to $a_, \div \bar{\mathrm{i}}$ when the intercept is measured along OY. The signs of the three first indices are therefore all changed, and also the cyclical order; and the symbol of the face through the trace $K_,HL_,$ is $(\bar{\mathrm{h}}\mathrm{i}\bar{\mathrm{k}}\mathrm{l})$. This face is by successive rotations, each of 60°, brought into five other positions, in which the cyclical order of the first three indices remains the same. The six faces have therefore the following equivalent symbols:—

$$\left.\begin{matrix} hlk & rqp & khl & prq & lkh & qpr \\ \bar{\mathrm{h}}\mathrm{i}\bar{\mathrm{k}}\mathrm{l} & \mathrm{ikhl} & \bar{\mathrm{k}}\bar{\mathrm{h}}\mathrm{il} & \mathrm{hikl} & \mathrm{i}\bar{\mathrm{k}}\bar{\mathrm{h}}\mathrm{l} & \mathrm{khil} \end{matrix}\right\}\ldots(\mathbf{f}).$$

The two sets of six faces given in (e) and (f) occur together or alone, according as the crystal has, or has not, the planes of symmetry S and Σ. They may be also associated with similar sets of six when the crystals are centro-symmetrical; or (e) may be associated with the set of faces parallel to (f) in the trapezohedral class, or vice versâ.

The sets of faces parallel to those given in (e) and (f) are found by changing the signs of all the indices together.

10. We now proceed to determine the relation between the symbols (hkl) and (hkil) representing the same face of a dihexagonal pyramid or scalenohedron.

In Chap. XVI, Art. 42, the scalenohedral face (hlk) is drawn through a point $\delta_{\prime\prime}$ or $A_{\prime\prime}$ at distance a on the axis OU; and the points H and K, in which it meets OM and $OM_{\prime}$, Fig. 348, are given by equations (45) and (47) of that Article; viz.

$$OH = \frac{2n}{3n+1}\,OM = \frac{2na\sqrt{3}}{3n+1},$$

and

$$OK_{\prime} = \frac{2n}{3n-1}\,OM = \frac{2na\sqrt{3}}{3n-1};$$

where n has the value given in (61) of Art. 44.

Similarly, the face (hkl) is drawn through A' to meet OM'' at L, where $OL = OK_{\prime} = \frac{2na\sqrt{3}}{3n-1}$. By shifting the face parallel to itself until it passes through δ' in Figs. 399 and 400, the ratios of the intercepts on OM, $OB_{\prime}$, and OV^{n} on the axis OZ are changed in the ratio $O\delta' \div OA' = \beta$ (say).

Let the new apex be at V^{p}, where $OV^{p} = \beta \,.\, OV^{n}$, and let, in Fig. 401, the trace of (hkl) on the equatorial plane be given by $EL\delta'Hd_{\prime\prime}$, in which the intercepts OH and OL given by the above equations have to be multiplied by β. Then, since the triangles $\delta'OL$ and LOE make up the whole triangle $\delta'OE$,

$$(O\delta' \,.\, OL + OL \,.\, OE)\sin 30° = O\delta' \,.\, OE \sin 60°;$$

$$\therefore\ \frac{1}{OE} + \frac{1}{O\delta'} = \frac{\tan 60°}{OL} = \frac{3n-1}{2na\beta}.$$

$$\therefore\ \frac{1}{OE} = \frac{3n-1}{2na\beta} - \frac{1}{a\beta} = \frac{n-1}{2na\beta} \ldots\ldots (5).$$

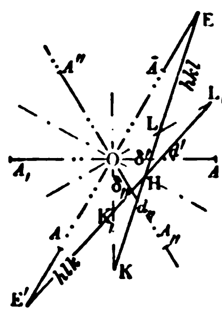


FIG. 401.

Similarly, the triangle $\delta' Od_{\prime\prime}$ is divided into two by OH, and we have

$$(O\delta' . OH + OH . Od_{\prime\prime}) \sin 30^\circ = O\delta' . Od_{\prime\prime} \sin 60^\circ;$$

$$\therefore \frac{1}{Od_{\prime\prime}} + \frac{1}{O\delta'} = \frac{\tan 60^\circ}{OH} = \frac{3n+1}{2na\beta}.$$

$$\therefore \frac{1}{Od_{\prime\prime}} = \frac{3n+1}{2na\beta} - \frac{1}{a\beta} = \frac{n+1}{2na\beta} \text{ ....(6).}$$

Hence

$$\left.\begin{aligned} \beta\mathrm{h} &= \frac{\beta . OA}{-OE} = \frac{1-n}{2n} \\ \beta\mathrm{k} &= \frac{\beta . OA'}{O\delta'} = 1 = \frac{2n}{2n} \\ \beta\mathrm{i} &= \frac{\beta . OA''}{-Od_{\prime\prime}} = -\frac{1+n}{2n} \\ \beta\mathrm{l} &= \beta\frac{OC}{OV^p} = \frac{OC}{OV^n} = \frac{2}{2mn} \end{aligned}\right\} \text{ ....(7).}$$

OE and $Od_{\prime\prime}$ are both treated as positive lengths in (5) and (6), but for the particular face taken they are measured in the negative directions on the axes OX and OU: hence the introduction of the minus signs in (7).

We therefore have

$$\frac{\mathrm{h}}{1-n} = \frac{\mathrm{k}}{2n} = \frac{\mathrm{i}}{-(1+n)} = \frac{\mathrm{l}}{2 \div m} \text{ .... (8).}$$

These equations give the hexagonal symbols corresponding to a face of the scalenohedron mRn. In Chap. XVI, Art. 44, the values of m and n in terms of the Millerian indices h, k, l are given by equations (60) and (61); viz.

$$\frac{1}{m} = \frac{h+k+l}{h-2k+l},$$

and

$$n = \frac{h-l}{h-2k+l}.$$

Hence,

$$1 - n = \frac{2l-2k}{h-2k+l};$$

$$-(1+n) = \frac{2(k-h)}{h-2k+l},$$

$$\therefore \frac{\mathrm{h}}{l-k} = \frac{\mathrm{k}}{h-l} = \frac{\mathrm{i}}{k-h} = \frac{\mathrm{l}}{h+k+l} \text{ ....(9).}$$

If k and l are interchanged, *i.e.* if we take the face (hlk), it is clear from (9) that the signs of h, k, i are changed, and also their

cyclical order; but not their numerical values. Thus the denominator of the last term is unchanged, that of the first becomes $k-l$, the second $h-k$, and the third $l-h$. But these four numbers are proportional to 1, $\bar{h}$, $\bar{i}$, $\bar{k}$, respectively. Hence, ($\bar{h}\bar{i}\bar{k}$l) is the equivalent symbol of (hlk). This can also be seen from the geometry of Fig. 401. For the trace of (hlk) is given by $E'\delta_{\prime\prime}d'$; where $OE' = -OE$, $Od' = -Od_{\prime\prime}$, and $O\delta_{\prime\prime} = -O\delta'$. Hence, as in (7), the hexagonal indices are given by:

$$\begin{aligned} \beta . OA \; &\div OE' = (n-1) \div 2n = \beta\bar{h}, \\ \beta . OA' &\div Od' = (1+n) \div 2n = \beta\bar{i}, \\ \beta . OA'' &\div O\delta_{\prime\prime} = -1 \qquad\quad\; = \beta\bar{k}, \\ OC \; &\div OV^{*} = 1 \div mn \qquad\; = \beta l. \end{aligned}$$

Equations (9) enable us therefore to find the hexagonal symbols of all the faces of $\{hkl\}$.

It is also easy to show that by a change in the signs of h, k, i, but not in their cyclical order, equations (9) give the equivalent symbol for the opposite face (pqr); and that therefore {hkil} is the equivalent of $\{hkl, pqr\}$. For

$$p = 2(h+k+l) - 3h = 2\theta - 3h; \quad q = 2\theta - 3k; \quad r = 2\theta - 3l.$$

$$\begin{aligned} \therefore \quad r - q &= 2\theta - 3l - (2\theta - 3k) = 3(k-l), \\ p - r &= 2\theta - 3h - (2\theta - 3l) = 3(l-h), \\ q - p &= 2\theta - 3k - (2\theta - 3h) = 3(h-k), \\ p + q + r & \qquad\qquad\qquad\qquad\quad\;\; = 3(h+k+l). \end{aligned}$$

Hence from (9),

$$\frac{-\mathrm{h}}{r-q} = \frac{-\mathrm{k}}{p-r} = \frac{-\mathrm{i}}{q-p} = \frac{1}{p+q+r} \quad \ldots\ldots\ldots\ldots (10).$$

Therefore ($\bar{h}\bar{k}\bar{i}$l) is the equivalent symbol for (pqr); and similarly, (hikl) for (prq).

It is therefore immaterial which of the faces of $\{hkl\}$ or of $\{pqr\}$ is employed to determine the equivalent hexagonal symbol, for {hkil} includes both $\{hkl\}$ and $\{pqr\}$.

11. Equations (9) and (10) can, by addition and subtraction of the numerators and denominators, be transformed so as to give (hkl) and (pqr) when either of the equivalent symbols (hkil) or ($\bar{h}\bar{k}\bar{i}$l) is known. For each term of (9) is equal to

$$\frac{\mathrm{l}+\mathrm{k}-\mathrm{i}}{h+k+l+h-l-k+h} = \frac{\mathrm{l}+\mathrm{k}-\mathrm{i}}{3h} = \frac{\mathrm{l}+\mathrm{i}-\mathrm{h}}{3k} = \frac{\mathrm{l}+\mathrm{h}-\mathrm{k}}{3l}.$$

Hence, $$\frac{h}{\mathrm{l}+\mathrm{k}-\mathrm{i}} = \frac{k}{\mathrm{l}+\mathrm{i}-\mathrm{h}} = \frac{l}{\mathrm{l}+\mathrm{h}-\mathrm{k}} \quad \ldots\ldots\ldots\ldots\ldots (11).$$

Again, each term of (9) is equal to

$$\frac{1-k+i}{h+k+l-h+l+k-h} = \frac{1-k+i}{2(h+k+l)-3h} = \frac{1-i+h}{2(h+k+l)-3k}$$
$$= \frac{1-h+k}{2(h+k+l)-3l}.$$

But the denominators of these equations are p, q and r respectively. Hence the opposite face (pqr) is given by

$$\frac{p}{1-(k-i)} = \frac{q}{1-(i-h)} = \frac{r}{1-(h-k)} \quad \ldots\ldots\ldots\ldots\ldots(12).$$

These latter equations accord with the values derived in a similar manner from equations (10); and they can be derived from (11) by changing the signs of h, k, i.

As, however, both (hkl) and (pqr) are included in every hexagonal pyramid, equations (11) and (12) can be given in the following form:

$$\frac{h}{1 \pm (k-i)} = \frac{k}{1 \pm (i-h)} = \frac{l}{1 \pm (h-k)} \quad \ldots\ldots\ldots\ldots(13);$$

where the plus signs give the symbol of one face and the minus signs that of the dirhombohedral or opposite face.

In obtaining the above equations the face (hkl) was supposed to make the least intercept on OA' (OY); but the equations are perfectly general, and it is immaterial which face of a form is taken in order to obtain the equivalent symbol.

12. In computations such as the determination of zone-indices, the indices of a face which is common to two zones, or the anharmonic ratio of four tautozonal faces, we omit one of the equatorial axes. We then have three axes OX, OY, OZ (say) which may be regarded as a set of axes of an oblique crystal in which $XOY = 120°$, and $ZOX = ZOY = 90°$. When the indices referring to the three axes have been obtained from the general relations mentioned, the index referring to the fourth axis OU is obtained from the relation $h + k + i = 0$.

13. The equations of the normal OP, or of the pole P, are obtained in a manner similar to that given in Chap. IV, Art. 15. Taking OA to be unity, they are:

$$OP = \frac{\cos XP}{h} = \frac{\cos YP}{k} = \frac{\cos UP}{i} = \frac{c \cos ZP}{l}$$
$$= \frac{\cos XP + \cos YP + \cos UP}{h+k+i} \quad \ldots\ldots\ldots\ldots(14).$$

We may take P to be the pole (hkl) of a rhombohedral scalenohedron, Fig. 402 (see also Figs. 412 and 413, p. 454). Using the notation of Chap. XVI, Arts. 48—50, we have:

$XP = aP = 90° + \xi$, $YP = a'P = \zeta$,
$UP = a''P = 90° + \eta$, and $ZP = CP$.

Hence, $\cos XP = -\sin\xi$,

$\cos YP = \cos\zeta$, $\cos UP = -\sin\eta$.

But by equation (68) of that Chapter

$$\cos\zeta - \sin\eta - \sin\xi = 0;$$

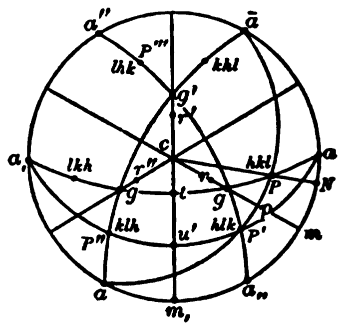


FIG. 402.

hence, the numerator of the last term of (14) is zero, and also the denominator.

$$\therefore\ \mathrm{h} + \mathrm{k} + \mathrm{i} = 0,$$

the result already given in (4).

But introducing into equations (74) of Chap. XVI the values of ξ, η, ζ, we have

$$\frac{\cos XP}{l-k} = \frac{\cos YP}{h-l} = \frac{\cos UP}{k-h} = \frac{\sqrt{3}\sin CP}{\sqrt{2\{(k-h)^2+(h-l)^2+(l-k)^2\}}} \ldots(15).$$

Dividing each term of these equations by the corresponding one of (14), we have

$$\frac{\mathrm{h}}{l-k} = \frac{\mathrm{k}}{h-l} = \frac{\mathrm{i}}{k-h} = \frac{\mathrm{l}\sqrt{3}\tan CP}{c\sqrt{2\{(k-h)^2+(h-l)^2+(l-h)^2\}}} \ldots\ldots(16).$$

By adding the squares of the numerators of the first three terms of (16) and those of the denominators, and then extracting the square root of the ratio, a term is obtained equal to each of the ratios; viz.

$$\sqrt{\frac{\mathrm{h}^2+\mathrm{k}^2+\mathrm{i}^2}{(k-h)^2+(h-l)^2+(l-k)^2}}.$$

Hence,

$$\tan CP = \frac{\sqrt{2(\mathrm{h}^2+\mathrm{k}^2+\mathrm{i}^2)}}{\mathrm{l}\sqrt{3}}\,c = (\text{from }(1))\ \frac{\sqrt{\mathrm{h}^2+\mathrm{k}^2+\mathrm{i}^2}}{\mathrm{l}\sqrt{2}}\tan D \ldots..(17).$$

If in (16), the value of h : l given in (9) is introduced, we have

$$\tan CP = \frac{\sqrt{(k-h)^2+(h-l)^2+(l-h)^2}}{(h+k+l)\sqrt{2}}\tan D \ldots\ldots(17^*).$$

Again, N being the pole in which the zone $[CP]$ intersects the zone $[mm']$, we have from (70) of Chap. XVI,

$$\tan YN = \tan a'N = \frac{1}{\sqrt{3}}\,\frac{\theta-3k}{h-l} = (\text{from }(11))\ \frac{\mathrm{h}-\mathrm{i}}{\mathrm{k}\sqrt{3}} \ldots\ldots(18).$$

From the right-angled triangle YPN, Fig. 413, p. 454, we have

$$\cos YP = \cos PN \cos YN = \sin ZP \cos YN.$$

Therefore from the second and fourth terms of (14),

$$\cos YN = \cos YP \div \sin ZP = \frac{\mathrm{k}}{\mathrm{l}} c \cot ZP = \frac{\mathrm{k}}{\mathrm{l}} \cos 30^\circ \tan D \cot ZP$$

$$= (\text{from (9)}) \frac{h-l}{h+k+l} \cos 30^\circ \tan D \cot CP \;\ldots\ldots\ldots\ldots (19).$$

The equations given above together with the relations holding between tautozonal poles suffice for the solutions of most of the problems which the student is likely to meet with.

14. In the preceding Articles, and also in the discussion of the forms characteristic of classes I to V, it is assumed that the face (100) is the same as $(01\bar{1}1)$; but as pointed out in Art. 5, the same pyramid is not always selected by different crystallographers to give the crystal element D, or c. Thus it frequently happens that, as in the case of apatite, Miller's (210) is Dana's $(01\bar{1}1)$. If it is desired to find the equivalent symbols of the forms corresponding to the different pyramid adopted, the transformation may be carried out (i) in two steps, or (ii) in a single step.

i. We first transform the Millerian symbol (hkl) to its equivalent $(h'k'l')$ in the same notation when three co-polar faces of a form $\{efg\}$ interchangeable by rotations of 120° about the principal axis become (100), (010) and (001), respectively; the parametral plane remaining unaltered. But (efg) being known to be the face represented by $(01\bar{1}1)$ in hexagonal notation, the values of h', k', l' are now introduced into equations (9), and give the equivalent symbol (hkil). In the transformation from hexagonal to Millerian symbols, we should first find the symbol $(h'k'l')$ corresponding to (hkil) from equations (11); and we should then transform from one set of Millerian axes to another.

Assume (efg) to become (100), then (gef) becomes (010), and (fge) (001); the parametral plane (111) remaining (111). Then $[u_1v_1w_1]$, &c., of Chap. VIII, Art. 20, are:

$$[u_1v_1w_1] = [e^2 - fg,\ f^2 - ge,\ g^2 - ef] = \text{new } [100];$$
$$[u_2v_2w_2] = [g^2 - ef,\ e^2 - fg,\ f^2 - ge] = \;\;,, \;\; [010];$$
$$[u_3v_3w_3] = [f^2 - ge,\ g^2 - ef,\ e^2 - fg] = \;\;,, \;\; [001].$$

Hence by equations (31) of Chap. VIII,

$$\left.\begin{aligned} h' &= \frac{h(e^2-fg)+k(f^2-ge)+l(g^2-ef)}{e^2+f^2+g^2-fg-ge-ef}; \\ k' &= \frac{h(g^2-ef)+k(e^2-fg)+l(f^2-ge)}{e^2+f^2+g^2-fg-ge-ef}; \\ l' &= \frac{h(f^2-ge)+k(g^2-ef)+l(e^2-fg)}{e^2+f^2+g^2-fg-ge-ef}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (20).$$

The values of h', k', l' obtained from equations (20) are now used in equations (9) instead of h, k, l; and the equivalent symbol (hkil) is determined.

Thus, taking the case of apatite in which Miller's (210) is Dana's ($01\bar{1}1$), we first change the Millerian axes, (210) becoming (100). Hence in equations (20), $e=2$, $f=1$, $g=0$; and we have:

$$\left.\begin{aligned} 3h' &= 4h+k-2l; \\ 3k' &= -2h+4k+l; \\ 3l' &= h-2k+4l. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21).$$

The equivalent symbols for the faces given in the first and fourth columns of the following table are now determined and are those occupying the third and sixth columns.

	Miller's.	New.		Miller's.	New.
x	210	100	$x_{,}$	012	$\bar{1}22$
r	321	411	$r_{,}$	123	011
y	$31\bar{1}$	$5\bar{1}\bar{1}$	$y_{,}$	$\bar{1}13$	$\bar{1}11$
v	411	$5\bar{1}2$	v	011	$\bar{1}52$
$s_{,}$	100	$4\bar{2}1$	s	$\bar{1}22$	$\bar{2}41$
u	$32\bar{2}$	$20\bar{1}$	$u_{,}$	$\bar{1}04$	$\bar{4}25$
n	$42\bar{3}$	$8\bar{1}\bar{4}$	$n_{,}$	$05\bar{2}$	$\bar{2}12$
m	$2\bar{1}\bar{1}$	$1\bar{1}0$			
a	$01\bar{1}$	$11\bar{2}$.			

The symbols of the faces being now known when the Millerian axes are parallel to the edges of intersection of alternate co-polar faces of the pyramid adopted by Professor Dana to be $\{01\bar{1}1\}$, we can apply equations (9) to find the hexagonal symbols. They are:

	New.	Dana's.		New.	Dana's.
x	100	$01\bar{1}1$	$x_{,}$	$\bar{1}22$	$0\bar{1}11$
r	411	$01\bar{1}2$	$r_{,}$	011	$0\bar{1}12$
y	$5\bar{1}\bar{1}$	$02\bar{2}1$	$y_{,}$	$\bar{1}11$	$0\bar{2}21$
v	$5\bar{1}2$	$11\bar{2}2$	v	$\bar{1}52$	$\bar{1}\bar{1}22$
$s_{,}$	$4\bar{2}1$	$11\bar{2}1$	s	$\bar{2}41$	$\bar{1}\bar{1}21$
u	$20\bar{1}$	$\bar{1}3\bar{2}1$	$u_{,}$	$\bar{4}25$	$1\bar{3}21$
n	$8\bar{1}\bar{4}$	$\bar{1}4\bar{3}1$	$n_{,}$	$05\bar{2}$	$1\bar{4}31$
m	$1\bar{1}0$	$11\bar{2}0$			
a	$11\bar{2}$	$\bar{1}100$.			

ii. In practice, time is saved by introducing the values of h', k', l' given by such equations as (20) and (21) into equations (9); and afterwards introducing the particular values of h, k, l corresponding to each form. The general formulæ, though cumbrous, are very symmetrical. Thus, since h', k', l' occur in the first degree in each denominator of (9), the denominators of the values given in (20) may be cancelled. Taking now the first term in (9), we have

$$\frac{\mathbf{h}}{h(f^2-g^2+ef-ge)+k(g^2-e^2+fg-ef)+l(e^2-f^2+ge-fg)}$$
$$=\frac{\mathbf{h}}{(e+f+g)\{h(f-g)+k(g-e)+l(e-f)\}}.$$

The second and third terms of (9) are, from the symmetry, clearly similar; and the last term is

$$\frac{\mathbf{l}}{(h+k+l)(e^2+f^2+g^2-fg-ge-ef)}$$
$$=\frac{2\times\mathbf{l}}{(h+k+l)\{(e-f)^2+(f-g)^2+(g-e)^2\}}.$$

The final equations are therefore

$$\frac{\mathbf{h}}{h(f-g)+k(g-e)+l(e-f)}=\frac{\mathbf{k}}{h(e-f)+k(f-g)+l(g-e)}$$
$$=\frac{\mathbf{i}}{h(g-e)+k(e-f)+l(f-g)}$$
$$=\frac{2(e+f+g)\mathbf{l}}{(h+k+l)\{(e-f)^2+(f-g)^2+(g-e)^2\}}\quad\ldots\ldots\ldots(22).$$

In the particular instance of apatite, in which $e=2$, $f=1$ and $g=0$, we have

$$\frac{\mathbf{h}}{h-2k+l}=\frac{\mathbf{k}}{h+k-2l}=\frac{\mathbf{i}}{-2h+k+l}=\frac{\mathbf{l}}{h+k+l}\quad\ldots\ldots(23).$$

Equations (23) serve *generally* to transform from Miller's symbols to Dana's when Miller's (210) is the latter's (01$\bar{1}$1).

I. *Acleistous hexagonal class*; $\tau\{hkl,\ pqr\} = \tau\{\mathbf{hkil}\}$.

15. In this class the hexad axis is the only element of symmetry present; and every form, with the exception of the pedions, has six like and interchangeable faces. When the faces of the form are parallel to the hexad axis, they form a hexagonal prism; and when they are inclined to the axis, they form an acleistous hexagonal pyramid, Fig. 392. As stated in the preceding

Articles the alternate faces of one of the possible pyramids are selected to give the Millerian axes, and the faces have the symbols (100), (010), (001); the three remaining faces being $(\bar{1}22)$, $(2\bar{1}2)$, $(22\bar{1})$: the symbol of the fundamental pyramid is $\tau\{100, \bar{1}22\}$.

The pedion, having its face perpendicular to the hexad axis, is (111) or $(\bar{1}\bar{1}\bar{1})$, according as it meets the hexad axis on the same side of the origin as the apex V of the pyramid $\tau\{100, \bar{1}22\}$, or on the opposite side.

In hexagonal notation the fundamental pyramid has the symbol $\tau\{01\bar{1}1\}$; and the face $(01\bar{1}1)$ meets the axis OY at distance a from the origin, and the axis OZ at distance c; where $c = a \cos 30° \tan D$. The equivalent symbols of the six faces of the pyramid in both notations are given in table **b**. The pedions are (0001) and $(000\bar{1})$ respectively.

16. The hexagonal prisms have the symbols: $\{2\bar{1}\bar{1}\} = \{01\bar{1}0\}$ (table **c**), $\{1\bar{1}0\} = \{11\bar{2}0\}$ (table **d**); and $\tau\{hkl\} = \tau\{\mathrm{hki0}\}$, in which $h + k + l = 0$. In the first two prisms the Greek prefix is omitted, for they are common to all classes of the system. The prism $\tau\{hkl\} = \tau\{\mathrm{hki0}\}$ has the following faces:

$$\left.\begin{array}{cccccc} hkl & \bar{k}\bar{l}\bar{h} & lhk & \bar{h}\bar{k}\bar{l} & klh & \bar{l}\bar{h}\bar{k} \\ \mathrm{hki0} & \mathrm{\bar{k}\bar{i}\bar{h}0} & \mathrm{ihk0} & \mathrm{\bar{h}\bar{k}\bar{i}0} & \mathrm{kih0} & \mathrm{\bar{i}\bar{h}\bar{k}0} \end{array}\right\} \quad \ldots\ldots\ldots\ldots(\mathbf{g}).$$

There is, however, no essential difference between the three prisms, for the symbols depend on the pyramid selected as the fundamental one. The ratios of the indices of the prism given by (**g**) can be determined by equation (18).

17. The acleistous hexagonal pyramid $\tau\{hkl, pqr\} = \tau\{\mathrm{hkil}\}$ has six faces, the symbols of which are given in table **e**.

When however the pyramid makes equal intercepts on two of the equatorial axes, we have two cases, according as one of the faces—supposed to be shifted parallel to itself so as to pass through A'—meets (i) the adjacent axis OU at $A_{\prime\prime}$ or (ii) OX at A.

i. In this case the trace in the equatorial plane is parallel to OX, and the first index is zero: the corresponding face is therefore $(\mathrm{0h\bar{h}l})$, and is tautozonal with pairs of opposite faces of the fundamental pyramid; and its Millerian symbol is (hll). The pyramid has therefore the equivalent symbols $\tau\{hll, prr\} = \tau\{\mathrm{0h\bar{h}l}\}$; and includes the faces:

$$\left.\begin{array}{cccccc} hll & rrp & lhl & prr & llh & rpr \\ \mathrm{0h\bar{h}l} & \mathrm{\bar{h}h0l} & \mathrm{\bar{h}0hl} & \mathrm{0\bar{h}hl} & \mathrm{h\bar{h}0l} & \mathrm{h0\bar{h}l} \end{array}\right\} \quad \ldots\ldots(\mathbf{h}).$$

ii. In the second case the equatorial trace is perpendicular to OU and bisects $OA_{\prime\prime}$: the corresponding face has the symbol $(h, h, 2\bar{h}, l)$, and is tautozonal with $(11\bar{2}0)$ and (0001). Since $(11\bar{2}0)$ is $(1\bar{1}0)$ and (0001) is (111), the face has the symbol (hlk), where $h + l - 2k = 0$. Here (lhk) is the opposite face (prq) of the general case; and if the indices h, k, l are taken in the two cyclical orders, the form may be represented by a symbol containing only the indices of a single face. For the sake of uniformity, we shall however denote the form by $\tau\{hkl, lkh\}$; for the general rule of the class does not admit of opposite cyclical orders.

The pyramid $\tau\{hkl, lkh\} = \tau\{\bar{h}, 2h, \bar{h}, l\}$ has the faces:

$$\left.\begin{array}{cccccc} hkl & khl & lhk & lkh & klh & hlk \\ \bar{h}, 2h, \bar{h}, l & 2h, h, h, l & \bar{h}, \bar{h}, 2h, l & h, 2\bar{h}, h, l & 2h, \bar{h}, \bar{h}, l & h, h, 2\bar{h}, l \end{array}\right\} \ldots (\mathrm{J}).$$

The pyramids of the two series $\tau\{0h\bar{h}l\}$ and $\tau\{\bar{h}, 2h, \bar{h}, l\}$ are not in this class special forms; the exceptional relation between the indices is due to their horizontal traces being respectively parallel and perpendicular to the edges which have been selected arbitrarily to give the directions of the equatorial axes. Further, the angles of a pyramid of one series can never be equal to those of a pyramid of any other of the possible series. If, for instance, the face $(h, h, 2\bar{h}, l)$ were inclined to the pedion (0001) at the same angle as a face $(0h_{\prime}\bar{h}_{\prime}l_{\prime})$, it would be the same as if the latter face were turned through 30° about the hexad axis. Assuming the two faces to pass through a common apex and A', the face $(0h_{\prime}\bar{h}_{\prime}l_{\prime})$ would after rotation meet OB of Fig. 392 at a distance equal to OA, and $OB \div OA$ would, by the law of rational indices, be commensurable. But $OB = OA' \cos 30°$, and $OB \div OA$ is incommensurable. The two faces cannot therefore have the same inclination to the equatorial plane or pedion. Since the inclinations to the pedion are different, the angles over the polar edges of a pyramid of one series must be different from those of any pyramid of the other series.

A similar proof can be given in the case of a face [illegible], only in this case the angle of supposed rotation would be [illegible]

18. The hexad axis is one of [illegible] symmetry and should be a pyroelectric axis.

The crystals may [illegible]

case, the crystals may be expected to rotate the plane of polarization of light transmitted along the hexad axis.

19. The crystals of lithium potassium sulphate, $LiKSO_4$, of $Li(NH_4)SO_4$, of $LiRbSO_4$ and of $LiKSeO_4$ are placed in this class; for the above sulphates rotate the plane of polarization, and the seleniate shows on the pyramid-faces the same kind of unsymmetrical corrosion-figures as the sulphates.

Lithium potassium sulphate. $D = 62° 39'$; $c = 1·6743$. The crystals are easily obtained by the evaporation of a solution containing the two sulphates, and occur as apparently simple hexagonal prisms terminated by both pedions and by similarly developed hexagonal pyramids at both ends. The habit of the crystals is therefore similar to that of the crystals of apatite and mimetite shown in Fig. 405. But the crystals are twins, occasionally of a simple character, but frequently of great complexity (Dr H. Traube. *N. Jahrb. f. Min.* II, 1892, p. 58; I, 1894, p. 171). The opposite ends of the crystals are often positively electrified with falling temperature, whilst a broad central zone is negatively electrified. Such crystals are twins joined along the face $(\bar{1}\bar{1}\bar{1})$, which is at the analogous pole of the two components. The latter moreover rotate the plane of polarization in opposite directions; and occasionally Airy's spirals have been seen in a plate cut from such a crystal. Other apparently simple crystals, showing opposite electrifications at the two ends, have been shown to be complex twins. The amount of rotation for a plate 1 mm. thick is given by Traube as 3·44°, but this may be too low; for it is possible that no plate has been obtained free from twin-lamellæ of opposite rotations.

The crystals of nepheline, $Na_8Al_8Si_9O_{34}$, of strontium- and lead-antimonyl dextro-tartrates are also placed in this class; for the corrosion-figures on the prism-faces are unsymmetrical trapezia, but those on adjacent faces are congruent when the crystals are turned through 60° about the principal axis. The tartrates are also pyro-electric; but neither their crystals nor those of nepheline show any rotation of the plane of polarization.

Strontium-antimonyl dextro-tartrate, $Sr_2(SbO)_2(C_4H_4O_6)_2$. A crystal after Traube (*N. Jahrb. f. Min.* Beil. Bd. VIII, 1893, p. 270) is shown in Fig. 403. $D = 44° 16·5'$, $c = ·8442$. The forms shown are: $m\{2\bar{1}\bar{1}\} = \{01\bar{1}0\}$, $o = \tau\{100, \bar{1}22\} = \tau\{01\bar{1}1\}$, $x = \tau\{1\bar{1}\bar{1}, \bar{5}11\} = \{02\bar{2}\bar{1}\}$. The measured angle $xx = 1\bar{1}\bar{1} \wedge 1\bar{5}1 = 52° 50'$ was used to determine the element. If we denote by x' the pole $5\bar{1}\bar{1}$, by x'' the pole $(1\bar{1}1)$ and by C the pole $(111) = (0001)$, we have, from the isosceles spherical triangle $Cx'x''$ and the A. R. $\{Cox'm\}$, the following equations:

$$\sin Cx' = \sin(x'x'' \div 2 = 26° 25') \div \sin 30°;\ \therefore\ Cx' = 62° 51';$$
$$\tan D = \tan(Cx' = 62° 51') \div 2;\ \therefore\ D = 44° 16·5';$$
$$c = \cos 30° \tan(D = 44° 16·5') = ·8442.$$

The obtusely terminated end at which o is developed is the antilogous pole, that at which x appears is the analogous pole. The corrosion-figures on two of the adjacent prism-faces are shown in Fig. 403. They are four-sided pits, having two sides parallel to the edges $[mo]$, but otherwise unsymmetrical; and those on adjacent faces are congruent when the crystals are turned through 60°. No evidence of twinning similar to that in lithium potassium sulphate was observed in the pyro-electric characters or in the arrangement of the corrosion-figures on the prism-faces.

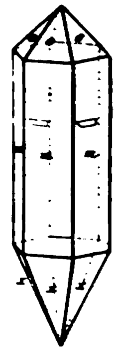


FIG. 403.

A crystal of *lead-antimonyl dextro-tartrate* after Traube (*loc. cit.*) is shown in Fig. 404. $D = 44° 9'$, $c = \cdot 84064$. The forms are:

$$m = \{2\bar{1}0\} = \{01\bar{1}0\},\ o = \tau\{100, \bar{1}22\} = \tau\{01\bar{1}1\},$$
$$x = \tau\{1\bar{1}\bar{1}, \bar{5}11\} = \tau\{02\bar{2}\bar{1}\}.$$

Traube gives as measured angles those in column (2) of the following table of angles, and computes the element and the angles in column (3) from the measured angle $xx = 1\bar{1}\bar{1} \wedge 1\bar{5}1 = 52° 47'$. Assuming $ox = 100 \wedge 1\bar{1}\bar{1} = 72° 51'$—the sum of the last two measured angles of column (2)—to be as trustworthy as xx, we may find the angles in column (4) by equation (21) of Chap. VIII, Art. 18. For in the A.R. $\{Comx\}$ we know the angles $Cm = 90°$ and ox; and the transformation needed to find om or Co falls under case (e) of that Chapter. Hence equation (21) of p. 102 becomes

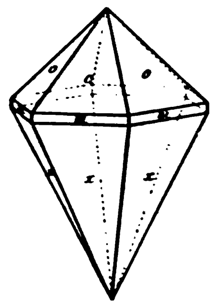


FIG. 404.

$$\mathsf{m} \cos(90° + ox - 2om)$$
$$= (\mathsf{m} - 1)\cos(90° - ox) + \cos(90° + ox),$$
$$\therefore\ \mathsf{m} \sin(2om - ox) = (\mathsf{m} - 2)\sin ox.$$

$$\text{Also}\quad 1 \div \mathsf{m} = \text{A. R. } \{Comx\} = \begin{vmatrix} 111 \\ 100 \\ \bar{1}\bar{1}1 \\ 2\bar{1}\bar{1} \end{vmatrix} \div \begin{vmatrix} 1\bar{1}\bar{1} \\ 100 \\ \bar{1}\bar{1}1 \\ 2\bar{1}\bar{1} \end{vmatrix} = 1 \div 3,$$

$$\therefore\ \mathsf{m} = 3.$$

$$\therefore\ \sin(2om - ox) = \sin 72° 51' \div 3,$$

$$\therefore\ 2om - 72° 51' = 18° 34\cdot4', \text{ and } om = 45° 42\cdot7'.$$

(1)	(2)	(3)	(4)
$1\bar{1}\bar{1} \wedge 1\bar{5}1$	*52° 47′		52° 50·4′
$100 \wedge 2\bar{1}2$	40 49	40° 46′	40 52
$100 \wedge 2\bar{1}\bar{1}$	45 41	45 51	45 42·7
$2\bar{1}\bar{1} \wedge 1\bar{1}\bar{1}$	27 10	27 15	27 8·3
$111 \wedge 100\ (=D)$		44 9	44 17·3

The value of c corresponding to D of column (4) is ·8448. The method of computation by equation (21) of Chap. VIII can also be applied to the strontium salt, and gives results in close accordance with the measured angles.

The pyro-electric and optical characters, and also the corrosion-figures, are similar to those of crystals of the isomorphous strontium salt.

Pyramidal Hemihedral

II. *Diplohedral hexagonal class;* $\pi\{hkl, pqr\} = \pi\{\mathrm{hkil}\}$.

20. If the hexad axis is associated with a centre of symmetry, there will also be a plane of symmetry Π perpendicular to the axis (Chap. IX, Prop. 4). The hexagonal pyramid of the last class will then be associated with a like one, having each of its faces parallel to a face of the first pyramid. The bipyramid $\pi\{hkl, pqr\} = \pi(\mathrm{hkil}\}$ consists therefore of the following faces:

$$\left.\begin{array}{cccccc} hkl & qrp & lhk & pqr & klh & rpq \\ \mathrm{hkil} & \mathrm{\bar{k}\bar{i}\bar{h}l} & \mathrm{ihkl} & \mathrm{\bar{h}\bar{k}\bar{i}l} & \mathrm{kihl} & \mathrm{\bar{i}\bar{h}\bar{k}l} \\ \bar{p}\bar{q}\bar{r} & \bar{k}\bar{l}\bar{h} & \bar{r}\bar{p}\bar{q} & \bar{h}\bar{k}\bar{l} & \bar{q}\bar{r}\bar{p} & \bar{l}\bar{h}\bar{k} \\ \mathrm{hki\bar{l}} & \mathrm{\bar{k}\bar{i}\bar{h}\bar{l}} & \mathrm{ihk\bar{l}} & \mathrm{\bar{h}\bar{k}\bar{i}\bar{l}} & \mathrm{kih\bar{l}} & \mathrm{\bar{i}\bar{h}\bar{k}\bar{l}} \end{array}\right\}\ldots\ldots\{\mathbf{k}\}.$$

In the above table the two faces in each column are symmetrically placed with respect to the equatorial plane Π; and the parallel faces are derived the one from the other by changing the signs of all the indices.

The particular case $\{100, \bar{1}22\} = \{01\bar{1}1\}$, which may be represented by Fig. 393, differs in no essential respect from $\pi\{\mathrm{hkil}\}$. The particular values of the indices in the two notations result from the arbitrary selection of this pyramid for the fundamental one. The element is obtained from this pyramid in the manner already given.

Similarly, the two series of hexagonal bipyramids $\{0\mathrm{h}\bar{\mathrm{h}}\mathrm{l}\}$ and $\{\bar{\mathrm{h}}, 2\mathrm{h}, \bar{\mathrm{h}}, \mathrm{l}\}$, which can be derived by the addition of parallel faces to those given in tables **h** and **j**, are not special forms. The symbols of the new faces, being obtained by changing the signs of all the indices of each face in these tables, need not be given here.

The Greek prefix is omitted in these cases, for the pyramids are geometrically identical with similar pyramids of class IV.

21. The hexagonal prisms of the last class belong also to this, so that there is no geometrical distinction between $\pi\{\mathrm{hki0}\}$ and $\tau\{\mathrm{hki0}\}$.

The pinakoid $\{111\} = \{0001\}$ has its faces perpendicular to the axis and parallel to the plane of symmetry: it consists of the two faces, 0001 and $000\bar{1}$.

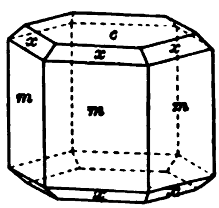


Fig. 405.

22. *Apatite*, $3Ca_3P_2O_8 \cdot Ca(Cl, F)_2$, affords a good illustration of forms of this class. In the crystal shown in Fig. 405 the forms are: $c\ \{0001\} = \{111\}$, $m\ \{01\bar{1}0\} = \{2\bar{1}\bar{1}\}$, and $x\ \{01\bar{1}1\} = \{100, \bar{1}22\}$. The relative magnitudes of the faces of the several forms vary greatly; the habit being sometimes prismatic

owing to the predominance of the prism-faces, and sometimes tabular through the predominance of the pinakoid.

A crystal having several additional forms is represented by Figs. 406—8. It can be completely determined by the observation of zones and by measurement of the following angles in the three zones $[m''am_,]$, $[crxym]$, $[mnusx_,]$:

$$\text{i.}\ \left[\begin{array}{lrr} m_{,,}a & 30^\circ & 0' \\ am_, & 30 & 0 \end{array}\right] \qquad \text{ii.}\ \left[\begin{array}{lrr} cr & 22^\circ & 59' \\ cx & {}^*40 & 18\cdot\dot{3} \\ cy & 59 & 29 \\ cm & 90 & 0 \end{array}\right] \qquad \text{iii.}\ \left[\begin{array}{lrr} mn & 22^\circ & 41' \\ mu & 30 & 20 \\ ms & 44 & 17 \\ mx_, & 71 & 8 \end{array}\right]$$

i. The angles in the zone $[ma]$ are fixed, as was shown in Chap. XVI, Art. 27.

Take m to be $(2\bar{1}\bar{1})=(01\bar{1}0)$, and x to be $(100)=(01\bar{1}1)$, then, by (1) of Art. 3, the parameter

$$c=\cos 30^\circ \tan 40^\circ\ 18\cdot\dot{3}'=\cdot 7346.$$

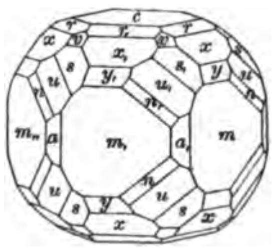


FIG. 406.

ii. Again, let p be any face $(0\mathrm{h}\bar{\mathrm{h}}\mathrm{l})$ in the zone $[cxm]$; then, from $A.R.\ \{cxpm\}$,

$$\tan cp \div \tan D=\begin{vmatrix} 0001 \\ 0\mathrm{h}\bar{\mathrm{h}}\mathrm{l} \\ \hline 0001 \\ 01\bar{1}1 \end{vmatrix} \div \begin{vmatrix} 01\bar{1}0 \\ 0\mathrm{h}\bar{\mathrm{h}}\mathrm{l} \\ \hline 01\bar{1}0 \\ 01\bar{1}1 \end{vmatrix} = \frac{\mathrm{h}}{\mathrm{l}} \div \frac{1}{1} = \frac{\mathrm{h}}{\mathrm{l}}.$$

Hence, when p coincides with r,

$$cp=cr=22^\circ\,59', \text{ and } \frac{\mathrm{h}}{\mathrm{l}}=\frac{1}{2};$$

$\therefore \mathrm{h}=1,\ \mathrm{l}=2.$ The face r is $(01\bar{1}2)$, and the form is $\{01\bar{1}2\}$ equivalent, from (11), to $\{411,011\}$.

Similarly, when p coincides with y, $cp=cy=59^\circ\,29'$, and $\mathrm{h}\div\mathrm{l}=2$; therefore y belongs to the form $\{02\bar{2}1\}=$ (from (11)) $\{5\bar{1}\bar{1},\bar{1}11\}$.

iii. In the remaining measured zone we know m to be $(01\bar{1}0)=(2\bar{1}\bar{1})$, the face $x_,$ on its right to be $(\bar{1}101)=(22\bar{1})$, and $m_{,,}$ in $[cx_,]$ to be $(\bar{1}100)=(11\bar{2})$.

The arcs joining the poles m, $x_,$, $m_{,,}$ of these faces, Fig. 407, form a right-angled triangle. Hence, by Napier's mnemonic,

$$\cos x_,mm_{,,}=\tan 60^\circ \cot 71^\circ\,8'.$$

Let us now determine the arc-distance from m of the pole σ in which the zones $[ca_,]=[111,\bar{1}01]$ and $[mx_,]=[2\bar{1}\bar{1},22\bar{1}]$ intersect. From Weiss's law we find the symbol of σ to be $(41\bar{2})$.

Since, moreover, the zone $[ca_,]$ meets $[mm_,]=[111]$ at a', where $ma'=30^\circ$, we have from the triangle $\sigma ma'$, $\cos\sigma ma'=\tan 30^\circ \cot m\sigma$.

But $\wedge\,\sigma ma'=\wedge\,x_,mm_{,,}$. $\therefore \tan 30^\circ \cot m\sigma=\tan 60^\circ \cot 71^\circ\,8'$; and

$$\cot m\sigma=\tan^2 60^\circ \cot 71^\circ\,8'.$$

$\therefore m\sigma=44^\circ\,17'$, and σ coincides with $s\,(41\bar{2})$. The faces s, $s_,$ belong therefore to the hexagonal bipyramid $\{41\bar{2},\bar{2}14\}=\{\bar{1}2\bar{1}1\}$.

Even if σ had not coincided with $s\,(\bar{1}2\bar{1}1)$, the labour of the computation would not have been entirely wasted; for it gives the angle which a third known pole in the zone makes with m and $x_,$. We shall use it to find the symbols of n and u from the A. R. $\{mPsx_,\}$, where P is any pole (hkil) in $[mx_,]$.

$$\therefore \text{A.R.}\{mPsx_{,}\}=\frac{\sin mP}{\sin ms}\div\frac{\sin x_{,}P}{\sin x_{,}s}=\frac{\begin{vmatrix}01\bar{1}0\\ hkil\end{vmatrix}}{\begin{vmatrix}01\bar{1}0\\ \bar{1}2\bar{1}1\end{vmatrix}}\div\frac{\begin{vmatrix}\bar{1}101\\ hkil\end{vmatrix}}{\begin{vmatrix}\bar{1}101\\ \bar{1}2\bar{1}1\end{vmatrix}}$$

$$=\text{(taking the middle pair) }\frac{k+i}{-i}=\text{(taking the last pair) }\frac{l}{-i}.$$

Hence, when P coincides with n, $mP=22^\circ\,41'$ and $x_{,}P=48^\circ\,27'$.

$$\therefore \frac{k+i}{-i}=\frac{l}{-i}=\frac{\sin 22^\circ\,41'\sin 26^\circ\,51}{\sin 44^\circ\,17'\sin 48^\circ\,27}=\frac{1}{3},$$

by computation.

$$\therefore\ 3k+4i=0,\ \text{and}\ 3l+i=0.$$

Now l is clearly positive, and i must therefore be negative. The two equations are satisfied by making $l=1$, $i=-3$ and $k=4$; and as these are the lowest whole numbers, they may be taken as the indices. The value of h is then found from equation (4) to be $\bar{1}$.

Hence n is $(\bar{1}4\bar{3}1)$; and the form has the symbol

$$\pi\{\bar{1}4\bar{3}1\}=\text{(from (11)) }\pi\{8\bar{1}\bar{4},\ \bar{2}12\}.$$

Similarly, when P is made to coincide with u, $mP=30^\circ\,20'$, and the

$$\text{A.R.}\{musx_{,}\}=1\div 2.$$

$$\therefore\ 2k+3i=0,\ 2l+i=0.$$

Hence u is $(\bar{1}3\bar{2}1)$, and the form is $\pi\{\bar{1}3\bar{2}1\}=\text{(from (11)) }\pi\{20\bar{1},\ \bar{4}25\}$.

iv. The faces v truncate the edges $[xx_{,}]$ of adjacent faces of the bipyramid x, and they lie in zones $[ca_{,}]$, &c. Hence by Weiss's zone-law, the symbol of v is $(52\bar{1})$, and the form is $\{\bar{1}2\bar{1}2\}=\{52\bar{1},\bar{1}25\}$.

The face $v(\bar{2}112)$ might be used with m and x' to give an A.R. from which the symbols of n and u can be determined; and, since $\wedge\, mv=90^\circ$, the computation would be simpler.

v. The stereograms, Figs. 407 and 408, give the positions of the poles of all the faces in Fig. 406 which lie above the equatorial plane; and the symbols in Millerian and hexagonal notations of many of them. The poles of the faces below the equatorial plane would be given by circlets surrounding each dot.

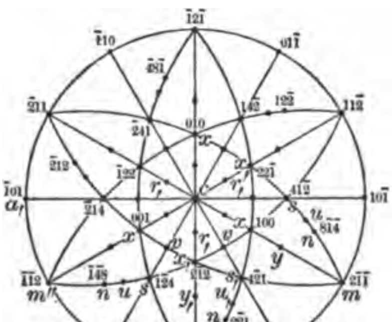


FIG. 407.

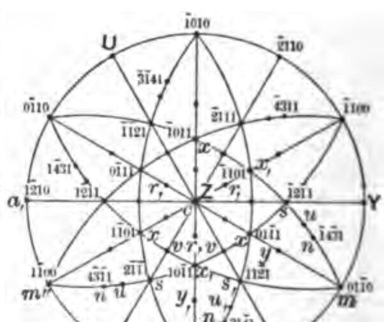


FIG. 408.

The figures are constructed by marking off arcs of 30° on the primitive, and the diameters through these points give the zones [cxm], [cvs,], &c. On [cm], the points r, x and y are determined by the construction of Chap. VII, Prob. 1. The zones [mns], &c., can then be quickly introduced, and the positions of all the poles found. The Millerian symbols given in this Article are not those given in Brooke and Miller's *Mineralogy*; for Miller adopted s as the fundamental pyramid {100, $\bar{1}$22}, and x is then {201}. The transformation from the symbols of Miller to those used in this Article has already been given in Art. 14.

Pyromorphite, $3Pb_3P_2O_8 . PbCl_2$, and *vanadinite*, $3Pb_3V_2O_8 . PbCl_2$, are both isomorphous with apatite; and generally occur in hexagonal prisms {$2\bar{1}\bar{1}$} terminated either by {111} alone or by {111} and x{100, $\bar{1}$22}. The crystals therefore resemble Figs. 328 and 405. They are optically uniaxal and negative.

Mimetite, $3Pb_3As_2O_8 . PbCl_2$, was long regarded as belonging to this class of crystals; but it has been found to be optically biaxal, and appears to be a mimetic twin (see Chap. XVIII). The crystals have the habits shown in Figs. 328 and 405; but some of the crystals from Wheal Alfred, Cornwall, in the Cambridge Museum, show only the prism {$2\bar{1}\bar{1}$} and the hexagonal bipyramid x{100, $\bar{1}$22}: measurement of these crystals gave for $2\bar{1}\bar{1} \wedge 1\bar{2}1$ values varying from 59° 55′ to 60° 4′, $100 \wedge \bar{1}22 = 80° 3′$, and $100 \wedge 2\bar{1}2 = 37° 30′$. Hence $D = 111 \wedge 100 = 40° 1·5′$.

III. *Acleistous dihexagonal class*; $\mu\{hkl, pqr\} = \mu\{hkil\}$. Hemimorphic

23. In crystals of this class the hexad axis is associated with six planes of symmetry intersecting in the axis at angles of 30°. It was shown in Chap. IX, Prop. 9, that, if a plane of symmetry is parallel to a hexad axis, there must be six planes of symmetry all parallel to the axis; and further, in Prop. 8, Cor. IV, it was seen that three planes Σ intersecting at angles of 60° are like and interchangeable, and that the angles between them are bisected by three other like and interchangeable planes S, also at 60° to one another. In Prop. 5 of the same Chapter it was proved that 30° is the least angle possible between any planes of symmetry: hence there can be no other planes of symmetry parallel to the axis. We shall suppose the Millerian axes to lie in the planes Σ, and the equatorial axes to be the intersections of the S planes with the equatorial plane.

The hexad axis is uniterminal, and should be a pyro-electric axis.

24. The special forms are:

1. Pedions having the symbols

$$\mu\{111\} = \mu\{0001\} \text{ and } \mu\{\bar{1}\bar{1}\bar{1}\} = \mu\{000\bar{1}\},$$

according as the base meets the hexad axis on the positive or negative side of the origin. These faces are perpendicular to all the planes of symmetry.

2. Hexagonal prisms:

i. $\{2\bar{1}\bar{1}\} = \{01\bar{1}0\}$, the faces of which are perpendicular each to one of the planes Σ. The planes S pass through pairs of opposite vertical edges, and are parallel each to a pair of the faces. The faces may be supposed to pass through the edges $AA_{,,}$, &c., of the hexagonal base in Fig. 409.

ii. $\{10\bar{1}\} = \{\bar{1}2\bar{1}0\}$, having its faces parallel to the Σ planes, and perpendicular each to an S plane. The faces of the one prism truncate the edges of the other.

3. A dihexagonal prism $\{\mathrm{hki0}\} = \{hkl\}$, where $h + k + l = 0$. This prism is geometrically identical with that described in Chap. XVI, Art. 27; and the angles are for any specified values of the indices found by equation (20) of that Article. The symbols of the faces are:

$$\left.\begin{array}{cccccc} hkl & \bar{l}\bar{k}\bar{h} & \bar{k}\bar{l}\bar{h} & khl & lhk & \bar{h}\bar{l}\bar{k} \\ \mathrm{hki0} & \mathrm{ikh0} & \mathrm{\bar{k}\bar{i}\bar{h}0} & \mathrm{\bar{k}\bar{h}\bar{i}0} & \mathrm{ihk0} & \mathrm{hik0} \\ \bar{h}\bar{k}\bar{l} & lkh & klh & \bar{k}\bar{h}\bar{l} & \bar{l}\bar{h}\bar{k} & hlk \\ \mathrm{\bar{h}\bar{k}\bar{i}0} & \mathrm{\bar{i}\bar{k}\bar{h}0} & \mathrm{kih0} & \mathrm{khi0} & \mathrm{\bar{i}\bar{h}\bar{k}0} & \mathrm{\bar{h}\bar{i}\bar{k}0} \end{array}\right\} \cdots\cdots\ (\mathrm{m}).$$

4. Hexagonal pyramids, one of which (i) has its faces perpendicular to the Σ planes and may be represented by Fig. 409; the other (ii) has its faces perpendicular to the S planes.

i. Pairs of the traces in the equatorial plane are parallel to one or other of the equatorial axes; and the pyramid has the symbol $\mu\{\mathrm{0h\bar{h}l}\} = \mu\{hll,\ prr\}$. The faces may be supposed to pass through the lines $AA_{,,}$, &c., of Fig. 409; and to meet the principal axis at V^m, where $OV^m = \mathrm{hc} \div \mathrm{l} = mc$ of Chap. XVI, Art. 35. The symbols of the faces are given in table **h**.

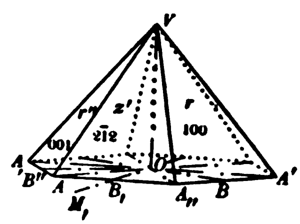


FIG. 409.

One of the possible pyramids of this series is selected as fundamental pyramid $\mu\{01\bar{1}1\}$.

ii. One of the faces of a pyramid of this second series may be drawn through the points A, A' to meet the principal axis at V, where $OV = \mathrm{hc} \div \mathrm{l}$. The trace AA' bisects $OA_{,,}$ at right angles, and the symbol of the face is $(\mathrm{h, h, 2\bar{h}, l})$. The faces of $\mu\{\mathrm{\bar{h}, 2h, \bar{h}, l}\}$ are given in table **j**.

The two series of hexagonal pyramids are geometrically the same as those described in Art. 17, where they were however only particular cases of the general form: in this class they are special forms, and their faces are perpendicular each to one of the planes of symmetry. They have the same general aspect, but the angles over the polar edges of a pyramid of one series are not equal to those of a pyramid of the other series, and the inclinations of the faces to the pedion are also different.

25. The general form consists of an acleistous dihexagonal pyramid, the faces of which all meet the hexad axis at the same point. The form may be supposed to consist of the twelve faces of Fig. 416, p. 456, which meet either at V or at $V_{\prime}$; it is geometrically similar to the figure produced by a combination of the two dirhombohedral pyramids $\mu\{hkl\}$ and $\mu\{pqr\}$ of class V of the rhombohedral

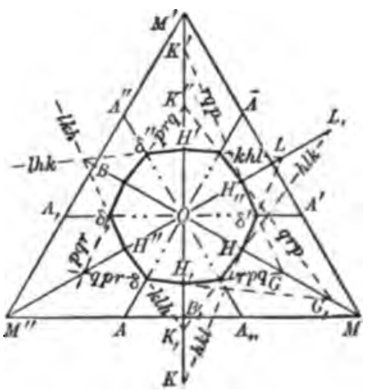


FIG. 410.

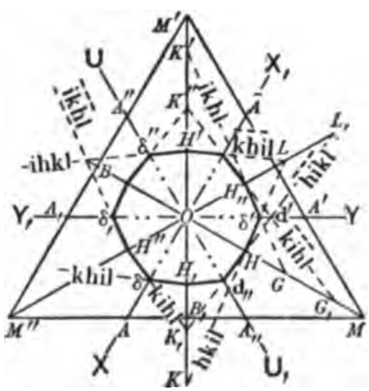


FIG. 411.

system. Hence, the Millerian symbol of the form is $\mu\{hkl, pqr\}$, where $p = 2\theta - 3h$, $q = 2\theta - 3k$, $r = 2\theta - 3l$. The equivalent hexagonal symbol $\mu\{\mathrm{hkil}\}$ is found by equations (9); or can be deduced independently from the geometry of the pyramid: the relation between the symbols of a pair of faces symmetrical with respect to a plane through the hexad axis was given in Art. 9. Hence the form $\mu\{hkl, pqr\} = \mu\{\mathrm{hkil}\}$ consists of the following faces:

$$\left.\begin{matrix} hkl & rqp & qrp & khl & lhk & prq \\ \mathrm{hkil} & \mathrm{ikhl} & \mathrm{\bar{k}i\bar{h}l} & \mathrm{\bar{k}\bar{h}il} & \mathrm{ihkl} & \mathrm{hikl} \\ pqr & lkh & klh & qpr & rpq & hlk \\ \mathrm{\bar{h}\bar{k}\bar{i}l} & \mathrm{\bar{i}\bar{k}\bar{h}l} & \mathrm{kihl} & \mathrm{khil} & \mathrm{\bar{i}\bar{h}\bar{k}l} & \mathrm{\bar{h}\bar{i}\bar{k}l} \end{matrix}\right\} \;\cdots\cdots\cdots\; (\mathrm{n}).$$

The arrangement of the faces and poles is indicated in Figs. 410–413. In the former pair the traces in the equatorial plane of most of the faces are labelled, and in the latter the symbols of the poles are given in both notations. The angles over adjacent polar edges, lying respectively in a Σ and S plane, are always different; for, were they to become equal, the lengths $O\mathfrak{S}$, OH would be equal, and therefore equal parameters might be taken on both OM and OA. But $OB = OA \cos 30°$, and OH is a commensurable submultiple of OB, and $O\mathfrak{S}$ of OA: the assumption therefore contravenes the law of rational indices.

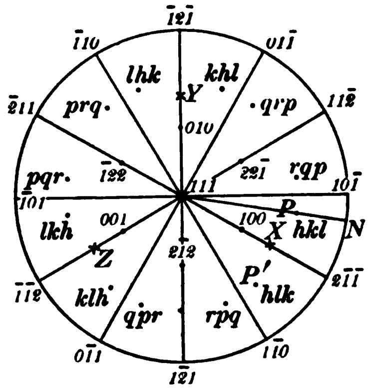


Fig. 412.

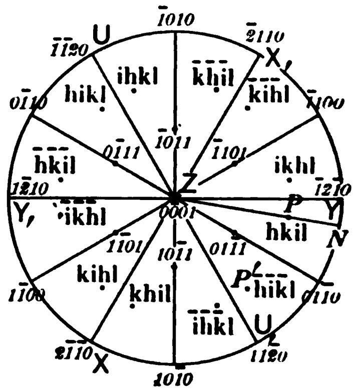


Fig. 413.

26. Silver iodide (*iodyrite*), AgI, is the only substance which has been placed in this class, and even for this the data are inadequate; for no

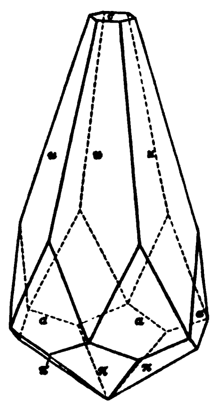


Fig. 414.

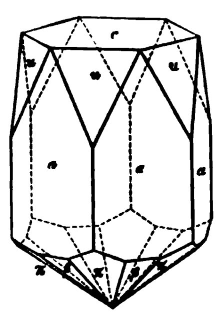


Fig. 415.

general forms have been observed, and no corrosion-figures obtained. The crystals may belong to class I of this system; or may possibly have to be placed in class V of the rhombohedral system, as has been done in the case of greenockite, *CdS*, which was also at one time regarded as belonging to this class. Figs. 414 and 415 show two crystals (after von Zepharovich, *Zeitsch. f. Kryst. u. Min.* IV, p. 119, 1879). The element $D=43^\circ\,25{\cdot}3'$, $c={\cdot}8196$. The forms are: $a\,\{01\bar{1}\}=\{\bar{1}2\bar{1}0\}$, $c=\mu\{111\}\doteq\mu\{0001\}$, $u=\mu\{3\bar{1}\bar{1},\bar{7}55\}=\mu\{04\bar{4}1\}$, $\pi=\mu\,\{1\bar{3}\bar{3};\ \bar{1}\bar{3},\bar{1},\bar{1}\}=\mu\,\{04\bar{4}5\}$, $\beta=\mu\,\{7,\overline{20},\overline{47};\ \overline{47},\overline{20},7\}=\mu\,\{\bar{9},18,\bar{9},\overline{20}\}$.

Holohedral IV. *Diplohedral dihexagonal class;* $\{hkl,\ pqr\}=\{\mathrm{hkil}\}$.

27. When a centre of symmetry is associated with the planes of symmetry of the last class, there must be a dyad axis perpendicular to each of the planes Σ and S, and a plane of symmetry Π perpendicular to the hexad axis. The crystals of this class have therefore the following elements of symmetry:

$$H,\ 3\Sigma,\ 3S,\ \Pi,\ C,\ 3\delta,\ 3\Delta.$$

It is clear that the dyad axes δ perpendicular each to a Σ plane must be like, for they are interchangeable by rotations of 60° about the hexad axis: each of them is also the intersection of the plane Π with one of the S planes. Similarly, the Δ axes are like and interchangeable, and are the lines of intersection of Π with each of the Σ planes: further, each of them is perpendicular to a plane S, and bisects the angle between pairs of the S planes and of the δ axes.

No other element of symmetry can be added to the assemblage given in the above table; for we cannot have more than six tautozonal planes of symmetry (Chap. IX, Prop. 5), and the axes of symmetry must be dyad axes all perpendicular to H, or we should have more than one hexad axis. The crystals have the greatest symmetry possible in the hexagonal system, and have therefore been described as holohedral. The general form is a dihexagonal bipyramid, Fig. 416, and the class will be denoted as the *diplohedral dihexagonal* class.

28. The special forms of the last class which have parallel faces are common also to this: they are the hexagonal prisms $\{01\bar{1}\}=\{\bar{1}2\bar{1}0\}$ and $\{21\bar{1}\}=\{01\bar{1}0\}$, and the dihexagonal prism $\{hkl\}=\{\mathrm{hki0}\}$.

The pedion becomes a pinakoid $\{111\}=\{0001\}$.

All the hexagonal pyramids become diplohedral; and therefore

$\{hll,\ prr\} = \{0h\bar{h}l\}$ is a hexagonal bipyramid similar geometrically to that of class II: the faces are now perpendicular to the planes of symmetry Σ, and the edges lie in the planes S and Π. Again, the hexagonal bipyramid $\{\bar{h}, 2h, \bar{h}, l\} = \{hkl\}$, where $h - 2k + l = 0$ is geometrically the same as $\{hkl\}$ of Chap. XVI, Art. 53, and also as the similarly placed pyramid of class II of this system. The faces are perpendicular to the planes S, and the edges lie in Σ and Π.

29. The dihexagonal bipyramid $\{hkl,\ pqr\} = \{hkil\}$, Fig. 416, has twenty-four faces, which are equal and similar scalene triangles. Since the edges of each face lie in dissimilar planes of symmetry, the angles over the three edges are unequal, although a case may possibly occur in which the angle over the median edge is for some particular temperature equal to that over one of the polar edges. But even in such an exceptional case a change of temperature would disturb the equality, for the coefficient of thermal expansion along the principal axis is different from that along a line at right angles to it. The faces of the bipyramid have the following symbols:

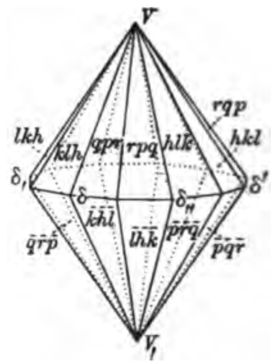


Fig. 416.

hkl	rqp	qrp	khl	lhk	prq	pqr	lkh	klh	qpr	rpq	hlk	
hkil	ikhl	$\bar{k}\bar{i}\bar{h}l$	$\bar{k}\bar{h}\bar{i}l$	ihkl	hikl	$\bar{h}\bar{k}\bar{i}l$	$\bar{i}\bar{k}\bar{h}l$	kihl	khil	$\bar{i}\bar{h}\bar{k}l$	$\bar{h}\bar{i}\bar{k}l$	(p).
$\bar{p}\bar{q}\bar{r}$	$\bar{l}\bar{k}\bar{h}$	$\bar{k}\bar{l}\bar{h}$	$\bar{q}\bar{p}\bar{r}$	$\bar{r}\bar{p}\bar{q}$	$\bar{h}\bar{l}\bar{k}$	$\bar{h}\bar{k}\bar{l}$	$\bar{r}\bar{q}\bar{p}$	$\bar{q}\bar{r}\bar{p}$	$\bar{k}\bar{h}\bar{l}$	$\bar{l}\bar{h}\bar{k}$	$\bar{p}\bar{r}\bar{q}$	
$hki\bar{l}$	$ikh\bar{l}$	$\bar{k}\bar{i}\bar{h}\bar{l}$	$\bar{k}\bar{h}\bar{i}\bar{l}$	$ihk\bar{l}$	$hik\bar{l}$	$\bar{h}\bar{k}\bar{i}\bar{l}$	$\bar{i}\bar{k}\bar{h}\bar{l}$	$kih\bar{l}$	$khi\bar{l}$	$\bar{i}\bar{h}\bar{k}\bar{l}$	$\bar{h}\bar{i}\bar{k}\bar{l}$	

The figure is constructed by joining each of the points H, δ', &c., of Figs. 410 and 411 to points V, $V_{,}$ on the hexad axis at distance $c \div l$, $O\delta'$ being $OA \div k$.

30. *Beryl*, $Be_3Al_2(SiO_3)_6$, forms crystals belonging to this class.

$$D = 29^\circ\ 56{\cdot}5',\quad c = {\cdot}4988.$$

The habit is that of long hexagonal prisms $m\{2\bar{1}\bar{1}\} = \{01\bar{1}0\}$, terminated by the pinakoid $c\{111\} = \{0001\}$: such crystals are represented by Fig. 328. The edges $[cm]$ are often modified by one or two narrow faces; and occasionally the coigns. When the edges $[cm]$ are modified by faces of a single form $\{01\bar{1}1\}$, the top of the crystal resembles that of apatite shown in Fig. 405. A combination of the two prisms m and $a\{10\bar{1}\} = \{\bar{1}2\bar{1}0\}$, having the horizontal edges modified by p and s is shown in Fig. 417.

The pyramid p being selected as $\{100, \bar{1}22\}=\{01\bar{1}1\}$ gives the element D, and also enables us to determine the symbol of s. For each face s is common to two slanting zones $[psm]$. Hence s has the symbol $\{4\bar{2}1\}=\{11\bar{2}1\}$: to this form Miller assigns the symbol $\{100, \bar{1}22\}$. Miller's (210) is therefore Dana's $(01\bar{1}1)$, and equations (23) for the transformation from Millerian to hexagonal symbols hold also for the forms observed in beryl. The crystals are optically negative, but give very irregular and anomalous bands of colour in the polariscope which indicate that the crystals are in a state of strain: the rectangular cross also breaks into two brushes on turning the plate round in its own plane.

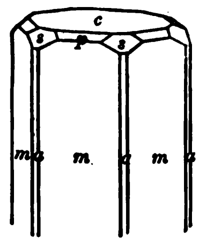


Fig. 417.

V. *Trapezohedral class*; $a\{hkl, pqr\}=a\{\text{hkil}\}$.

31. The crystals of this class have six dyad axes associated with the hexad axis, but no other element of symmetry. The dyad axes are all at right angles to the hexad axis; and three of them, inclined to one another at angles of 60°, are like and interchangeable, the other three are also like and interchangeable and bisect the angles between pairs of the first triad: the first set will be denoted by Δ and may be taken to occupy the positions OM, OM', OM'' of Fig. 410, the second set, having the directions OA, $OA_{\prime\prime}$, OA', will be denoted by δ.

32. The general form consists of six co-polar faces meeting at an apex V associated with six similar faces meeting at the opposite apex $V_{\prime}$, which are so arranged that a face of the one set is interchangeable with any one of the second by a semi-revolution about one or other of the dyad axes. Geometrically, the form may be derived from the dihexagonal bipyramid of the last class by selecting six co-polar faces at one end which are interchangeable by rotations of 60° about the hexad axis, and the six non-parallel faces which meet at the other apex.

Thus the co-polar faces may be taken to be those passing through V and the alternate traces $H\delta'$, $G_{\prime}H_{\prime\prime}K''$, $H'\delta''$, &c. of Figs. 410 and 411, when their symbols are:

$$\begin{array}{cccccc} hkl & qrp & lhk & pqr & klh & rpq \\ \text{hkil} & \bar{\text{k}}\bar{\text{i}}\bar{\text{h}}\text{l} & \text{ihkl} & \bar{\text{h}}\bar{\text{k}}\bar{\text{i}}\text{l} & \text{kihl} & \bar{\text{i}}\bar{\text{h}}\bar{\text{k}}\text{l} \end{array}$$

But the dyad axis OM of these figures interchanges opposite apices, and the traces $H\delta'$ with $HK_{,}$, $H_{,,}K''$ with $H_{,}\delta$, $H'\delta''$ with $H''\delta_{,,}$ &c. Since $HK_{,}$ is parallel to $\delta''K''$, the face through $V_{,}$ and $HK_{,}$ is parallel to that through V and $\delta''K''$, *i.e.* to (prq): the face $V_{,}HK_{,}$ is therefore $(\bar{p}\bar{r}\bar{q}) = (\bar{h}\bar{i}\bar{k}\bar{l})$. The edge in which (hkl) and $(\bar{p}\bar{r}\bar{q})$ meet, is at right angles to OH and is bisected at H where it crosses the equatorial plane; for a semi-revolution about OM, interchanging the two faces, must interchange the two ends of the edge of intersection. Similarly, a semi-revolution about $O\delta'$ interchanges the apices and the traces $H\delta'$, $\delta'H_{,,}$. But $\delta'H_{,,}$ is parallel to $\delta_{,}H''$, and the new face through $V_{,}$ and $\delta'H_{,,}$ is parallel to (lkh): it is therefore $(\bar{l}\bar{k}\bar{h})$. This face meets (hkl) in an edge which crosses the equatorial plane and is at right angles to, and bisected by, $O\delta'$. The face (hkl) is consequently a trapezium bounded by two like and interchangeable co-polar edges, and by two dissimilar skew median edges, each bisected by a dissimilar dyad axis. But all the faces are similar and equal; and the form is a hexagonal trapezohedron, Fig. 418, to which we may assign the symbol $a\,\{hkl,\ pqr\} = a\,\{\mathrm{hkil}\}$. It consists of the twelve faces:

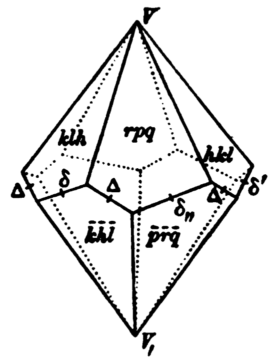


FIG. 418.

hkl	qrp	lhk	pqr	klh	rpq	
hkil	$\bar{\mathrm{k}}\bar{\mathrm{i}}\bar{\mathrm{h}}\mathrm{l}$	ihkl	$\mathrm{h}\bar{\mathrm{k}}\mathrm{il}$	kihl	$\bar{\mathrm{i}}\bar{\mathrm{h}}\bar{\mathrm{k}}\mathrm{l}$	.... (q).
$\bar{p}\bar{r}\bar{q}$	$\bar{l}\bar{k}\bar{h}$	$\bar{q}\bar{p}\bar{r}$	$\bar{h}\bar{l}\bar{k}$	$\bar{r}\bar{q}\bar{p}$	$\bar{k}\bar{h}\bar{l}$	
$\bar{\mathrm{h}}\bar{\mathrm{i}}\bar{\mathrm{k}}\bar{\mathrm{l}}$	ikhl	$\bar{\mathrm{k}}\mathrm{hi}\bar{\mathrm{l}}$	$\mathrm{hik}\bar{\mathrm{l}}$	$\mathrm{i}\bar{\mathrm{k}}\mathrm{h}\bar{\mathrm{l}}$	$\mathrm{khi}\bar{\mathrm{l}}$	

The angles over the polar and median edges are all different, and can be determined from the equations given in Art. 13, and simple spherical triangles.

33. If the faces of table **p** which do not occur in **q** are taken together, we obtain the complementary hexagonal trapezohedron, Fig. 419, the faces of which are parallel each to a face of the trapezohedron of the last Article. The new form may be represented by the symbol $a\,\{\bar{h}\bar{k}\bar{l},\ \bar{p}\bar{q}\bar{r}\}$, or by $a\,\{lkh,\ rqp\} = a\,\{\mathrm{ikhl}\}$. The symbols of the faces to the front of Fig. 419 are inscribed on the diagram and there is no need to give them in further detail.

The student can easily pick them out from table p. The complementary forms are enantiomorphous, for they can be placed so that the one is the reflexion of the other in any one of the planes perpendicular to an axis of symmetry. The crystals may therefore be expected to rotate the plane of polarization.

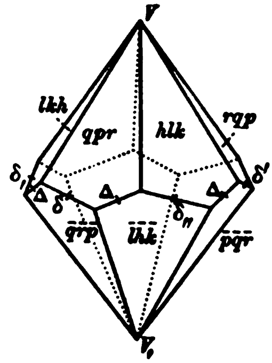


Fig. 419.

34. Projecting, Fig. 410, or Fig. 411, in the horizontal plane $D'CA'$ of Fig. 51 in the manner described in Chap. VI, Art. 19, the trapezohedra can be drawn; the apices V and $V_{\prime}$ being given by $CV = C\gamma \cos 30^\circ \tan D \div 1$. The polar edge $[hkl, rpq]$ of Fig. 418 joins V to the point of intersection of the traces $KH\delta'$ and $H_{\prime}G$; and the co-polar edges are similarly found by taking the homologous pairs of traces. The inferior co-polar edges are found in a similar manner: thus the direction of $[\bar{p}\bar{r}\bar{q}, \bar{k}\bar{h}\bar{l}]$ is the same as that of the line joining V to the point of intersection of the traces $\delta''K''$ (prq) and $H'L$ (khl).

The median edge $[hkl, \bar{l}\bar{k}\bar{h}]$ through δ' is parallel to $[hkl, lkh]$; and its direction VK is that of the line joining V to K, the point of intersection of the traces KH and $H''\delta_{\prime}$. Similarly, the direction of the edge $[rpq, \bar{p}\bar{r}\bar{q}]$ through $\delta_{\prime\prime}$ is found by marking off on OM'' a length equal to OK and joining the point to V; and so on for all the homologous edges: the edges are drawn through the points in which δ', $\delta_{\prime\prime}$, &c., of Fig. 410 are projected.

The remaining median edges pass through the points Δ of Fig. 418, in which the points H of Fig. 410 have been projected; and join each the points in which the median edges through the adjacent points δ meet the nearest polar edges of the two faces meeting in the edge required. Or the edges may be found in a manner similar to that employed for the edges through the points δ. Thus the edge $[hkl, \bar{p}\bar{r}\bar{q}]$ is parallel to the line $[hkl, prq]$; and its direction is found by joining V to the point of intersection of the trace $H\delta'$ (hkl) and $\delta''K''$ (prq): this point may be very distant, and the method would be inconvenient in practice.

In the complementary trapezohedron $\alpha\{hlk, prq\}$, the polar edges through V are parallel to those of the first drawn through $V_{\prime}$, and *vice versâ*. Similarly, opposite median edges are interchanged; thus the edge through $\delta_{\prime}$ of Fig. 419 is parallel to that through δ' in Fig. 418. Hence, when one trapezohedron is drawn, the complementary one is quickly formed.

35. The special forms are all geometrically identical with those of the preceding class, for their faces are parallel to one or other of

the axes of symmetry. Since these axes are all of even degree, any face parallel to one of them must be associated with a parallel face.

Hence, the pinakoid $\{111\} = \{0001\}$; the hexagonal prisms

$$\{01\bar{1}\} = \{\bar{1}2\bar{1}0\}, \quad \{2\bar{1}\bar{1}\} = \{01\bar{1}0\};$$

the dihexagonal prism $\{hkl\} = \{hki0\}$; the hexagonal bipyramids $\{hll, prr\} = \{0h\bar{h}l\}$ and $\{\bar{h}, 2h, \bar{h}, l\} = \{hkl\}$, where $h - 2k + l = 0$; are all forms of this class.

36. The crystals of the following isomorphous double salts have been provisionally placed in this class by Dr H. Traube (*N. Jahrb. f. Min.* I, 1894, p. 245):

Barium-antimonyl dextro-tartrate + potassium nitrate,

$$Ba(SbO)_2(C_4H_4O_6)_2 . KNO_3. \quad D = 74° 2.5', \quad c = 3{\cdot}0285.$$

The forms usually present are $m\{2\bar{1}\bar{1}\} = \{01\bar{1}1\}$, $c\{111\} = \{0001\}$, $r = \{100, \bar{1}22\} = \{01\bar{1}1\}$, $d = \{5\bar{1}\bar{1}, \bar{1}11\} = \{02\bar{2}1\}$, $v = \{13, \bar{4}, \bar{4}; \bar{3}22\} = \{05\bar{5}1\}$. The crystals therefore have somewhat the habit of those crystals of sapphire on which steep pyramids are developed.

Lead-antimonyl dextro-tartrate + potassium nitrate,

$$Pb(SbO)_2(C_4H_4O_6)_2 . KNO_3. \quad D = 76° 27', \quad c = 3{\cdot}5936.$$

The crystals are hexagonal prisms $\{2\bar{1}\bar{1}\} = \{01\bar{1}1\}$ terminated by the pinakoid $\{111\} = \{0001\}$, and by narrow faces $r\{100, \bar{1}22\} = \{01\bar{1}1\}$ modifying the edges [cm].

The physical characters of the two salts are much the same. The crystals show no pyro-electric poles, nor do they rotate the plane of polarization. The determination of the class rests solely on the character of the corrosion-figures observed on the faces of the pinakoid and prism. When plates parallel to the pinakoid are observed in parallel light between crossed Nicols, they show segments similar to those characteristic of triplets of biaxal crystals such as witherite, &c. (see Chap. XVIII); and the characteristic figure of a biaxal crystal can occasionally be seen in places, when convergent light is used. But the segmentation is irregular, and the internal structure manifested is very complex. Again, plates cut from near the surface sometimes give a central portion which is uniaxal.

Plates parallel to the prism-faces examined between crossed Nicols are also seen to have a lamellar structure; and occasionally, when convergent light is used, the brushes characteristic of a biaxal crystal may be seen. The structure is sometimes that of plates bounded by parallel planes, and sometimes such as is found in bodies having a fibrous structure. The determination of the class, and even of the system, is therefore doubtful.

CHAPTER XVIII.

TWIN-CRYSTALS AND OTHER COMPOSITE CRYSTALS.

General Introduction.

1. COMPOSITE crystals often occur, in which the several portions have different orientations governed by regular and definite laws. When the crystallization of a substance held in solution is hurried by rapid evaporation of the solvent, the crystals usually grow together in groups, in which the arrangement of the several members is purely accidental. But it was observed at a very early date that crystals of certain minerals, in particular those of cassiterite and spinel, are joined together in a regular and constant manner to form a well-defined individual. Such regularly formed composite crystals will be the subject of this Chapter.

Romé de l'Isle was the first to attempt an explanation of the composite character of the crystals of spinel and cassiterite; and he introduced the word *macle* to denote the kind of composite crystal which we now call a twin. Werner employed the word *zwilling* (= *twin*) at present used by German crystallographers; and later on Haüy introduced the word *hemitrope* (from ἡμι- = half, and τρόπος = a turn); for he perceived that the orientation of the two portions of every well-defined twin known to him is given by the following law. A complete crystal, bounded by the forms observable on the twin, is divided along a central plane which is parallel to a possible face; and the half on one side of the plane is then turned through 180° about its normal, the two halves remaining in contact to form the twin. This law gives in very many cases the relative orientations of the two portions united together in a twin-crystal; it offers no suggestion as to the cause of twinning, and supplies no explanation of the growth of the twin.

A crystal or form of normal character and uniform orientation will be said to be *simple*, when it is necessary to distinguish it from a composite crystal or a twinned form.

2. With very few exceptions, which however are important, twins consist of portions, the relative orientations of which are such that a semi-revolution of one portion about a line having a definite direction brings the rotated part into the same orientation as the fixed part. The line of rotation we shall call the *twin-axis*: its direction may be that of (i) the normal to a possible face, (ii) a zone-axis, or (iii) a line lying in a crystal face perpendicular to a zone-axis (this last is very doubtful). The twin-axis has clearly the same relation to both portions. Such twins we may designate *hemitropic* twins, when we require to distinguish them from those rare composite crystals in which the orientation of the portions can only be given by regarding the one as the reflexion of the other in a definite plane; these latter being then called *symmetric* twins.

Taking the hemitropic twins, in which the orientation of the portions can be connected by an axis of rotation, we can divide them into two main classes according to the manner in which the portions are united: *1. Juxtaposed* twins, in which the portions are united along a plane surface, and lie on opposite sides of it. *2. Interpenetrant* twins, in which the portions are intimately commingled without any regular surface of separation between their matter.

1. Juxtaposed twins. The plane surface along which the two portions of a juxtaposed twin are united will be called the *combination-plane.* It may be (i) parallel to a possible face which is perpendicular to the twin-axis, or (ii) it may be parallel to the twin-axis.

i. This group of twins includes those called by Haüy hemitropes, in which the twin-orientation is fully expressed by the single statement that the *twin-face* is that particular face parallel to which the portions are united after the one has been turned through 180° about the face-normal. For example, in spinel the twins have a face of the octahedron for twin-face. Provided the two portions belong to a centro-symmetrical crystal, they are situated symmetrically to the combination-plane; and, if they are also equal, the coigns will lie at equal distances on straight lines perpendicular to this plane. Instances of equably and symmetrically developed twins are shown in the drawings of spinel, cassiterite and calcite.

ii. When the combination-plane is parallel to the twin-axis, the directions of both must be expressly specified. This arrange-

ment is common when the twin-axis is a zone-axis which is not perpendicular to a possible face; *e.g.* the so-called Carlsbad twin of orthoclase, and several of the twins of anorthite and the other plagioclastic felspars. In these twins the two portions do not, after a semi-revolution of one of them about the twin-axis, form a complete crystal. They are two like halves of two separate crystals placed in parallel orientation. The hemitropes placed under (i) may, as we shall show, in many cases be referred to a twin-axis parallel to the combination-plane, but the statement of the twin-orientation is more precise when they are referred to a twin-face.

In many cases the twins of classes i and ii cross one another in the middle, so that portions at the opposite ends and on opposite sides of the combination-plane are in like orientation, those at the same end are in twin-orientation. They are generally called *intercrossing twins.* The surface at which the individuals seem to cross is often fairly well-defined. In drawings it is often taken to be a plane perpendicular to the combination-plane.

2. *Interpenetrant twins.* However intimate the intergrowth of interpenetrant twins may seem to be, it has been found by examining the cleavages and other physical characters, and more especially by examining in plane-polarised light plates cut across the twins, that the matter of the different portions remains distinct and separate. Just at the boundary of the individuals the optical phenomena are often indistinct in consequence of the interlocking and overlapping of the matter of different portions. Fluor gives a good instance of interpenetrating twins.

Symmetric twins. A very large number of the hemitropic twins falling under the preceding subdivisions are symmetrical to certain definite planes. Thus, in most of the juxtaposed twins of subclass (i), the portions on opposite sides of the combination-plane are reciprocal reflexions in this plane. But a few composite crystals are known in which the physical structure of the two portions can *only* be connected by regarding the one as the reflexion of the other in certain definite planes. Such composite crystals, of which instances occur in sodium chlorate, sodium periodate and quartz will be distinguished as *symmetric* twins.

The composite crystals known as complementary and mimetic twins fall under one or other of the preceding divisions. By *complementary* twin is meant one composed of two individuals which belong to a class of inferior symmetry in the system; the two

individuals being combined in such a way that the homologous faces of the two individuals taken together produce a form identical with that of greatest symmetry in the same system. Thus the two interpenetrating pentagonal dodecahedra of pyrites, Fig. 431, p. 476, make a complementary twin, in which the faces taken together would compose a tetrakis-hexahedron of class II of the cubic system. Similarly, the juxtaposed twin of pyrargyrite, Fig. 499, p. 527, is another complementary twin.

These complementary twins serve as an introduction to the complex twins of harmotome and phillipsite, in which an apparently simple prismatic crystal consists of intercrossing twinned individuals of the oblique or anorthic system. These prismatic crystals are again twinned so as to approximate in external appearance to a tetragonal, or even to a cubic, crystal. These complex twins and other similar twins, which we shall discuss further on, give an insight into those curious cases known as *mimetic* twins in which apparently simple crystals, having the external form characteristic of a class with complex symmetry, are composed of a number of portions of crystals of inferior symmetry.

The statement which defines the twin-orientation in a given case is called the *twin-law*; for instance, that of spinel is:—twin-face a face of the octahedron.

Multiple twins. In certain substances, calcite, labradorite, &c., the twinning is sometimes repeated several times parallel to the same twin-face, and the crystal consists of a series of thin plates twinned according to the same law. The lamellæ and twins are said to be *polysynthetic*. The physical characters of homologous faces and edges being the same, twinning may occur at the same time with respect to each of the homologous faces of a form to which a twin-face belongs, and with respect to all the homologous lines of which the twin-axis is one. Such multiple twins occur among the crystals of some substances, *e.g.* rutile and cassiterite; but they are, as a rule, far less common than the twins which are composed of only two portions. A twin of two individuals we shall call a *doublet*, when we wish to emphasize the fact that it consists of only two portions; one consisting of portions of three crystals a *triplet*; one of four different portions a *quartet*; and so on for five, six and eight.

3. *Tests of twinning.* Many twins are distinguished by the

presence of re-entrant angles[1], which may be made by like or unlike faces. The angles between the faces meeting in such edges will be indicated by affixing a minus sign; thus, in Fig. 420, p. 466, the angle $111 \wedge (\bar{1}\bar{1}\bar{1}) = -38° 56'$. Re-entrant angles may, however, occur on untwinned crystals, and be due either to an irregularity in the deposition of the matter or to the presence of some obstacle. It is therefore necessary to determine whether the portions forming the re-entrant angle have a parallel or a twin- orientation. Inspection generally suffices to show whether all the like faces in the neighbourhood of the re-entrant angle are parallel or not.

Optical test. When the crystals are translucent, the identity or difference of orientation of the several parts can in most cases be decided by examination in plane-polarised light; this method fails, however, in the case of isotropic (cubic) crystals, and also in twins of the rhombohedral system which have the triad axis for twin-axis, and in a few cases in other systems. The optical investigation in polarised light is the most delicate test that can be applied. By its means, for instance, a twinned structure has been revealed in mimetite which had not been previously suspected, and of which there is no external evidence.

Striæ and corrosion. The presence of twinning is sometimes shown by barbed striæ, meeting in a line on a face of an apparently simple crystal. It is also sometimes revealed by a difference of orientation of the corrosion-figures observed on the same face of a crystal; thus, etched crystals of succin-iodimide have been observed showing at one end of a prism-face triangular pits orientated as shown in Fig. 215, p. 269, whilst at the other end of the same face, when composite, the apices are all turned the other way.

Electrical test. The difference of orientation is sometimes rendered manifest by the electrification resulting from change of temperature; thus, the opposite ends of the twinned crystals of succin-iodimide just mentioned are both analogous poles, and are negatively electrified with falling temperature. The apparently simple hexagonal crystals of lithium potassium sulphate (Chap. XVII, Art. 19) often reveal a twinned structure by the fact that, as the temperature falls, the two similarly developed ends become positively electrified whilst a central zone is negatively electrified.

[1] The edges at which faces meet in re-entrant angles will be indicated by lines of interrupted strokes, except when the figure is taken without modification from a memoir on the particular twin, or from Dana's *Mineralogy*.

4. Before a group, composed of differently orientated portions, can be accepted as a twin, the association must be definite and regular, and occur with sufficient frequency to prove that it is not merely accidental. Possibly a single case of a well-formed twin of simple law may be accepted, provided the law itself is not uncommon in the system to which the crystal belongs.

i. TWINS OF THE CUBIC SYSTEM.

5. In this system the twin-axis is usually one of the triad axes, *i.e.* the normal to a face of the octahedron or tetrahedron; and the juxtaposed twins are generally combined along a plane perpendicular to the twin-axis. Such twins are hemitropes having a face of {111} for twin-face. A few instances, some of which are given in Art. 11, have been described in which combination takes place along one of those faces of the form {211} which are parallel to the twin-axis. Some twins of crystals of class V require a normal (211) for twin-axis; for instance, diamond. The above twins will form the main subject of sub-section *A*.

Complementary twins, having for twin-axis the normal to a dodecahedral face and an angle of rotation of 180° about the axis, will be discussed in sub-section *B*.

The only instance of a hemitropic twin, which does not fall under subdivisions *A* and *B*, is a twin of galena having a face of {441} for twin-face. This law gives rise to a lamellar structure due to repetition of the twinning parallel to the same face. Repeated twinning also occurs in the same crystal parallel to two or more faces of the form.

A. *Twin-axis a triad axis.*

6. *Spinel.* Let us suppose an octahedron of spinel or magnetite to be divided into two equal parts by a plane parallel to the face

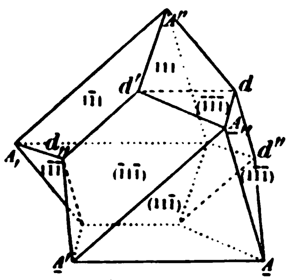


FIG. 420.

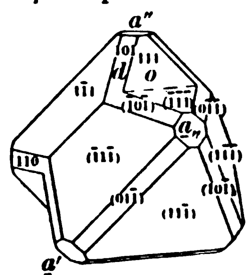


FIG. 421.

$(11\bar{1})$, and the front half to be turned through 180° about the normal to the plane of section; the two portions being then joined together along this plane, the twinned crystal, Fig. 420, is constructed. The edges in the plane of section form a regular hexagon, the sides of which are parallel to the dyad axes of the crystal: these edges are the intersections of portions of opposite faces which are parallel in the simple crystal; and the angles over opposite edges are equal salient and re-entrant angles. Thus the salient angle over the edge $d_{,,}d'$ is $1\bar{1}1 \wedge (\bar{1}1\bar{1}) = 180° - 2 \times 70° 32' = 38° 56'$; the re-entrant angle over the opposite edge is equal to that over *dd'* and is $-38° 56'$. The twin often acquires a more or less strongly marked tabular habit by the disproportionate development of the faces parallel to the combination-plane. Twins of this habit also occur in crystals of gold and diamond.

Galena. Fig. 421 represents a similar twin of galena in the Cambridge Museum, in which the forms are: $o\{111\}$, $d\{101\}$ and $a\{100\}$. In Chap. xv, Art. 17, it was shown that six faces of the dodecahedron are parallel to each triad axis; opposite pairs of these faces being parallel, and adjacent faces making angles of 60° with one another. Hence the twin-axis being $[11\bar{1}]$, the pairs of parallel faces 101, $(\bar{1}0\bar{1})$; 110, (110); 011, $(0\bar{1}1)$ are co-planar after a semi-revolution of one portion about the twin-axis. This is well seen in the specimen, and is indicated in the drawing by the omission of a dividing line between the co-planar portions of faces of the two individuals, such as 101 and (101).

In these figures, and in those of other twins, we enclose in parentheses the symbols of the faces on the portion which is supposed to have been rotated. When the twin is a triplet, quartet, &c., the face-symbols of the other portions are enclosed in brackets, braces, &c. When letters are used to indicate the faces, those of the rotated portion are usually underlined, and the additional portions of multiple twins are indicated either by different type or by affixing to the letter an index-number. The different portions are also sometimes indicated by capital Roman numbers.

7. To determine the twin-law of the spinel and diamond doublets.

The determination of the twin-law in simple cases, such as those represented by Figs. 420 and 421, is easy; for two octahedral faces on different portions of the twin are parallel, as can be proved by

measurement of the angles in the zone of which dd' is the direction of the zone-axis. The twin-axis must therefore be either parallel or perpendicular to these faces. By trial with a cardboard model of an octahedron bisected by a plane parallel to $(11\bar{1})$, the student can convince himself that a semi-revolution of one half about the triad axis perpendicular to the plane of section transforms the simple octahedron into the twinned form.

A semi-revolution about the line lying in the plane of section and bisecting at right angles the edge dd' in Fig. 420 brings the rotated portion into the position of the fixed half. By such a rotation the twin is represented as consisting of two like halves of separate crystals. Semi-revolutions about each of the two homologous lines in the plane $d_{\prime\prime}dd'$ bisecting, respectively, dd'' and $d'd_{\prime\prime}$ also bring the rotated half into the position of the fixed one. These positions of the twin-axis satisfy the geometry of the twin, and in the particular instances of spinel, magnetite, and galena, also the physical relations; and there is nothing to discriminate between them and the triad axis as the twin-axis. The latter is adopted for the sake of greater simplicity in the expression of the twin-law.

Diamond. The twin of diamond, Fig. 422, is best interpreted as having for twin-axis the line which lies in the combination-plane and joins an obtuse to an opposite acute coign. This line is the normal to one of the faces $11\bar{2}$, $1\bar{2}1$, or $\bar{2}11$, which are parallel to the triad axis [111]. The two portions are similar portions of separate crystals. A semi-revolution of the front half about the triad axis [111] brings similar faces on the two portions into parallelism; but opposite faces in a crystal of class V of the cubic system are dissimilar. A simple crystal is therefore not produced by such a rotation. The twin may also be described as a symmetric twin, in which the two portions are symmetrical to a combination-plane parallel to (111).

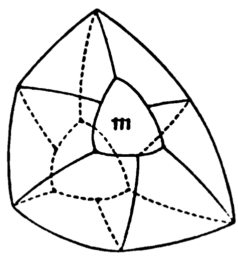


Fig. 422.

The same interpretation should be applied to the spinel-like twins of diamond, in which faces 111 alone occur.

8. To draw the spinel-doublet.

We begin with a drawing of the cube, Fig. 423, such as was described in Chap. VI; and we inscribe the octahedron by joining the middle points A, A', &c., of the cubic faces. Selecting for

twin-axis the diagonal $\rho''\rho_{,,}$, we determine the position of the cubic axes of the rotated half, and the points of bisection of the edges AA'', $AA_{,}$, $A_{,}A_{,,}$, &c. The edges dd', &c., of Fig. 420, forming the hexagonal section in the combination-plane, can then be drawn.

To determine the position of the cubic axes of the rotated portion we proceed as follows.

The plane $AA'A_{,,}$ meets the twin-axis at R, where $OR = O\rho_{,,} \div 3$. On the axis cut off a length $R\Omega$ equal to OR.

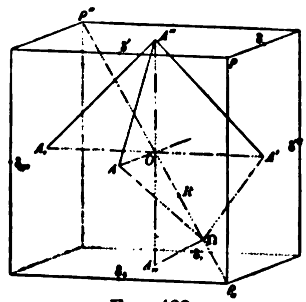


Fig. 423.

The point Ω is the origin of the rotated cubic axes, which are ΩA, $\Omega A'$, $\Omega A_{,,}$ in direction and magnitude; these being the reflexions of the original axes in the plane $AA'A_{,,}$.

To prove this: let Fig. 424 be part of a section of the cube and octahedron by the plane containing OA', $O\rho_{,,}$ and the dodecahedral axis $O\delta_4$; and let $d_{,}$ be the point in which $A'R$ meets $O\delta_4$. Produce $A'R$ to $\mathfrak{a}_{,}$, where $\mathfrak{a}_{,}R = RA'$; and join $O\mathfrak{a}_{,}$, $\Omega\mathfrak{a}_{,}$ and $\Omega A'$.

The four-sided figure, $OA'\Omega\mathfrak{a}_{,}$ is a rhombus, for the diagonals $O\Omega$ and $A'\mathfrak{a}$ are bisected at right angles at R. Hence $\Omega A' = \Omega\mathfrak{a}_{,} = OA'$. The angles $OA'\mathfrak{a}_{,}$, $\Omega A'\mathfrak{a}_{,}$ and $\Omega\mathfrak{a}_{,}A'$ are also equal; and so are the angles $A'\Omega R$, $\mathfrak{a}_{,}\Omega R$.

Hence a semi-revolution about $O\Omega$ interchanges $\Omega A'$ with $\Omega\mathfrak{a}_{,}$, which is equal and parallel to OA'. $\Omega A'$ is therefore the direction of the rotated axis of Y; but it is the negative direction, for, if the original axes are shifted without rotation so as to pass through Ω, OA' and $\Omega\mathfrak{a}_{,}$ are measured in opposite directions.

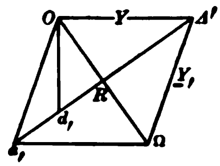


Fig. 424.

Similarly, it can be shown that ΩA is the rotated axis of X and the parametral length a measured in the negative direction; and that $\Omega A_{,,}$ is the rotated axis of Z and the length a measured on it in the positive direction.

The coigns $\underline{A}$, $\underline{A}'$, $\underline{A}_{,,}$ of the rotated portion of the doublet are now found by producing through Ω the axes ΩA, $\Omega A'$, $\Omega A_{,,}$ in Fig. 423 to points at the same parametral distance from Ω. The point $\underline{A}_{,,}$ is joined to d and d', and similarly for the other edges, so that Fig. 420 is quickly completed.

brings them into a position congruent with their original position. But, as shown in Fig. 426, those edges of the simple form which diverge from the opposite ditrigonal coigns ρ'' and $\rho_{\prime\prime}$ will, if produced, meet in the central plane perpendicular to $\rho''\rho_{\prime\prime}$. The two sets of faces meet in a hexagon in this plane; and the middle points of the sides of the hexagon lie in the dyad axes which are at right angles to the triad axis. The bi-pyramid so formed is similar to that of the rhombohedral system discussed in Chap. XVI, Arts. 53—55. A semi-revolution of the six faces meeting at either apex brings the matter into twin orientation, but leaves the geometrical aspect of the bipyramid unaltered. It is clearly immaterial which half of the twin is the rotated portion: the symbols inscribed on the faces correspond with a semi-revolution of the six faces meeting at ρ''. The twins often reveal their composite character by the presence of grooves modifying the edges and coigns which lie in the combination-plane. In the figure several of the edges of the tetrakis-hexahedron are shown by lines of short strokes.

11. Sadebeck (*Zeitsch. d. Deutsch. geol. Ges.* XXIV, p. 427, 1872, and XXVI, p. 617, 1874) has shown that twins of galena and fahlerz sometimes occur in which the twin-axis is a triad axis, and the combination-plane is one of those faces of the form {211} which are parallel to the twin-axis.

Galena. Fig. 427 is a copy of Sadebeck's diagram to illustrate the position of the two portions in the twin of galena: it is a plan of two cubo-octahedra placed in the twin-orientation and projected on a plane perpendicular to the twin-axis. In each crystal the edges parallel to the paper are at 60° to one another; each of them being the intersection of a cubic and an octahedral face, and the direction of a dyad axis. Those faces of {211} which are parallel to the twin-axis would, if developed, replace these edges, and would in each crystal compose a hexagonal prism. The two crystals having been placed side by side in similar orientations, that to the right has been turned through 180° about the twin-axis: the two crystals touch in a common face of the hexagonal prisms. The ideal twin is now obtained by bisecting both crystals by a plane

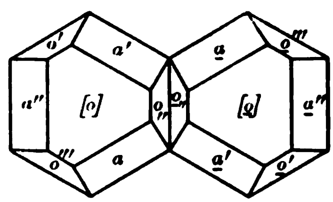


FIG. 427.

dodecahedron, inclined to one another at angles of 60°, are tautozonal and have the twin-axis $O\rho_{\prime\prime}$ for zone-axis. A semi-revolution of the front half brings the face $(0\bar{1}\bar{1})$ into the same plane as 011. Similarly, 101 and $(\bar{1}0\bar{1})$, $1\bar{1}0$ and $(\bar{1}10)$, &c., are co-planar. The twin therefore resembles a hexagonal prism, which is terminated at each end of the twin-axis by three faces of the dodecahedron belonging to separate portions. The faces at opposite ends are no longer parallel, but are symmetrically placed with respect to the combination-plane.

The crystals of blende are never simple rhombic dodecahedra, but are usually combinations of this form with $\mu\{111\}$, $\{100\}$, &c. Further, the twinning is not, as supposed in the ideal twin just described, limited to the production of a doublet, but is repeated several times. When the several portions are twinned parallel to the same face of the tetrahedron, the twin has six tautozonal dodecahedral faces, the portions of which belonging to the different individuals are co-planar and show no trace of twinning save at the edges in which they meet faces not belonging to their zone. When the faces of the cube and tetrahedron are largely developed, and the lamellæ are numerous and thin, the faces of these forms intersect at salient and re-entrant angles in a manner which renders it often difficult to discriminate between portions of a cubic and a tetrahedral face.

Further, the twinning is not, as a rule, limited to one twin-face but is repeated parallel to different tetrahedral faces. This is clearly seen on breaking some of the crystals, when interruptions in the cleavages due to twin-lamellæ having different orientations will be perceived.

10. *Copper.* Fig. 426 represents a rare twin of copper which resembles a doubly terminated hexagonal pyramid of the rhombohedral or hexagonal systems. This is due to the fact that the faces present are those of the tetrakis-hexahedron $\{210\}$, in which the angles between adjacent pairs of faces are all equal (p. 290). Hence a semi-revolution about $O\rho_{\prime\prime}$ of the six faces meeting at the ditrigonal coign $\rho_{\prime\prime}$ in Fig. 231

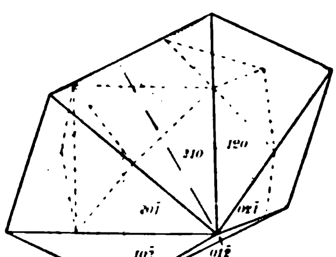


FIG. 426.

the student can easily make a diagram which gives fairly well the relations of the twin. The co-planar faces $\bar{1}\bar{1}1$ and $(\bar{1}\bar{1}1)$ are united in the short edges [do], and the common face resembles the figure produced by joining two deltas at their apices. The two pairs of tetrahedral faces, which in the first modification, meet in salient angles now meet in equal re-entrant angles of—56° 15′: the remaining pair of faces o do not meet.

Interpenetrant twins.

12. *Fluor.* Fig. 429 represents an ideal interpenetrant twin of cubes of fluor having $\rho''\rho_{,,}$ for twin-axis. Those cubic edges, which do not meet on $\rho''\rho_{,,}$, bisect one another at the points δ', $\delta_{,}$, &c., of Figs. 423 and 429. The edges of the rotated cube meeting at ρ'' and $\rho_{,,}$ are equal and parallel to the rotated axes in Fig. 423. The directions and length of these edges being known, all that is needed is to complete the parallelograms which represent the faces of the rotated cube. The individuals are so related that a wedge-like portion of one cube, such as $\rho_{,,}\delta\delta'$, protrudes from a cubic face of the other. The re-entrant angles at which the faces intersect are of two kinds; the angle $100 \wedge (00\bar{1}) = 010 \wedge (00\bar{1}) = -48° 11'$, and the angle $001 \wedge (00\bar{1}) = -70° 32'$.

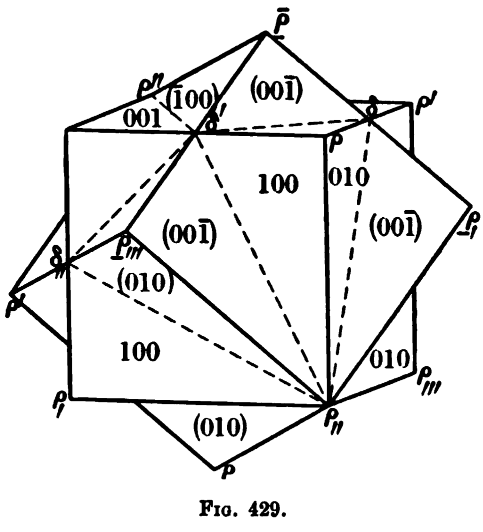


Fig. 429.

The crystals are not intergrown as regularly as is represented in the figure. Usually a small portion of a second individual protrudes

in twin-orientation from a face of the cube, and similar but apparently unconnected portions of different magnitudes often protrude from other faces. But however numerous and independent these several portions may seem to be, they are all twinned to the large individual about the *same* triad axis, and there is no repetition of the twinning about the other homologous axes.

13. *Sodalite.* A remarkable interpenetrant twin of sodalite, $Na_4(Al\,Cl)\,Al_2Si_3O_{12}$, shown in Fig. 430, is sometimes observed in crystals from Vesuvius. This twin consists of two rhombic dodecahedra twinned about one of the triad axes. As in the twin of blende, the six faces of both individuals parallel to the twin-axis coalesce to form a hexagonal prism. In this case, however, each of these faces is divided into four parts, which, for want of precise knowledge as to the boundaries of the individuals, we may consider to be equal; the boundary-lines join points like those marked xx' and $nn_{\prime}$ on one of the composite faces. Two portions of this face belong to 101 and two to $(\bar{1}0\bar{1})$, arranged cross-wise so that opposite segments belong to one individual and are in like orientation.

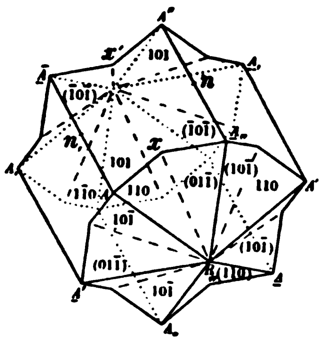


Fig. 430.

The terminal faces meet in two sets of six like edges. The one set of edges, like $AR_{\prime\prime}$, is shown by continuous lines: they are the edges of the rhombic dodecahedra, and the angle over each of them is 60°. These edges also lie in pairs, interchangeable by a semi-revolution about the twin-axis, in planes parallel respectively to two faces of the hexagonal prism; as, for instance, the pair $AR_{\prime\prime}$, $\underline{A}R_{\prime\prime}$, which are parallel to the face $A''\underline{A}_{\prime\prime}A'\underline{A}_{\prime}$. The other set of edges, like $R_{\prime\prime}x$, is formed by faces of different individuals, and the angles over them are re-entrant, each being $-38° 57'$.

In the diagram the dimensions are such that an equably developed rhombic dodecahedron can be formed by prolonging each edge of either individual whilst the cubic coigns remain unchanged. In the specimens at Cambridge the prism edges, such as $A'\underline{A}_{\prime}$, are much elongated as compared with the terminal edges, $AR_{\prime\prime}$, &c.; and only one end of the twins can be seen.

B. Twin-axis the normal to a face of {110}.

14. Twins of this kind cannot occur in crystals of classes I and II of the cubic system; for in these crystals the normals to the dodecahedral faces are dyad axes, and after a semi-revolution about any one of them the orientation is the same as at first. But in crystals of classes III, IV, and V, the only axes of symmetry associated with the four triad axes are three dyad axes parallel to the edges of the cube. Hence a semi-revolution of a crystal of these classes about the normal to a dodecahedral face brings it into a new orientation which is always physically different from the first, and is only geometrically alike in the cases of those special forms which are common to the class of the crystal and to classes I and II. For all other forms the orientation will be geometrically as well as physically different.

15. *Pyrites.* We take first the twinned dihedral pentagonal dodecahedra $\pi\{210\}$ of pyrites shown in Fig. 431. The two dodecahedra interpenetrate one another more or less regularly, so that pairs of the cubic edges cross one another at right angles. In the diagram, representing the individuals of equal size, each pentagonal face can be traced by the more strongly marked edges, the angles over which are salient. The re-entrant edges in which faces of different individuals meet are shown by faint lines, and are of two kinds. One set, to which $[e\underline{e}]$ belongs, consists of the edges of the auxiliary cube employed in Chap. xv, Arts. 37 and 38, for the construction of the complementary dodecahedra $\tau\{hk0\}$ and $\tau(kh0)$, which are geometrically the same as $\pi\{hk0\}$ and $\pi\{kh0\}$. The re-entrant angle over each of these edges is $-36° 52'$, being twice the angle the twin-axis makes with the adjacent normal (102). The second set of edges join each a triad coign to a point on the dyad axis in which two cubic edges intersect; and the angle over each of them is also $-36° 52'$.

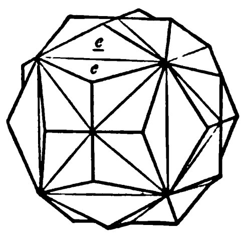


FIG. 431.

If we removed those portions of the dodecahedra which form irregular three-faced pyramids, protruding each beyond a face of the other dodecahedron, we should obtain a twenty-four-faced figure, bounded by the faint lines of the diagram, which is geometrically identical with the tetrakis-hexahedron {210} of Chap. xv, Art. 18. Each of the triangular faces of this latter form is a portion of one of

the pentagonal faces which serves as base for the protruding portion in the twin; and the twenty-four faces of the twinned dodecahedra make up the holohedral form. The twins have therefore been called *supplementary*, or *complementary*—terms which serve to indicate the geometrical relations of the faces to one another.

The twin-axis may be the normal to any one of the faces of $\{110\}$, and the angle of rotation 180°. The same orientation of the two individuals is also obtained by a rotation of 90° about any one of the cubic axes. These axes are dyad axes, and a quarter-revolution about one of them was shown in Chap. xv, Art. 38, to bring a dodecahedron $\tau\{kh0\}$ into the position of $\tau\{hk0\}$. The only difference between the two modes of stating the twin-orientation is that different faces of the individuals are brought after rotation into similar positions.

The dyad and triad axes of the two individuals have the same directions. Hence the cubic faces truncating the intersecting cubic edges of the twin would be parallel, and would be co-planar, if the individuals were, as in the figure, equably developed. If, further, these cubic faces were developed to such an extent as to pass through the faint cubic edges of Fig. 431, we should obtain a mimetic twinned cube, which could not be distinguished from a simple cube except by the fact that different portions of each cubic face would be striated in directions at right angles to one another. Each cubic face would consist of four isosceles triangles; the equal sides joining the centre of the face to the trigonal coigns.

Similarly, the octahedral faces modifying each trigonal coign are, when present, co-planar. If developed to such an extent as to obliterate the other faces, they would form an octahedron. Each face of this octahedron would, however, consist of six triangular portions; the portions being separated by the lines through the centre of the face and each coign. Alternate triangles would belong to the same individual, for they would be interchanged by rotations of 120° about the triad axis perpendicular to the composite face.

From what precedes it follows that mimetic twins of pyrites are possible, which may have the external form of the tetrakis-hexahedron $\{210\}$, of the cube $\{100\}$, or of the octahedron $\{111\}$.

16. *Eulytine.* Fig. 432 (after vom Rath) represents two interpenetrating triakis-tetrahedra $\mu\{211\}$ of eulytine twinned according to the same law as pyrites. The individual having one of its faces labelled $\underline{n}$ is in the position of $\mu\{2\bar{1}1\}$, and can be

brought into the same orientation as the other by (i) a semi-revolution about the normal to any face of {110}, or (ii) by a rotation of 90° about one of the dyad axes which pass through the points of intersection of pairs of opposite tetrahedral edges of the two forms. In the figure the equably developed inter-crossing individuals are represented as meeting in the three axial planes. The triad axes of the two individuals are in the same lines, but the obtuse ditrigonal coign of the one would, if developed, lie between the centre and the acute ditrigonal coign of the other: we may express this by the statement that the acute ditrigonal coign of the one is superimposed on the obtuse ditrigonal coign of the other.

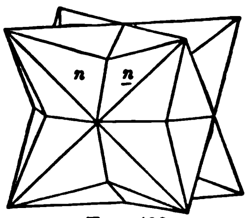


FIG. 432.

M. Bertrand (*Bull. Soc. franç. de Min.* IV, p. 61, 1881) found that plates of eulytine, cut parallel to a face (111) close to the surface of apparently simple crystals, on which the tetrahedron $\mu\{111\}$ is largely developed, give in convergent light between crossed Nicols the black cross and a series of circular rings characteristic of a uniaxal crystal. He therefore interprets the apparently simple crystal of eulytine as a mimetic quartet, consisting of four pyramids of a rhombohedral crystal. Neglecting any small modifying faces, we may represent the mimetic tetrahedron by Fig. 433. Each individual of the quartet consists of the trigonal pyramid formed by joining the centre O to three coigns ρ, and having for base the equilateral triangle made by these coigns. Such a pyramid $O\rho'\rho'''\rho_{,,}$ resembles the pyramid $OXYZ$ of Fig. 309: Or is its triad axis, and the pyramid-faces are $O\rho'\rho'''$, $O\rho'\rho_{,,}$, $O\rho_{,,}\rho'''$. These faces are parallel to three faces of the rhombic dodecahedron; and a pyramid exactly equal to $O\rho'\rho_{,,}\rho'''$ of Fig. 433 is made by the portion of Fig. 434 which is intercepted by the plane $DD'D''$ and has its apex at ρ. Four such pyramids as $\rho DD'D''$ will, when united in pairs along their pyramid-faces and meeting at ρ, exactly fill the space enclosed by their four bases. If isotropic, such a crystal will be a tetrahedron of the cubic system. Treated as a mimetic

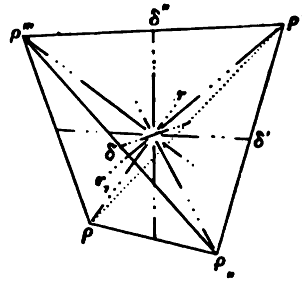


FIG. 433.

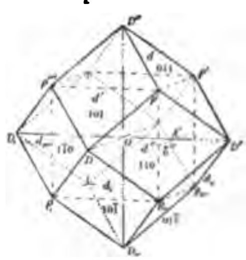


FIG. 434.

quartet of four pyramids of the rhombohedral system, the angular element may be taken to be 35° 16′, the inclination of a tetrahedral face to an adjoining face of the rhombic dodecahedron.

According to this view the faces of the cube become those of the rhombohedron $\{\bar{1}11\}$; the three faces of the cubic $\mu\{211\}$ adjoining the base (111) become faces of the rhombohedron $\{110\}$; and the three faces of the complementary tetrahedron $\mu\{11\bar{1}\}$ equally inclined to the face $DD'D''$ become three faces of the rhombohedron $\{3\bar{1}\bar{1}\}$. Attention was, in Chap. XVI, Art. 59, called to the way in which these rhombohedra form a series such that the faces of one truncate the edges of a following one; the same is true of the corresponding cubic faces. They are given in succession in the following table[1]:

Cubic crystal.			*Rhombohedral crystal.*		
211	truncates	[110, 101]	011	truncates	[001, 010].
110	„	[100, 010]	001	„	$[\bar{1}11, 1\bar{1}1]$.
100	„	$[11\bar{1}, 1\bar{1}1]$	$\bar{1}11$	„	$[\bar{1}\bar{1}3, \bar{1}3\bar{1}]$.

From what precedes, it is clear that, in a section near the centre of the tetrahedron, portions of different rhombohedral pyramids will overlap, causing an indistinctness in the optic phenomena: hence the necessity of a section from near the surface.

Further, M. Bertrand states that in certain crystals, the external shape of which he does not give, but in which probably the faces of $\mu\{211\}$ are largely developed, each of the rhombohedral pyramids constituting the mimetic tetrahedron is itself made up of three separate portions which have their optic axes substantially parallel. Professor Klein and other crystallographers maintain that the crystals are truly cubic, and that the optic phenomena in these crystals, as also the similar phenomena (described as *anomalous*) in crystals of leucite, garnet, boracite, analcime, &c., are the result of a strained structure. This interpretation has been supported by finding that some of the substances become isotropic when the temperature is raised, and by the production of similar phenomena in isotropic substances by mechanical strains; the phenomena in the latter being found to vary with the external shape.

If M. Bertrand's interpretation be admitted, it follows that the twin of eulytine represented in Fig. 432 consists of twenty-four (or possibly of forty-eight) different portions. Assuming that they all meet regularly in the planes joining the tetrahedral and reentrant edges to the centre, three (or six) meet at each acute ditrigonal coign. There are not enough data to settle to which

[1] The student will do well to make a stereogram, like Fig. 265, p. 312, in which the cubic symbols of the poles in the table are inscribed in black ink, and the corresponding rhombohedral symbols of the same poles in red ink.

class of the rhombohedral system the individual should belong; but the class is probably one in which there are neither dyad axes nor a centre of symmetry.

17. *Sodium chlorate.* When the crystals are deposited from aqueous solution at temperatures below 0° C., they are frequently deltoid dodecahedra $\tau\{332\}$, or $\tau\{332\}$ with $\tau\{111\}$, or tetrahedra $\tau\{111\}$ twinned in a manner geometrically the same as the twins of eulytine (Groth, *Pogg. Ann.* CLVIII, p. 216, 1876). The twinned tetrahedra are represented in Fig. 435, in which the dyad axes of the two forms are given by the lines joining pairs of opposite points A, *e.g.* $A'A_,$. Examination of plates cut from the twinned crystal shows that the individuals rotate the plane of polarization in opposite directions, the one being dextro- the other laevogyral.

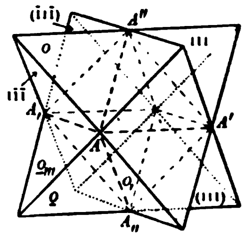


FIG. 435.

According to Fresnel's theory, light is propagated in such crystals by circular oscillations, so that the particles of ether of which the motion constitutes a ray of light are, at any given instant, arranged in a helix described on a cylinder of circular section which has its axis in the line of propagation of the light: the intensity of the light is measured by the square of the radius of the circular section. This helix is right-handed and resembles a corkscrew, if the light consists of a right-handed circularly polarised beam; and is a left-handed screw in the case of a left-handed circularly polarised beam. When plane-polarised light falls on the crystal, it is resolved into two circularly polarised beams of equal intensity, but opposite directions of rotation. Further, in dextrogyral crystals the right-handed beam is the faster, in lævogyral crystals the left-handed. But by reflexion in a plane a right-handed helix is reproduced in a left-handed helix, and *vice versâ*; whilst by a semi-revolution about a line a helix is reproduced in one of the same kind. It follows therefore that the two individuals in the twin of sodium chlorate are optically as well as crystallographically reciprocal reflexions in the planes containing the dyad axes of the twinned individuals. It is highly probable that the constraint, which in enantiomorphous crystals limits the motion of the ether to circular oscillations, is connected with the arrangement of the particles in the two correlated bodies; the arrangement in the one being to that in the other as an object is to its reflexion in a mirror, placed parallel to certain faces of both crystals.

The optical relations of the individuals united in the twin of sodium chlorate show that a semi-revolution about a line parallel to

one of the tetrahedral edges, or a quarter-revolution about one of the dyad axes does not bring the matter of one into a state of similar orientation with that of the other; and such a twin cannot be explained by means of a twin-axis of rotation. Geometrically and optically the two individuals are reciprocally symmetrical to each of the planes $A'AA_{,}$, &c., which contain two dyad axes of both individuals: they form a symmetric twin defined in Art. 2.

18. *Diamond and haüyne.* Octahedra of these substances are found, in which the edges are replaced by grooves as shown in Fig. 436. The crystals of diamond are placed in class V of the cubic system, partly because a few crystals having the form of hexakis-tetrahedra of this class have been observed, partly to account for the grooves: those of haüyne, for the latter reason and from their close affinity to crystals of sodalite. When etched by a solution containing 12 per cent. of hydrogen chloride, each face of the rhombic dodecahedron of sodalite shows triangular pits, which are symmetrical to the dodecahedral plane of symmetry perpendicular to the corroded face. This fact is consistent with the presence of the dodecahedral planes of symmetry, but involves the absence of the cubic planes of symmetry.

If two tetrahedra are twinned like those shown in Fig. 435, they intersect in re-entrant edges AA', &c.; and the trigonal pyramids projecting beyond these edges belong to separate individuals. If now planes are drawn through the re-entrant edges cutting off these pyramids, an octahedron like that in Fig. 437, the

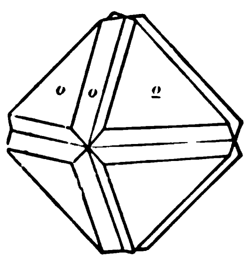


Fig. 436.

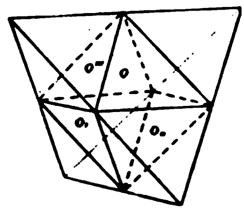


Fig. 437.

faces of which are labelled o, $o_{,,}$, $o_{,}$, o''', will be obtained. Such an octahedron includes portions of both individuals, and is a mimetic twin similar to that described as possible in pyrites. In this twin, however, the portion in each octant which belongs to one individual

fills the space included between the axial planes, and each face belongs to a single individual. By drawing the truncating planes further away from the centre, so that only a portion of each of the projecting pyramids is cut off, a mimetic octahedron having grooved edges, like Fig. 436, is produced: the sides of the grooves are faces of the tetrahedron which is complementary to that to which the octahedral face belongs.

The same explanation may be given of the crystal of diamond, Fig. 294. This can be derived from twinned hexakis-tetrahedra which would somewhat resemble the twin of eulytine, Fig. 432. Each ditrigonal coign is then modified by a tetrahedral face (111), which cuts off the greater portion of the protruding pyramid.

ii. Twins of the tetragonal system.

A. *Twin-axis the normal to a face of* $\{h0l\}$ *or* $\{hhl\}$.

19. *Cassiterite and rutile.* These twins have a face of the pyramid {101} for twin-face. In the twin of cassiterite, Fig. 438, a face *a* of {100} of both individuals is co-planar: the twin-axis is therefore parallel or perpendicular to this face. But the normal to *a* is a dyad axis; and a semi-revolution about it gives an orientation identical with the first. The twin-axis is therefore parallel to the face, and, since the crystal is symmetrical to the face and to a centre, the twin-axis may be either perpendicular or parallel to the combination-plane. By a semi-revolution about the normal $(0\bar{1}1)$, the faces on the lower half of Fig. 438 are brought into parallelism with those on the upper half. By a semi-revolution about the line in the combination-plane parallel to $[as] = [100, 1\bar{1}1]$, the lower half is brought into congruence with the upper half, and the twin is represented as if formed by the union of two like halves of separate crystals. The twin is also symmetrical with respect to the combination-plane. We shall, however, speak of it as a hemitrope with $0\bar{1}1$ for twin-face.

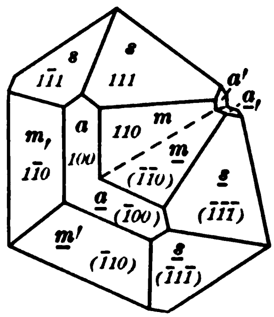


Fig. 438.

In both minerals the faces *a* and *m* are usually well developed.

In cassiterite the faces in their zone are striated vertically: this character enables us to distinguish them from the faces *s*, which have a somewhat glazed aspect. In Fig. 438 the faces *s* present make salient angles with one another; but when the prism-edges are short, faces *s* make a trough with two re-entrant angles in the neighbourhood where $m_,$, $\underline{m}'$ and $a_,$ meet. The prism-faces *m* make two salient and two re-entrant angles of equal magnitude.

20. To determine the direction of the twin-axis in a doublet similar to Fig. 438.

i. When the faces *a* of both portions are seen to be co-planar, measurement of the salient or re-entrant angle $a' \wedge \underline{a}_,$ suffices. For C in Fig. 439 being 001 and $T(0\bar{1}1)$, we have $a' \wedge \underline{a}_, = 2CT$; and in cassiterite $CT = 33° \, 55'$, and in rutile $32° \, 47{\cdot}25'$. Therefore $a' \wedge \underline{a}_,$ is $67° \, 50'$ or $65° \, 34{\cdot}5'$.

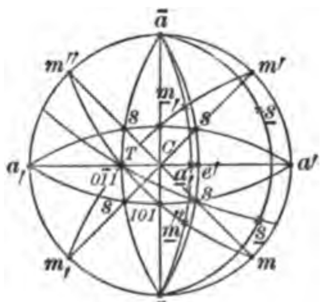


FIG. 439.

Measurement of any one of the angles, $m \wedge \underline{m}$, $m_, \wedge \underline{m}'$, or $s \wedge \underline{s}$ also suffices; for

$$m_, \wedge \underline{m}' = -m \wedge \underline{m} = 180° - 2m_,T, \quad \text{and} \quad s \wedge \underline{s} = 180° - 2sT.$$

For cassiterite, $m_,T = 66° \, 45'$; $m_, \wedge \underline{m}' = 46° \, 29'$; $s \wedge \underline{s} = 38° \, 29'$;
„ rutile, „ $= 67 \;\; 29$, „ $= 45 \;\; 2$; „ $= 42 \;\; 39$.

ii. When however the faces *a* are not present, measurement of two of the above angles is needed to establish the position of the twin-axis. Thus, in Fig. 439, measurement in the drawn zones $[Ts\underline{s}]$, $[m_, T\underline{m}_,]$ of the angles $s \wedge \underline{s}$ and $m' \wedge \underline{m}_,$ gives the sides sT and $m_,T$ of the spherical triangle $Tsm_,$; and by preliminary examination, the student is supposed to have identified *s* and $m_,$ and to know their position on the sphere of projection. Hence T can be projected by the method given in Chap. VII, Art. 19; and its position with respect to the fixed crystal is determined.

In Fig. 439, the poles of the rotated portion are represented by stars; and the same method of distinguishing the poles of the rotated crystal will be used in other cases.

In determining the direction of the twin-axis, the student is sometimes puzzled which of two possible lines to select as twin-axis; viz. OT the perpendicular to the combination-plane, the position of

which is given by the pole $T'(0\bar{1}1)$ in Fig. 439, or $OT_{,}$ lying in the combination-plane, the extremity of which bisects the arc $a'\underline{a}_{,}$ in the stereogram. Both give correctly the orientation of this twin, but OT is a possible normal, whilst $OT_{,}$ is a possible zone-axis, a semi-revolution about which gives like halves of separate crystals. The difficulty mainly arises from inability to see clearly the position of the combination-plane, and from the attempt to assign indices to $T_{,}$ as the pole of a possible face. If the measurements are only approximate, the difficulty may be great; and in default of other evidence the line which has the lowest indices is selected.

In the case of the two minerals cassiterite and rutile, it is easy to distinguish between the two representations, and to show that $OT_{,}$ is not a possible normal. Thus, knowing $a'\underline{a}_{,}$ to be 67° 50′ in cassiterite and 65° 34·5′ in rutile, we have for CT 33° 55′ and 32° 47·25′. Hence T is in both cases the pole $(0\bar{1}1)$, and the expression for the twin-law is simple and the same for both substances; viz. twin-face $(0\bar{1}1)$. Let us now suppose $T_{,}$ bisecting the arc $a'\underline{a}_{,}$ in Fig. 439 to be a possible pole $(0kl)$. Then $\wedge TT_{,} = 90°$; for T and $T_{,}$ are the external and internal bisectors of the arc $a'\underline{a}_{,}$. But e' is (011), and

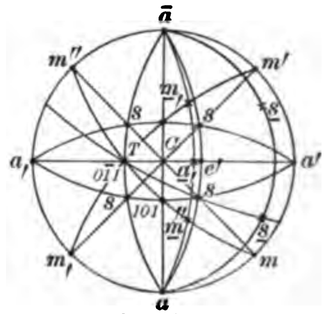


Fig. 439.

$\tan Ce' = \tan E = c \div a$ (Chap. XIV, Art. 8). Again, from the A. R. $\{Ce'T_{,}a'\}$, we have

$$\tan Ce' \div \tan CT_{,} = l \div k.$$

But $$CT_{,} = 90° - CT = 90° - Ce' = 90° - E.$$

$$\therefore\ l \div k \text{ (a rational number)} = \tan^2 E = c^2 \div a^2.$$

Hence the ratio of the squares of the parameters must be rational. But we know that these parameters vary by insensibly small increments with the temperature, and that the coefficient of thermal expansion along the principal axis differs from that in directions at right angles to it. Hence, even if for some particular temperature $c^2 \div a^2$ is rational, it cannot remain so as the temperature changes; and $l \div k$ cannot be a fixed commensurable ratio. Introducing in the above equation the values of E for cassiterite and rutile, we have:

for cassiterite, $k \div l = \cot^2 33° 55' = 2·2$, nearly $= 11 \div 5$ (say);
for rutile, „ $= \cot^2 32° 47·25' = 2·4$, nearly $= 5 \div 2$ (say).

Hence, approximately, $T_{,}$ is (0,11,5) in cassiterite, and (052) in rutile.

Calculating now from the A. R. the positions of the poles (0,11,5) and (052), we find $C \wedge 0,11,5 = 55° 56·5'$, and $0\bar{1}1 \wedge 0,11,5 = 89° 51·5'$; and

$C \wedge 052 = 58° 9·75'$ and $0\bar{1}1 \wedge 052 = 90° 57'$. The pole (052) of rutile is displaced by a very appreciable angle from the position required for $T_{,}$, and 0,11,5 are unusually high indices for a normal, and more especially for one which is a twin-axis.

21. Crystallographers at one time attempted to represent the faces of the rotated portion as possible faces of the fixed one; and determined the relations which should exist between the elements of the crystal for this to be possible. Thus, the condition, that the poles $\underline{m}$ and $\underline{s}$ in Fig. 439 should be possible poles of the tetragonal crystal given by the poles a, m, s, e, is that $c^2 \div a^2 = \tan^2 E$ should be rational. We have already seen that this is nearly satisfied in rutile and cassiterite. Naumann makes frequent use of the idea in the drawing of twin-crystals; and we shall give an instance of its use in describing one of the twins of staurolite.

We can easily find the condition that the faces $\underline{a}_{,}$, $\underline{m}_{,,}$, $\underline{s}$ should be possible faces of the fixed crystal, the poles of which are a, $m_{,,}$, s, &c.

Suppose $\underline{a}_{,}$ to be $(0kl)$, and take the A. R. $\{C\underline{a}_{,}e'a'\}$.

Then $$\tan E \div \tan C\underline{a}_{,} = l \div k.$$

But $$C\underline{a}_{,} = Ta_{,} - CT = 90° - 2E;$$

$$\therefore \tan 2E \tan E = l \div k;$$

$$\therefore \cot^2 E = \frac{l+2k}{l} = 2·2 = 11 \div 5, \text{ for cassiterite};$$

$$= 2·4 = 5 \div 2, \text{ for rutile};$$

taking the same approximate values as before.

Hence in cassiterite $\underline{a}_{,}$ becomes (035), and in rutile (034).

Again $\underline{C}$ being the position of (001) in the rotated crystal, $\underline{C}T = TC$, and $C\underline{C} = 2E$.

Hence $\underline{C}$ being $(0\bar{k}l)$, we have

$$l \div k = \tan E \div \tan 2E = (1 - \tan^2 E) \div 2.$$

Hence, taking the same approximations, $\underline{C}$ is $(0,\bar{1}\bar{1},3)$ for cassiterite, and $(0,\bar{1}\bar{0},3)$ for rutile.

The symbols of the other poles can now be found by the law of zones. Thus $\underline{m}''$ lies in $[a\underline{a}_{,}] = [100, 035] = [05\bar{3}]$, or $= [100, 034] = [04\bar{3}]$, and in $[mT] = [\bar{1}11]$. Hence $\underline{m}''$ is (835) in cassiterite, and (734) in rutile.

Again $\underline{s}$ lies in the zones $[sT]$ and $[\underline{C}\underline{m}'']$.

For cassiterite, $[sT]$ is $[2\bar{1}\bar{1}]$ and $[\underline{C}\underline{m}''] = [\bar{8}, 3, 11]$; $\therefore \underline{s}$ is (471).

" rutile " " $[21\bar{1}]$ " " $= [\bar{7}, 3, 10]$; $\therefore$ " " (7,13,1).

The above symbols, depending on the approximate value adopted for $\cot^2 E = a^2 \div c^2$, are only approximate; and, since E and the parameters change with the temperature by insensibly small increments, the faces of the rotated portion cannot have rational indices when they are referred to the axes and parametral plane of the fixed portion.

22. To find the position of the axes of the rotated portion; and to draw the doublet of cassiterite.

A projection of the axes of the crystal in conventional position is first obtained from that of a set of cubic axes by changing the length of the vertical axis in the ratio $\tan E : 1$. Thus, in cassiterite $OC = OA'' \times \cdot 6723$; and in rutile $OC = OA'' \times \cdot 644$.

The position in the projected plane COA' of the normal NON'' to the twin-face $(0\bar{1}1)$ or $(01\bar{1})$ has now to be found.

In the plane of the paper, Fig. 440, construct a right-angled triangle $OA_{,}D$ having the angle at $A_{,} = E$ (33° 55′ for cassiterite, 32° 47·25′ for rutile). With $A_{,}O$ as diameter, describe a circle cutting $A_{,}D$ at M. Through M a line is then drawn parallel to DC to meet $A_{,}C$ in the point N'' required.

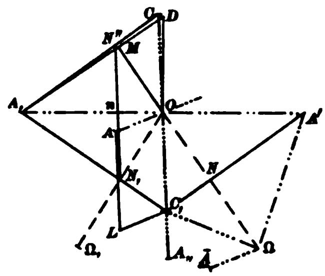


FIG. 440.

It is clear that the angle at M, being the angle in a semicircle, is a right angle. Also

$$A_{,}M \div OM = \cot DA_{,}O = \cot E;$$

and $$OM \div MD = \cot (DOM = DA_{,}O) = \cot E.$$

$$\therefore\ A_{,}M \div MD = \cot^2 E.$$

But in the perspective of the drawing, $A_{,}OC$ is a right-angled triangle having $\wedge\, OA_{,}C = E$, and $A_{,}OC = 90°$. And, since the lines MN'' and DC are parallel,

$$A_{,}N'' : N''C = A_{,}M : MD = \cot^2 E : 1.$$

The point N'' has therefore in the hypotenuse of the projected triangle the same relation that M has in the triangle $A_{,}OD$. N'' is therefore the foot of the perpendicular drawn from O on $CA_{,}$.

It is not necessary that the right-angled triangle in the paper having an angle E at $A_{,}$ should have $OA_{,}$ for side. The triangle may have any position and any size. Thus, by drawing a line through M parallel to $DC_{,}$, the point $N_{,}$ is found which is the point at which the normal $(0\bar{1}\bar{1})$ meets $A_{,}C_{,}$. For

$$A_{,}N_{,} : N_{,}C_{,} = A_{,}M : MD = \cot^2 E : 1.$$

The twin-axis ON is in the next place produced to Ω where $N\Omega = ON = ON''$. The lines $\Omega C_{,}$ and $\Omega A'$ drawn to the axial points

where $(01\bar{1})$ meets the axes are two of the axes of the rotated portion both in direction and magnitude. The triangles $ONC_,$ and $\Omega NC_,$ are equal. Hence $\Omega C_, = OC_,$, and $\wedge C_,\Omega N = \wedge C_,ON$. Rotation through 180° about ON will therefore bring $\Omega C_,$ parallel to $OC_,$. The signs will however be different; for $OC_,$ is measured downwards, and $\Omega C_,$ would in its rotated position be measured upwards. The same proof holds for $\Omega A'$. The third axis $\Omega \underline{A}$ is parallel to OA, both being parallel to the twin-face, and the parametral length is equal to OA.

Similarly, if $(0\bar{1}\bar{1})$ is the twin-face, $\Omega_,$ is the origin of the rotated axes, where $\Omega_,N_, = ON_,$. In this case the axes of the rotated portion are $\Omega_,C_,$, $\Omega_,A_,$, and $\Omega_,\underline{A}$ parallel and equal to OA. The same process will give Ω' and Ω'' in the upper quadrants and the corresponding rotated axes when the faces (011) and $(0\bar{1}1)$ are the twin-faces.

In drawing Fig. 438, the portion in conventional position is first drawn. Any point in the edge $[am]$ is taken, which gives to the prism-edges a length corresponding approximately to those of the twin; and the lines of intersection $[m\underline{m}]$, $[a\underline{a}]$, $[m_,\underline{m}']$, &c., in which the prism-faces meet the combination-plane passing through the selected point in $[am]$, are determined. Since these edges lie in $(01\bar{1})$, their directions are those of the zone-axes $[110,\ 011]$, $[100,\ 01\bar{1}]$, $[1\bar{1}0,\ 01\bar{1}]$, &c., of the fixed crystal. Draw in Fig. 440 the lines AL, $C_,L$ parallel respectively to OZ and OX. The face (110) may be supposed to pass through AL and A', $(1\bar{1}0)$ through AL and $A_,$, and $(01\bar{1})$ through $C_,L$ and A'. Hence the edge $[m\underline{m}]$ parallel to $[110,\ 01\bar{1}]$ is given by $A'L$, $[a\underline{a}]$ by $A'C_,$, and $[m_,\underline{m}']$ by a line joining A' to a point $L_,$ on $C_,L$ produced, where $C_,L_, = C_,L$. The edge $[a'\underline{a}_,]$ is parallel to OA. The edges and the line of construction $[a\underline{a}]$ are now drawn and terminate the fixed portion of the twin.

The prism-edges $[\underline{ma}]$, &c., of the rotated portion are now drawn parallel to $\Omega C_,$ through the points in which the prism-edges of the fixed crystal meet the combination-plane. The positions of the coigns on these new edges are determined by drawing lines parallel to $O\Omega$ through the corresponding coigns on the fixed crystal. The edges $[\underline{as}]$, $[\underline{ms}]$, &c., are then drawn; and the pyramid-edges are got by finding the apex $\underline{C}_,$ of the lower individual. This apex is the point of intersection of $C_,\Omega$ of Fig. 440 with the line through the apex C of the fixed crystal, parallel to $ON\Omega$.

23. We shall now describe some of the multiple twins of rutile, and shall consider first those in which the several individuals have a common face a of the prism $\{100\}$, and the twin-axes and principal axes all lie in the same plane, $A'OC$ of Fig. 440. The prisms m and a are usually well developed, but they are sometimes replaced by faces l of $\{210\}$. Fig. 441 represents a triplet in which the faces of $\{210\}$ form the prism-zones. A similar triplet, having the same forms as those in Fig. 438, is shown in Fig. 442.

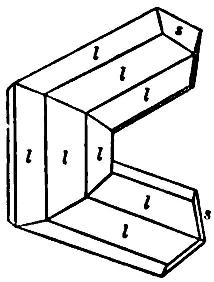


FIG. 441.

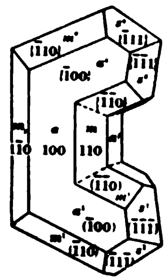


FIG. 442.

Another like individual can now be united with the upper one of the triplet, Fig. 442, taking $(0\bar{1}1)$ of this individual for twin-face. The two individuals are those distinguished in Fig. 443 by letters bearing the affixes 4 and 5, respectively. Similarly, a like individual can be twinned according to the same law to the lower individual in Fig. 442; the twin-face being $(0\bar{1}\bar{1})$ of the latter. The new member of the sextet is that bearing the affix 2. A sixth individual can now be united in like twin-orientation with either of the individuals having respectively the affixes 2 and 5; but it cannot be in true twin-association with both of them. In Fig. 443 the individual with affix 3 is twinned to that having the affix 2. We have already seen that the salient angle

$$a^2a^3 \text{ is } 2CT = 2 \times 32° \, 47{\cdot}25' = 65° \, 34{\cdot}5'.$$

$$\therefore \; a^3a^2 + a^2a^1 + \ldots + a^4a^5 = 5 \times 65° \, 34{\cdot}5' = 327° \, 52{\cdot}5'.$$

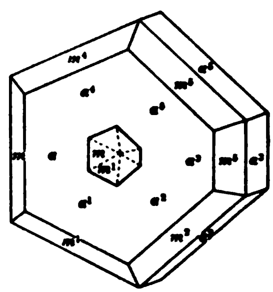


FIG. 443.

The angle a^3a^5 is therefore $32° \, 7{\cdot}5'$; but, if the two individuals were in twin-orientation, the angle would be $65° \, 34{\cdot}5'$. The face a^5, regarded as a face of the individual with

affix 3, is very nearly in the position of (052). The sextets and octets (Art. 24) of rutile from Graves' Mt., Georgia, U.S.A., were first described by G. Rose (*Pogg. Ann.* cxv, p. 643, 1862). He places a dimple formed, as shown in Fig. 444, by *m* faces in the centre of the common *a* face; but the specimen in the Cambridge Museum shows no sign of the dimple.

The similar multiple twins of cassiterite are not so regular; and it is rare to get twins in which more than three or four individuals have their principal axes co-planar, and the parallel faces *a* are very slightly, if at all, developed.

24. Another remarkable twin of rutile, an octet, was also (*loc. cit.*) described by Rose. A copy of his figure is given in Fig. 444, and a stereogram of the octet in Fig. 445, the primitive being the zone-

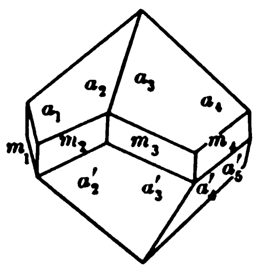


FIG. 444.

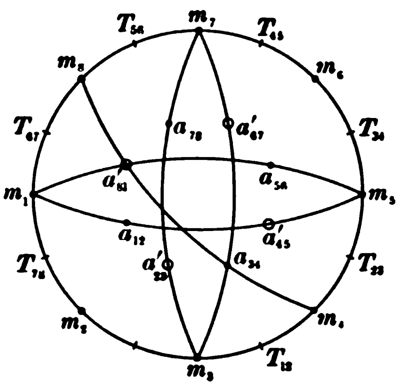


FIG. 445.

circle $[m\underline{m}''Tm'']$ of Fig. 439. In these diagrams the faces and portions of a face belonging to a single individual are denoted by letters carrying single index-numbers, and the poles of a face or axis common to two individuals by double index-numbers. In this twin each individual, except 1 and 8, is united to the two adjacent individuals in true twin-association; the twin-face on one side being $T(011)$, as in the doublet, and on the other side the homologous face (101) in the zone $[mT]$ of Fig. 439.

In Fig. 439 the arc $101 \wedge 0\bar{1}1$ is the hypotenuse of an isosceles right-angled triangle, of which CT is a side.

$$\therefore \quad \cos(101 \wedge 011) = \cos^2(CT = 32^\circ\ 47{\cdot}25');$$

$$\therefore \quad 101 \wedge 011 = 45^\circ\ 2', \text{ and } m''T = m \wedge 101 = 67^\circ\ 29'.$$

Hence $$m\underline{m}'' = 180^\circ - 2m''T = 101 \wedge 0\bar{1}1 = 45^\circ\ 2'.$$

The successive prism-faces m which lie in the zone containing all the twin-axes are therefore inclined to one another at angles of 45° 2′.

Since all the individuals from 1 to 8 are in strict twin-orientation, the angles $m_1m_2 = m_2m_3 = \&c. = m_7m_8 = 45° 2'$; and their sum is $7 \times 45° 2' = 315° 14'$. Hence $m_8m_1 = 360° - 315° 14' = 44° 46'$. The first and eighth individual are not therefore in twin-combination; and the face indicated in Fig. 445 by the circlet a'_{81} is not single, but should consist of two triangular portions separated by a re-entrant angle of 12′.

The twins of this kind are not often complete, nor do the components meet accurately in the edges $[a_2a_3]$, $[m_2m_3]$, &c., as shown in Fig. 444. The presence of other faces m and $e\{101\}$ near these edges much facilitates the detection of the twin character. The trapezohedral faces a_{12}, &c. are generally very even and smooth, and show little or no trace of their composite character; but one of the specimens in the Cambridge Museum shows a well-defined re-entrant angle. Unfortunately the faces of this specimen are too dull to admit of measurement, and it is thus impossible to ascertain whether the faces meeting in the edge make the angle 12′ between a_8 and a_1, which regular twinning from 1 to 8 requires.

Nest-like octets of the black variety (nigrine) found in Arkansas, U.S.A., also occur, in which the twin-axes have the same arrangement and lie in a zone $[mT]$. The difference in habit arises from the fact that the faces developed belong to the ditetragonal prism $\{210\}$—the form denoted by the letters l in Fig. 441. The faces l are deeply striated parallel to the principal axis, and these axes zig-zag across the plane $[mT]$ containing the twin-axes in the same way as the edges $[m_1a_1]$, &c., in Fig. 444. The prisms are usually slender, so that a depression is left in the middle of the twin.

25. Multiple twins of cassiterite in which individuals are twinned about homologous faces e of the pyramid $\{101\}$ also occur; but the twin-axes of the successive components rarely remain in the same plane. A frequent habit is that in which a prism $m\{110\}$ forms the larger portion. Suppose now a cup-like depression to be formed at one end of this prism by the four pyramid faces e parallel to those which in the simple crystal form the salient pyramid at the opposite end. Along each face of the tetragonal cup a small individual is united in twin-orientation with the larger prism. Four individuals are similarly united to the prism at the other end. We

thus get a twin of nine individuals; the four small individuals at each end are not in twin-association with one another, nor with those at the other end; but the pair of components situated at opposite coigns of the prism are in like orientation, and may, if they touch, be regarded as portions of one individual. Complete twins of this kind are very rare.

26. *Zircon.* Although the crystals of zircon are isomorphous with those of rutile and cassiterite, it was not known to twin until Herr O. Meyer discovered microscopic twins in rock-sections (*Zeitsch. d. Deutsch. geol. Ges.* XXX, p. 10, 1878). Shortly afterwards macroscopic twins, similar in habit to the doublet represented in Fig. 438, were discovered in Renfrew Co., Canada; and were almost simultaneously described by Mr. L. Fletcher (*Phil. Mag.* [v], XII, p. 26, 1881) and Mr. W. E. Hidden (*Am. Jour. of Sci.* [iii], XXI, p. 507, 1881). The twin-face is, as in the twins of rutile and cassiterite, a face of the pyramid $\{101\}$, and the principal axes of the two portions are inclined at 65° 16′ to one another. A similar twin and five other twins of zircon from Henderson Co., N. Carolina, U.S.A., have recently been described by Messrs. Hidden and Pratt (*Am. Jour. of Sci.* [iv], VI, p. 323, 1898); to whose courtesy and that of the Editor of the *Journal* I am indebted for the following figures illustrating the twins. The five twins are intercrossing doublets in which the twin-face is parallel to a face of different pyramids of the series $\{hhl\}$: a face m of both individuals is coplanar in all of them.

Fig. 446 represents an intercrossing doublet having $(01\bar{1})$ for

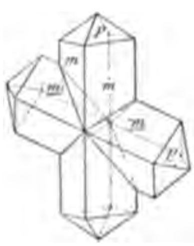


FIG. 446.

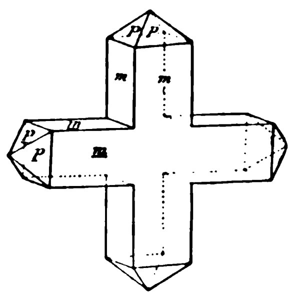


FIG. 447.

twin-face: the forms are $m\{110\}$ and $p\{111\}$. The twin from Renfrew Co. has the same twin-law, but the individuals do not

cross: it can be drawn by omitting the prolongations beyond the combination-plane of the individuals in Fig. 446.

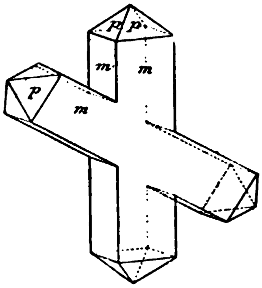


Fig. 448.

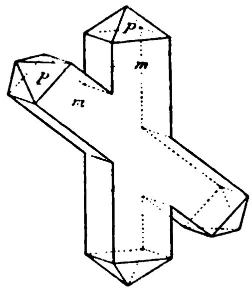


Fig. 449.

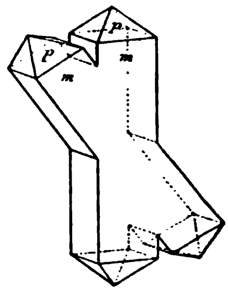


Fig. 450.

In Fig. 447 a pair of faces p, parallel to the combination-plane, and also a pair of faces m perpendicular to the former are co-planar. The normal to the prism-face m being a dyad axis, the twin-axis is the normal to the face $p(1\bar{1}1)$, and the combination-plane is parallel to this pyramid face. Since $Cp = 90° - mp = 42° \; 10'$, the two principal axes, given by the prism-edges $[mm]$ and $[\underline{mm}]$, are inclined to one another at angles of 84° 20′ and 95° 40′.

Fig. 448 represents a similar intercrossing doublet in which the angle between the edges $[mm]$ and $[\underline{mm}]$ was found by measurement to be 112° 23′ and 67° 37′. The authors take a face $(\bar{5}53)$ for twin-face, which corresponds with an angle between the tetrad axes of 112° 57′.

In Fig. 449 the twin-face is $(\bar{2}21)$, the calculated angle between the principal axes being 122° 12′ and the measured 122°.

In Fig. 450 the twin-face is $(\bar{3}31)$; the calculated and measured angles between the principal axes are 139° 35′, and 139° 10′.

The authors have not given a figure of the twin in which $(\bar{7}74)$ is twin-face. The corresponding angle between the principal axes is 115° 30′; and their measurements gave 115° 49′.

27. *Copper pyrites.* The simple crystals of this substance were described in Chap. XIV, Art. 19. The twins are combined according to three different laws: (*1*) juxtaposed and interpenetrant twins having the normal (111) for twin-axis, and in the former case the face (111) for combination-plane; (*2*) complementary interpenetrant twins geometrically very like the twinned tetrahedra, Fig. 435; (*3*) juxtaposed symmetric twins with a face of the pyramid {011} for combination-plane.

1. *Twin-face* $(11\bar{1})$. Since the angles of the sphenoids, $o=\kappa\{111\}$ and $\omega=\kappa\{11\bar{1}\}$, are very nearly the same as those of the regular tetrahedron, a simple octaid formed of the two sphenoids will closely resemble an octahedron: but since the physical characters of the faces of the complementary sphenoids are different, the octaid will resemble the crystals of blende in which both tetrahedra occur in nearly equal development. Suppose such an octaid to be bisected by a plane parallel to $(11\bar{1})$ and the lower half to be turned through 180° about the normal to the plane of section, and then attached to the fixed half in this plane. An asymmetric twin, closely resembling that of blende described in Art. 9, is obtained. Geometrically, the twin is very like that of spinel, Fig. 420: but the difference in physical character of the faces meeting in salient and re-entrant angles in the combination-plane, is often well seen in the twins. The twinning is often repeated parallel to the same or to different faces of the sphenoid.

2. A very rare complementary twin, formed of two interpenetrating sphenoids o, having the principal axes coincident, and similar in general appearance to the twinned tetrahedra, Fig. 435, has been described by Haidinger. The horizontal edges of the sphenoids cross, as do those of the tetrahedra in sodium chlorate, at right angles; but the other intercrossing edges are not exactly at right angles. The coigns are truncated by small faces ω of $\kappa\{11\bar{1}\}$. In this twin the axis of rotation may be taken to be a normal to any face of the prism $\{110\}$ with an angle of rotation of 180°, or the principal axis with an angle of rotation of 90°.

3. *Symmetric twins*, in which the individuals are united in a plane parallel to a face of the tetragonal pyramid $\{101\}$ were also described by Haidinger (*Mem. Wern. Nat. Hist. Soc. Edin.* IV, p. 1, 1822). In these twins the sphenoidal faces of the several individuals meeting in edges in the combination-plane, are physically similar. The exact relation of the two individuals was, however, only recently established by Mr. Fletcher (*Phil. Mag.* [V], XIV, p. 276, 1882). Suppose an octaid, to be made up of the faces o of $\kappa\{111\}$ and ω of $\kappa\{11\bar{1}\}$ in equable development; and suppose it to be divided by a central plane parallel to the pyramid-face $(0\bar{1}1)$ which truncates the edges $A_{,}C$ of Fig. 451. A semi-revolution of the lower half about the normal TT to the plane of section brings dissimilar faces o and ω to intersect in this plane, in the same way as occurs in the juxtaposed twins according to the first law. In the

composite figure the four faces which have $[CA_{,}]$ for zone-axis are single, and the four other faces are each composed of portions of dissimilar faces inclined to one another at very small salient and re-entrant angles. When the angle $\omega\omega'$ is 108° 40′, the crystal-element $= \wedge\, COT$ is 44° 34·5′: the angle To is 89° 18·25′, and the salient angle $o\underline{\omega}$ and the re-entrant angle $o_{,}\underline{\omega}'$ are both 1° 23·5′. The salient and re-entrant angles between the corresponding portions of faces at the back of the figure have the same values. A composite crystal, similar to the ideal one shown in the figure, is an asymmetric twin with $(0\bar{1}1)$ for twin-face: one instance of it from the Junge Hohe Birke Mine, Freiberg, has been described by Sadebeck.

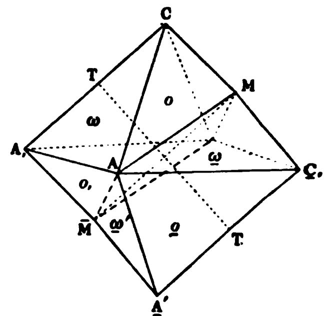


FIG. 451.

The orientation of the two portions in the above composite figure is also given by a semi-revolution about the line $M\overline{M}$, which is the zone-axis parallel to the edge $[CA_{,}]$.

But in the twin represented by Fig. 452 the faces meeting in the edges $[\omega\underline{\omega}]$, $[o\underline{o}]$ are like faces; and the twin is physically as well as geometrically symmetrical with respect to the combination-plane through these edges. Such an orientation of the portions composing the doublet can be obtained by taking two equal octaids placed with the faces o and ω of the one similarly situated respectively to the faces o and ω of the other. One of the octaids is then turned through 90° about its principal axis, when its faces ω are brought into a position occupied by the dissimilar faces o of the fixed one, and vice versâ. Suppose both octaids to be now bisected by planes $MA\overline{M}$, parallel to the face truncating the similarly placed edge $[CA_{,}]$; and let the lower half of the rotated octaid be again turned through 180° about the line $[TT]$ perpendicular to $(0\bar{1}1)$, and then united in the plane of section to the upper half of the fixed octaid. The composite figure produced will be geometrically the same as before; but the faces o and ω of the

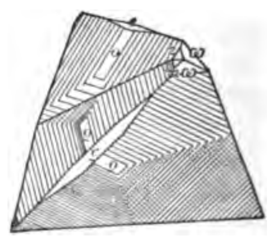


FIG. 452.

rotated portion are now interchanged. Hence the faces meeting in the salient and re-entrant edges, such as $[AM]$ and $[A\bar{M}]$ of Fig. 451, are now like faces, and a doublet, of which Fig. 452 is an instance, is produced. A doublet from Pool Mines, Cornwall, measured by Mr. Fletcher gave angles $\omega\omega = o\underline{o} = 1° \ 23'$, angles agreeing accurately with those required by the angular element $44° \ 34{\cdot}5'$. On another specimen the angles were found to be $2° \ 3'$; and these values agree exactly with those required by the measured tetrahedral angle $\omega\omega' = 108° \ 17{\cdot}25'$. This variation in the tetrahedral angle may be due to a variation in chemical composition. Two analyses of a portion of the twin were made, one agreed fairly with a composition $2CuFeS_2 + FeS_2$, the other with a composition $4CuFeS_2 + FeS_2$; but no great stress can be laid on this difference owing to the presence of iron pyrites, which could be detected by examination with the microscope.

The representation of the twin-orientation by axes of rotation requires, if angles of 90° and 180° are alone admissible, two axes and two different angles; viz. (*a*) a rotation of one portion of a crystal through 90° about its principal axis, and (*b*) a semi-revolution of the portion which is to be united to a portion of the fixed one about the normal to a face of the pyramid $\{011\}$. No other instance of this kind is known, and the explanation by axes of rotation is unsatisfactory. It is true that, as follows from Euler's theorem of successive rotations (Chap. IX, Art. 13), the above rotations are equivalent to a single rotation of $240° \ 29{\cdot}4'$ or $119° \ 30{\cdot}6'$ about the normal to a face $(1\bar{1}1)$ or $(\bar{1}\bar{1}1)$; but such an angle of rotation is unknown amongst twins.

The angle is readily found from the right-angled spherical triangle $ce_{,}\omega$ of Fig. 453, in which $\wedge e_{,}c\omega = 45°$ is one-half the angle of rotation about the principal axis, and $\wedge ce_{,}\omega = 90°$ is half the angle of rotation about the twin-axis. Hence ω $(1\bar{1}1)$ is the extremity of the equivalent axis of rotation; and

$$\cos e_{,}\omega c = \sin 45° \cos (ce_{,} = 44° \ 34{\cdot}5').$$

Therefore the internal angle $e_{,}\omega c$ at ω is $59° \ 45{\cdot}3'$. Twice the external angle is $240° \ 29{\cdot}4' = 360° - 119° \ 30{\cdot}6'$. The rotated portion is in the same position as if it had been turned once about the normal $(1\bar{1}1)$ through $119° 30{\cdot}6'$.

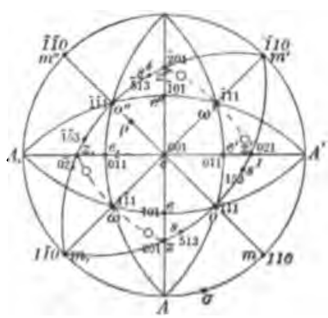


Fig. 453.

The twin-orientation is completely defined by the statement that the two portions are physically and geometrically symmetrical with respect to the combination-plane.

The twinning according to this law is often repeated, and figures somewhat resembling octrahedra can be obtained in which each face consists of portions of three like faces which are nearly, but not quite, co-planar. The portions of these composite faces are separated by grooves at which well-marked striæ meet in a manner similar to that shown between o and $\underline{o}$ in the doublet, Fig. 452.

B. *Twin-axis the normal to a face of* $\{100\}$ *or* $\{110\}$.

28. *Scheelite.* This substance affords good examples of simple juxtaposed twins and of complementary interpenetrant twins having the same axis of rotation. In these twins, which have been ably described by Professor Max Bauer (*Jahr. Ver. Württ.* p. 129, 1871), the axis of rotation may be taken to be parallel to any one of the horizontal edges of one or other of the tetragonal pyramids $e\{101\}$ and $o\{111\}$ (see Fig. 209, p. 262). For the sake of simplicity, we shall suppose the horizontal edge of $e\{101\}$, which was selected as the axis of X, to be the twin-axis.

Fig. 454 is a basal plan of two crystals having the faces given in p. 262 and placed side by side in twin-orientation. The two crystals are then reciprocal reflexions in a plane through the common edge parallel to the principal axes.

The individual to the right can be brought into a like orientation with that on the left by a semi-revolution about the common edge or about any of the edges in which the faces e and o meet the plane

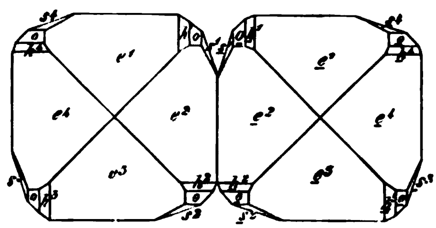


FIG. 454.

of the paper: it can also be brought into a like orientation by a quarter-revolution about its principal axis. As far as the twin-orientation is concerned, it is immaterial which of the axes is

selected, provided the rotation is 180° if about a horizontal axis, and 90° about the principal axis.

The juxtaposed twins are sometimes combined in a plane which is approximately the plane through OX and the principal axis. The faces *e* meeting in the combination-surface are co-planar; but the irregular line of separation traversing the face is easily recognised, being the axis of more or less strongly marked barbed striæ parallel respectively to the edges $[e^3s^3]$ of the two portions. Occasionally an indentation, formed by faces *o* and *s* of the two individuals, is seen where the line of separation of the portions e^3 and $\underline{e}^3$ meets their horizontal edge.

Fig. 455 is a plan on the common basal plane of an interpenetrant twin consisting of eight segments which are united together along vertical planes in the same way as the doublet. As in that twin, the faces *e* divided by interrupted lines are co-planar and frequently show fine barbed striæ meeting in the line of separation. The portions meeting in the polar edges of the geometrically simple pyramid *e* are also in twin-orientation to one another; and this is sometimes proved by the way in which the pyramid-coigns are modified by faces *h*, *o*, and *s* belonging to the forms $\pi\{313\}$, $\{111\}$ and $\pi\{131\}$. The faces *o* at each coign are co-planar; and the faces *s* form an indentation similar to that occasionally seen in the middle of the horizontal edge of a face *e*. The surfaces of separation are not, as a rule, true planes, and the lines of separation are somewhat irregular. Twins of this kind are not common; they are principally found in Schlaggenwald, Bohemia.

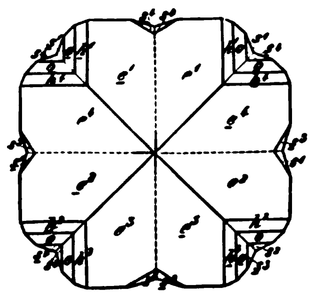


FIG. 455.

Professor Bauer describes twins of the same kind united along a surface more or less approximately parallel to the base. Further, the relative dimensions of the individuals are very unequal; and it often happens that a mere corner is, as it were, cut out of a crystal and replaced by an equal amount of matter in twin-orientation.

iii. TWINS OF THE PRISMATIC SYSTEM.

29. In crystals of this system the twin-axis is commonly the normal either of a prism $\{hk0\}$, or of one of the domes $\{0kl\}$ or $\{h0l\}$;

and twins of this kind are especially frequent when the faces of the form make angles of nearly 60° and 120° with one another. The twins are often juxtaposed with the plane perpendicular to the twin-axis for combination-plane: the twin-orientation may then be given by stating the twin-face.

As instances of substances, having for twin-face a face *m* of a prism {110} with angles of nearly 60°, we have: redruthite, Cu_2S, $mm_, = 60° 24'$; mispickel, FeAsS, $mm_, = 68° 13'$; stephanite, Ag_5SbS_4, $mm_, = 64° 21'$; aragonite, $CaCO_3$, alstonite, $(Ca,Ba)CO_3$, witherite, $BaCO_3$, strontianite, $SrCO_3$, and cerussite, $PbCO_3$, in which $mm_,$ varies between the limits 63° 48′ and 62° 15′; potassium sulphate, K_2SO_4, and ammonium sulphate, $(H_4N)_2SO_4$, $mm_, = 59° 36'$ for both.

As instances of domes with angles of nearly 60° between the faces, one of which is perpendicular to the twin-axis, we may mention: marcasite, FeS_2, $101 \wedge 10\bar{1} = 63° 40'$; mispickel, $101 \wedge 10\bar{1} = 59° 22'$; manganite, $Mn_2O_3 . H_2O$, $011 \wedge 0\bar{1}1 = 57° 10'$; chrysoberyl, $BeAlO_4$, $011 \wedge 0\bar{1}1 = 60° 14'$ and $031 \wedge 03\bar{1} = 59° 46'$. Kokscharow gave *i* (011) as the twin-face of chrysoberyl, Hessenberg and Cathrein make ρ (031) the twin-face. The angle $\rho_, i = 03\bar{1} \wedge 011 = 89° 46'$; and it is not easy to distinguish between the two representations of the law of twinning, when the crystals are developed, as is frequently the case, in large triplets or sextets, which do not admit of accurate measurement, and are not sufficiently translucent to enable us to determine the optical orientation of the several portions.

Another class of twins similar to the preceding is that in which the twin-axis is perpendicular to a face of a prism or dome, the faces of which are inclined to one another at angles of nearly 90°. Thus, bournonite forms multiple twins, like Fig. 456, with *m* (110) for twin-face, $mm_,$ being 86° 20′.

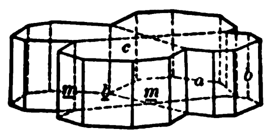


FIG. 456.

Staurolite, $H_4(Fe,Mg)_6(Al,Fe)_{24}Si_{11}O_{66}$ (?), is another instance.

Twins with the normal to a pyramid-face for twin-axis are rare. In Art. 41 we shall describe one of staurolite, in which a normal to (232) is the twin-axis.

Aragonite group.

30. The determination of a doublet of *aragonite*, such as those shown in Figs. 457 and 458, is easy; for the prism-edges are all parallel, and the basal planes of the two portions are, when present,

co-planar. The twin-axis is therefore parallel to the base; for the normal to the base is a dyad axis, a semi-revolution about which leaves a crystal in an orientation identical with the first. Again, it is easy to see in the crystal, as is evident from the basal plan, Fig. 457, that two prism-faces $m_,$ and $\underline{m}'$ are parallel: the axis of semi-revolution is therefore perpendicular to these faces. Further, since the crystals are, as a rule, translucent, it is often possible to see, by total reflexion from a plane surface traversing the doublet, that the individuals are combined along a plane parallel to $m_,$. Hence the doublet has the face $m_,(1\bar{1}0)$ for twin-face.

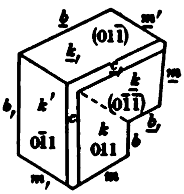


FIG. 457.

In the zone $[m_,m\underline{m}]$ we can measure the following angles; and confirm the accuracy of the determination of the twin-face made by inspection. $mm_, = \underline{m}\underline{m}' = 63° 48'$, $bm = b_,m_, = \underline{b}\underline{m}' = \underline{b}_,\underline{m} = 58° 6'$, all angles of the simple crystal. Again, we have the following angles between faces on different individuals:

$$b_,\underline{b} = 180° - 2 \times 58° 6' = 63° 48' = -b\underline{b}_,,$$

$$m\underline{m} = 180° - 2 \times 63° 48' = 52° 24',$$

and $$m\underline{b}_, = \underline{m}b = -5° 42'.$$

Provided the faces can be distinguished with certainty, measurement of any one of these angles suffices to give the positions of all the faces, and of the twin-axis.

The faces k and $\underline{k}$ of the domes $\{011\}$ make at each end of the prism a salient and re-entrant angle, both of equal magnitude. Since $m_,k' = 72° 1'$, $k'\underline{k}_, = -k\underline{k} = 180° - 2 \times 72° 1' = 35° 58'$.

31. To draw the basal plan, Fig. 457, two lines intersecting in O at right angles are taken in the paper, and convenient lengths OA, $OA_,$, and OB, $OB_,$ are measured off on them in the ratio $a : b$. A series of lines parallel to OA are drawn at distances apart corresponding fairly with the width of the faces k, k' and c, when the crystal is viewed endwise. Lines parallel to AB and $AB_,$ give the traces m, $m_,$; and complete the simple crystal. This is then bisected by a vertical plane passing through O parallel to $m_,$: the lines of section are shown in the edges $[k'\underline{k}_,]$, $[k\underline{k}]$.

To get the rotated axes, the twin-axis is drawn through the origin perpendicular to $BA_,$, and produced to an equal distance beyond $A_,B$. The point Ω so found is the origin of the rotated axes,

which are given in direction and magnitude by $\Omega A_{,}$, ΩB, and a length c on the vertical.

From the points in the trace of the bisecting plane the edges $[\underline{b}\underline{k}_{,}]$, $[\underline{k}\underline{b}_{,}]$, &c., are drawn parallel to $\Omega A_{,}$ to meet lines from the coigns bmk, &c., parallel to the twin-axis. Through the points so found, the traces $\underline{m}$, $\underline{m}'$ are now drawn, completing the figure.

To make the clinographic drawing of the doublet, Fig. 458, the cubic axes are projected in the way described in Chap. VI, Arts. 22 and 23, and lengths are measured off on them in the ratios $a : b : c$, as described in Chap. VI, Art. 14. The simple crystal represented by unbarred letters can then be drawn, and bisected by a central plane parallel to the twin-face $m_{,}(1\bar{1}0)$. The directions of the rotated axes of X and Y of the doublet are found in the same way as the rotated axes of Z and Y were found in Art. 22. The points B and $A_{,}$ on the axes of Y and X are joined, and a point M in $BA_{,}$ determined, where $BM : MA_{,} = \cot^2 F : 1 = \cot^2 31° 54' : 1$. The line OM is produced to Ω where $\Omega M = MO$. The rotated axes of X and Y are $\Omega A_{,}$ and ΩB; the axis OZ remains vertical. Lines are now drawn through the points in which the edges of the fixed portion meet the combination-plane parallel to $\Omega A_{,}$. They give the edges $[\underline{b}_{,}\underline{k}]$, $[\underline{b}_{,}\underline{k}']$, &c., of the rotated portion. The coigns on these edges, such as $\underline{b}_{,}\underline{m}\underline{k}$, are the points in which the edges meet lines parallel to the twin-axis drawn through the corresponding coigns of the fixed portion. The figure can then be completed by drawing the vertical edges.

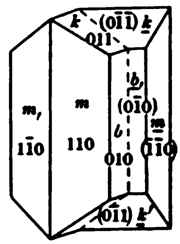


FIG. 458.

32. Multiple twins often occur, in which the same normal is the twin-axis of all the individuals. Three individuals thus twinned are shown in Fig. 459, in which individual II appears as a thin plate sandwiched between the two outer individuals which necessarily have the same orientation. Very thin lamellæ twinned in this way may be often perceived in apparently simple crystals by the linear interruptions which they produce on the faces k and b. They may also be discerned in sections by internal reflexions and by the confusion they produce in the optical phenomena seen in a polariscope.

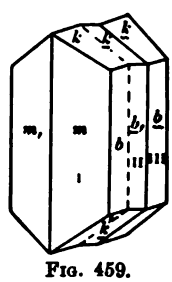


FIG. 459.

If four individuals are thus twinned the two outer portions will be in twin-orientation, and the twin will only differ from a doublet in having two twin-plates interposed. The process may be further repeated in a similar manner, so that the outer individuals are in similar orientation when the number of individuals is odd; and in twin-orientation when the number is even.

33. When the individuals in a doublet are continued beyond the twin-face, *i.e.* intercross, they are arranged in the way shown in the plan, Fig. 460. In this diagram the combination-plane is projected in the line $aa_{,}$; and there are two re-entrant angles $k\underline{k}$ and $k'\underline{k}_{,} = -35° 58'$ over this plane. The vertical plane containing the twin-axis $TT_{,}$ is not parallel to a possible face; and the re-entrant edges in which the faces $k\underline{k}_{,}$ and $\underline{k}k'$ meet are not straight and regular. The plane approximates to the face (130), for $b \wedge 130 = 28° 10'$ and $m_{,} \wedge 130 = 93° 44'$.

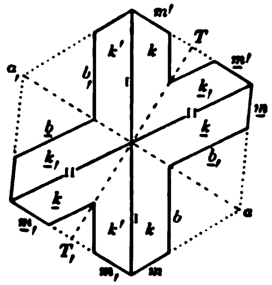


Fig. 460.

Were the intercrossing doublet formed of individuals having the forms {110} and {001} only, and were the prism-faces m so extended as to meet at the points $TT_{,}$ and $aa_{,}$ we should have a pseudo-hexagonal prism having four angles $mm_{,} = 63° 48'$ and two angles at a and $a_{,}$ each equal to $52° 24'$. The bases would be co-planar for they are parallel to the twin-axis: they would each consist of four segments divided by the vertical planes through $TT_{,}$ and $aa_{,}$. Usually, however, such pseudo-hexagonal composite crystals are formed of three, or more, individuals; and as shown in Fig. 461, the combination may occur very irregularly. In this figure the twin is regularly developed for five portions; but the sixth is replaced by two smaller individuals, which are in twin-orientation to the adjacent larger portions, but are not in regular association with one another.

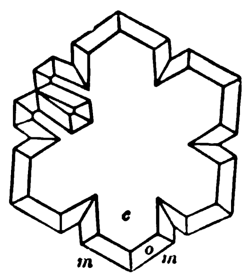


Fig. 461.

34. *Witherite* and *alstonite.* These crystals are all multiple twins conforming to the same laws of development as the pseudo-

hexagonal twins of aragonite. Witherite is usually in short pseudo-hexagonal prisms terminated by one or two pseudo-hexagonal pyramids. The section, Fig. 462 (after Des Cloizeaux), is cut perpendicularly to the common vertical axis. It shows that, in its main features, the crystal is an intercrossing triplet, in which the faces forming the hexagonal prism all belong to the pinakoid $b\{010\}$. The series of lines marked m indicate that each individual of the triplet is traversed by numerous lamellæ, each series being parallel to the same twin-face m of the form $\{110\}$ in the manner illustrated by Fig. 459. The lines connecting two circlets show the directions of the plane of the optic axes of the main portion of the segment in which each of them lies.

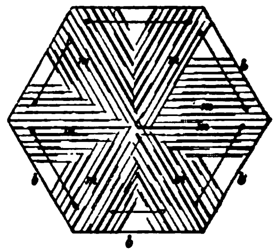


FIG. 462.

The crystals of alstonite appear as acute hexagonal pyramids with striated faces, on each of which an interruption, approximately in a plane corresponding to that marked $TT_{,}$ in Fig. 460, shows the composite nature of the face. A section parallel to the base shows that the crystal is an intercrossing sextet consisting of twelve segments. Each segment is bounded by a well-defined line, which lies in the vertical plane containing opposite edges of the pyramid. But the dividing lines joining the middle points of opposite horizontal edges are irregular. The line of extinction between crossed Nicols is in each segment inclined at nearly 30° to the horizontal edge of the pyramid, and the plane of the optic axes is parallel to (100) and Bx_a to OZ. The composite faces belong to a prismatic pyramid of the series $\{hhl\}$, and not, as in witherite, to a dome $\{0kl\}$.

35. *Cerussite.* Intercrossing triplets with $m(110)$ for twin-face are common, and often have the habit shown in Figs. 463 and 464. The faces $b\{010\}$ are often striated horizontally; and the shading in Fig. 463 has been employed to show the depth of the re-entrant angles.

In Fig. 463 the unlettered portions to the right and left belong to a single crystal having its face $b(010)$ parallel to the paper: the portion carrying $\underline{p}$ and $\underline{m}$ is twinned to this crystal about the normal to the prism-face lying to the right; and as shown by the absence of dividing lines between the pyramid-faces, the prism- and pyramid-faces to the right are co-planar. Similarly, the portion bearing p

and m is twinned to the unlettered crystal about the normal to the prism-face to the left; and the prism- and pyramid-faces to the left

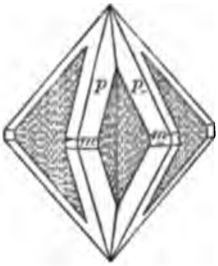


FIG. 463.

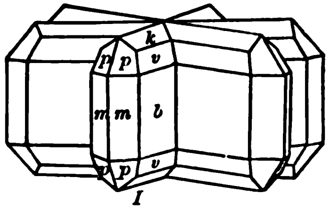


FIG. 464.

are co-planar. But the portions bearing letters are not in twin-orientation with one another; for the prism-angle $mm_{,} = 62° 46'$. Hence $m \wedge \underline{m} = -8° 18'$; and $p \wedge \underline{p} = -6° 44'$, $Cp = 001 \wedge 111$ being $54° 14'$.

In Fig. 464 the faces k belong to {011}, and v to {031}.

36. *Potassium sulphate.* Twins of this substance having much the appearance of hexagonal prisms and pyramids, similar to those of aragonite and witherite, are obtained by evaporation from aqueous solution. Sections of them are easily made, so that the character of the twin-orientation can be determined by the examination of plates in plane-polarised light.

Fig. 465 represents an actual doublet which was fully determined in the Cambridge Museum. The forms observed were: o {111}, a {100}, m {110}, f {130}, and two or three faces irregularly developed and alternating, so as to form step-like grooves where they met the combination-plane: they were q {011} with {021} and possibly {031}. The faces o, a, m and f were smooth and bright, and gave good images. The face f, $\underline{f}$ is composite, and the two portions make a very small salient angle with one another. The principal angles are: $am = 29° 48'$, $mf = 30° 0'$, $fm_{,} = 89° 36'$, $om = 33° 39'$ and $oo''' = 48° 52'$. Hence $a : b : c = \cdot5727 : 1 : \cdot7494$.

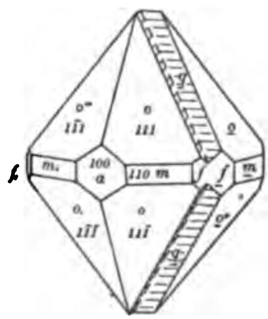


FIG. 465.

The plane of the optic axes is (100) and Bx_a is parallel to OZ.

Hence a plate, ground down on roughened glass perpendicularly to the edges [*amf*], gives an optic figure in convergent polarised light, in which the line joining the 'eyes,' shown in the diagrams by a line joining two circlets, is perpendicular to the edge [*bc*].

1. In the doublet the twin-axis is the normal to $(1\bar{1}0)$, and the two individuals are combined parallel to this face.

From the angles given above for the simple crystal, we can compute the angles $m\underline{m}$, $f\underline{f}$ and $o\underline{o}$: they are $m\underline{m} = 180^\circ - 2m\underline{m}_{,} = 60^\circ 48'$, $f\underline{f} = 180^\circ - 2 \times 89^\circ 36' = 48'$, and $o\underline{o} = 180^\circ - 2m_{,}o = 49^\circ 49{\cdot}5'$. The salient angle $f\underline{f}$ was found by measurement to give values varying between 35′ and 46′; the other measured angles agreed well with the theoretical values.

A section of the crystal, Fig. 466, was carefully prepared, leaving as far as possible the prism-faces and portions of the pyramid-faces intact, so that the position of the plane of the optic axes could be accurately correlated with the external form. The directions of the planes of the optic axes are shown by the lines joining circlets, and prove the twin-face to be $m_{,}$ $(1\bar{1}0)$: the trace of the combination-plane is shown by a line of alternating strokes and dots.

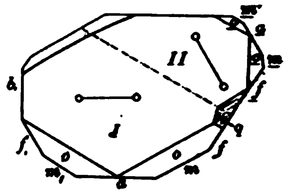
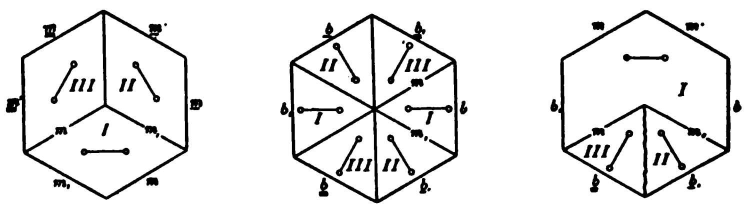


FIG. 466.

The triplets are sometimes very regular, and closely resemble a hexagonal prism terminated by a hexagonal pyramid. When sections of such triplets are made, several different arrangements of the component individuals are found.



(*a*) (*b*) (*c*)

FIG. 467.

i. The hexagonal section may consist of three rhombic segments, Fig. 467 (*a*), the combination-planes passing through alternate edges of the hexagonal prism and meeting in a central axis. The faces

of the pseudo-hexagonal prism are all prism-faces $m\{110\}$, and the pyramid is formed by o faces. The plane of the optic axes is in each segment parallel to the longer diagonal of the rhombus.

ii. The hexagonal section may consist of six similar triangular segments, Fig. 467 (*b*), and the combination-planes pass across the central axis and through opposite edges of the prism. The plane of the optic axes in opposite segments is coincident, and is, as shown by the line of eyes, perpendicular to the face of the bounding prism. These prism-faces must therefore all belong to the pinakoid $b\,\{010\}$, which is at right angles to the plane of the optic axes.

iii. Another common arrangement is shown by the section, Fig. 467 (*c*), in which two-thirds of the section belongs to a single crystal, to which two other small segments are twinned. Four of the external faces of the hexagonal prism belong to the pinakoid, and two to the prism m; but the prisms do not as a rule give very regular hexagons, although, as shown in the diagram, the faces m and b are sometimes very nearly equal in size. As indicated by the irregular line the two inserted segments are not in twin-combination with one another. The re-entrant angle $mm_{,}$ within the large individual is $180° - 59°\,36' = 120°\,24'$: if then two wedges, bounded by m faces, are inserted, the joint angle is $2 \times 59°\,36'$, and a thin wedge having an angle of $1°\,12'$, would be left vacant in the place of the irregular line. The filling up of this gap takes place irregularly and produces a want of distinctness in the optic character in its neighbourhood. The above and other sections made by the students in the Cambridge Museum were all found to obey the same twin-law.

2. Scacchi gives for potassium sulphate a second twin-law in which the twin-face is $f(130)$; and he states that the twin occurs frequently, when a small amount of sodium sulphate is added to the solution of potassium sulphate. Many efforts were made by the author to obtain such twins, but without success. In sections of regular sextets or intercrossing triplets according to this law, we should have a hexagon bounded by faces m as in Fig. 467 (*a*); but the combination-planes bisect opposite sides of the hexagon forming trapezia, which each include one of its corners. The planes of the optic axes in these segments are perpendicular to the line joining the edge $[mm_{,}]$ to the centre. In a doublet of this kind faces $m_{,}$ and $\underline{m}'$, o' and $\underline{o}_{,}$ become almost co-planar. On one side we should have a salient angle $m_{,}\,\underline{m}' = 48'$, and on the opposite side a re-entrant angle $m'\underline{m}_{,} = -48°$; and the adjacent faces o make similar angles of $40'$.

37. *Mimetite.* The twins of witherite and potassium sulphate which have been described in the preceding Articles enable us to explain the mimetic twins of mimetite, which so closely resemble crystals of the hexagonal system that their true character was only discovered in 1881. The common habit of the crystals is shown in Fig. 468, and the crystals are so perfect that they were regarded as an undoubted member of the isomorphous series of hexagonal crystals of class II formed by apatite, pyromorphite and vanadinite; and this belief was supported by the fact that the chloro-phosphate, pyromorphite, and the chloro-arsenate, mimetite can only be distinguished by analysis, and are often intermingled in one crystal. M. Bertrand (*Bull. Soc. franç. de Min.* IV, p. 36, 1881) discovered that sections, cut parallel to the face *c*, are composed of six triangular segments similar to those of witherite and potassium sulphate shown in Figs. 462 and 467 (*b*), respectively. In the crystals from Johanngeorgenstadt, Saxony, the plane of the optic axes in each segment is parallel to its external side, *i.e.* to the face *m* of the pseudo-hexagonal prism, and the section resembles that of witherite without the twin lamellæ; the acute bisectrix is parallel to the prism-edges [*mm*], and the angle of the optic axes is about 64° in air. The crystals are therefore sextets of the prismatic system similar to those of witherite and potassium sulphate, having a face of the pinakoid {010} for external face, and twinned about a face (110); the angle 010 $\wedge$ 110 being 60°, or very nearly 60°.

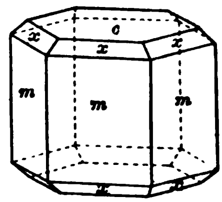


FIG. 468.

The crystals from Roughten Gill in Cumberland containing both phosphoric and arsenic acids were found to consist of a central prism of pyromorphite which is uniaxal, and of a surrounding zone of mimetite formed of six biaxal segments. The planes of the optic axes in these segments appear to be parallel to the diagonals of the hexagonal section, and not to the external sides of the segments.

A plate cut perpendicularly to the prism-edges of a crystal from Wheal Alfred, Cornwall, was found by the author, on examination in the polariscope, to be irregularly divided into segments, in which the light is extinguished in directions making nearly 12° with the prism-edges. In convergent light the hyperbolic brushes of a biaxal plate having a small angle between the optic axes are seen. Further,

the main segments are traversed by narrow plates in a manner similar to that in the section of witherite shown in Fig. 462.

38. *Chrysoberyl.* The orientation of a prismatic crystal adopted in its representation is arbitrarily chosen, and it happens that chrysoberyl has been so placed that the twin-axis is perpendicular to a face of a dome $\{0kl\}$, which may equally well be placed with its edges vertical, when it would be called a prism. Consequently, in the intercrossing triplets and sextets of chrysoberyl, Figs. 469 and 470,

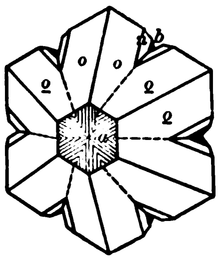


FIG. 469.

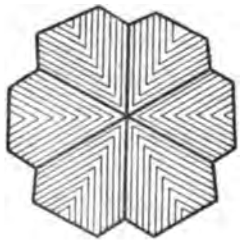


FIG. 470.

the twin-face is (031), although they are similar to the twins of witherite, alstonite and potassium sulphate discussed in the preceding Articles. In Fig. 469 the faces shown are; *a* {100}, *o* {111}, *n* {121}, *b* {010}. The faces *n* and *b* make in some cases indentations in the pyramid-edges as shown in the figure.

Now $010 \wedge 031 = 29° 53'$, $010 \wedge 011 = 59° 53'$, $a \wedge o = 43° 8'$, and $111 \wedge 1\bar{1}1 = 40° 7'$: hence $031 \wedge 0\bar{1}1 = 90° 14'$ and $031 \wedge 1\bar{1}1 = 90° 9'$. If then (031) is the twin-face, the portions *o* and $\underline{o}$ separated by lines of interrupted strokes are not co-planar, but include a re-entrant angle of 18′; and the pyramid-edges in the paper include a re-entrant angle of 28′. If the face (011) is the twin-face, then *o* and $\underline{o}$ are co-planar, and the pyramid-edges are co-linear.

Fig. 470 represents an intercrossing sextet with faces *a* {100} and *i* {011} alone developed. Each segment of the pinakoid having barbed striæ is a doublet with (031) for twin-face; and these doublets are united along faces of {011}.

39. *Mispickel* and the isomorphous glaucodote, (Co, Fe) AsS, afford good examples of substances twinned according to two different laws. In one a face *m* of {110} is the twin-face; in the other the twin-axis is perpendicular to a face *e* of {101}, the individuals

generally interpenetrating as shown in Fig. 472, and the twins being sometimes juxtaposed sextets.

Fig. 471 represents a large sextet or intercrossing triplet of mispickel from Cornwall in the Cambridge Museum, in which m(110) is the twin-face. The individuals having symbols in parentheses are twinned to the fixed individual, *i.e.* to that having unmodified letters and symbols, with $1\bar{1}0$ for twin-face; the individuals having their symbols in brackets are twinned to the fixed individual with 110 for twin-face. The basal pinakoid is common to all the individuals, and the lines of separation in the combination-planes are shown by straight lines. The individuals having differently modified symbols are not in twin-association with one another; and this is indicated in the figure by an irregular line of separation. The faces c of the form {001} and u of {014} are much striated and the edges [cu] are not so well defined as is implied in the figure.

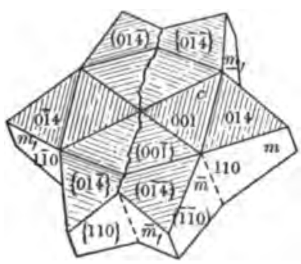


FIG. 471.

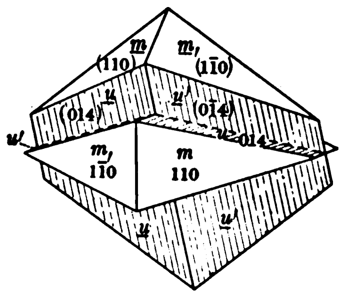


FIG. 472.

To show the appearance of the twin more clearly, the vertical axis of the twin has been put in the position $O\delta'$ of the cube, Fig. 226, p. 284: the axis of X of the fixed individual is in $O\delta^4$ and OY in OT'. Hence a length $O\delta' \times c \div \sqrt{2} = O\delta' \times \cdot 84$ gave the parametral length OC, and $O\delta^4 \times a \div \sqrt{2} = O\delta^4 \times \cdot 479$ gave OA. The positions of the twin-axes and of the rotated axes lying in the plane $T'O\delta^4$ were found by the construction given in Art. 22.

Fig. 472 represents an interpenetrant twin of mispickel in which the normal to e(101) is the twin-axis; the fixed individual being in the conventional position. The twinned individuals are usually of different sizes, and the axial plane XOZ of the one is rarely coincident, as represented in the figure, with that of the other.

40. *Redruthite* (*chalcocite*). This mineral is twinned according to three different laws, (i) twin-face $m(110)$, (ii) intercrossing twins with (032) for twin-face, and (iii) with twin-face $v(112)$. Fig. 473 represents a twin from Bristol, Conn., U.S.A., described by J. D. Dana, in which two laws occur. The basal pinakoid is placed vertically, and is striated parallel to the edge [bc]. In the larger crystal a wedge formed of two individuals twinned to the first crystal about faces m is inserted. The resulting twin is similar to that represented by Fig. 467 (c). To the large crystal another, in which the faces b and c are also vertical, is twinned with twin-face (032): the angle between the pinakoids c being 111° 0′ and 69° 0′.

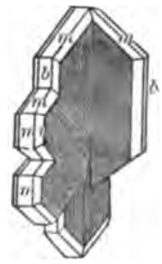


FIG. 473.

41. *Staurolite.* Figs. 474—479 serve to elucidate the twins of staurolite. The simple crystal is usually a stout prism $m\{110\}$, associated with the pinakoids $b\{010\}$ and $c\{001\}$, and occasionally with $r\{101\}$. The angles are $mm_{,} = 50° 40'$, $bm = 64° 40'$, $cr = 55° 16'$.

1. Twin-axis $x(032)$. Fig. 474 is an ideal representation of a common twin. It consists of two interpenetrating crystals, which cross one another nearly at 90°, in such a way that the faces b and c of the two individuals are tautozonal. The possible face with low indices which most nearly truncates the edge [bc], *i.e.* makes 45° with b and c, is $x(032)$; the angle cx being 45° 41′. Adopting, as has been done in Fig. 474, the normal to this face as the twin-axis with an angle of rotation of 180°, the pair of faces b make at the edge through g a re-entrant angle $010 \wedge (010) = 88° 38'$, and the faces c a salient angle $001 \wedge (001) = 91° 22'$. The prism-faces meet in two sets of re-entrant angles: the edges [da], [ae] are in the plane parallel to (032); the edges [ga] and [ah] are not in a plane parallel to a possible face. Usually one individual is much smaller than the other; and the prism-edges [$mm_{,}$], [mm'] do not intersect, as in the figure.

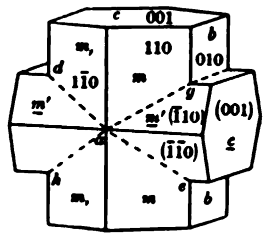


FIG. 474.

2. Twin-axis $\zeta(232)$. An ideal drawing of this twin is given in Fig. 476: in it the normal to a possible face $\zeta(232)$ is taken as the twin-axis; this normal makes angles of nearly 60° with $c(001)$

and b (010) (see table of angles, p. 511). The individuals are, as in the first twin, rarely equal; and seldom cross one another in the very symmetrical manner shown in the diagram.

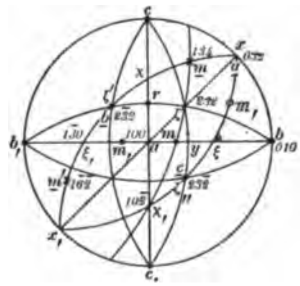


FIG. 475.

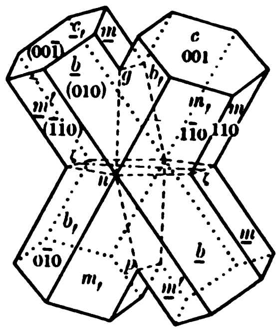


FIG. 476.

3. Twin-axis ξ (130). Fig. 477 represents a rare twin described by Professor E. S. Dana. The twin-axis may be taken to be (130) or (230). The former requires an angle $b\underline{b} = 70° \, 18'$, the latter an angle $70° \, 46'$, whilst measurement gave $70° \, 30'$.

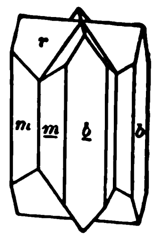


FIG. 477.

Professor Cesàro (*Bull. Soc. franç. de Min.* x, p. 244, 1887) has shown that the crystals may, in accordance with views propounded by Mallard in the same journal (VIII, p. 452, 1885) to explain the formation of twins, be regarded as pseudo-cubic; *i.e.* that the particles are arranged in a manner approximating to one of the arrangements possible in cubic crystals. Thus the crystal of staurolite is in some respects compared to one of pyrites; the axis OX of the former being parallel to a cubic edge of the latter. This axis is a pseudo-tetrad axis; and the zone [bc] is a pseudo-tetragonal zone, having $cx = 45° \, 41'$ and $bx = 44° \, 19'$ as approximations to 45°. Again, the poles of the octahedron {111} lie in zones each of which contains a pole of the cube and one of the rhombic dodecahedron, the angle ao being $54° \, 44'$. Now, the possible faces ξ (130) and χ (102), Fig. 475, of staurolite would together make a figure differing but little from the octahedron; for $a\xi = 54° \, 51'$, $a\chi = 54° \, 12'$ and $\xi\chi = 70° \, 19'$. The faces b and c then occupy the positions of the pair of dodecahedral faces parallel to the pseudo-tetrad axis OX. The remaining faces of the rhombic dodecahedron must lie in the zones [ax]; and their poles ζ in Fig. 475 must make angles of nearly 45° with a and 60° with b and c. Computations from these requirements and the angles already given prove ζ to be the poles of the possible pyramid {232}. The following tables give some of the

most important angles between the actual and possible poles of staurolite, and the angles between some of the corresponding cubic faces.

Staurolite.						*Cubic crystal.*	
				$x \wedge 134$	25° 30′	$100 \wedge 311$	25° 14′
am	25° 20′	ar	34° 44′	$x\xi_{,}$	54 12	,, ,, 211	35 16
ay (230)	35 23	$a\chi$	54 12	$x\chi$	55 29	,, ,, 111	54 44
$a\xi$	54 51	ac	90 0	$x\zeta'$	90 58	,, ,, 011	90 0
ab	90 0	$r\chi_{,}$	88 56	$\zeta' \wedge 134$	65 28	211 ,, $1\bar{1}1$	90 0
$y\xi_{,}$	90 14			$\zeta' \wedge 1\overline{62}$	68 53		
$m\zeta$	31 0						
$\zeta \wedge 134$	31 29·5	$b\zeta$	59 44	$a\zeta$	44 47	101 ,, 101	45 0
ξm	31 0	$c\zeta$	60 31	ζx	45 13	100 ,, 001	45 0
$m \wedge 134$	0 29·5						

By changing the parametral plane of staurolite, we can show the affinity to a cubic crystal in another way. If m is taken to be (*310*) and x (*011*), the faces have the following symbols: $r=(\mathit{201})$, $\chi=(\mathit{101})$, $\xi=(\mathit{110})$ and $\zeta=(\mathit{211})$. The parameters are now given by $\mathbf{c} \div \mathbf{b} = \cot bx = 1{\cdot}024$, and $\mathbf{a} \div \mathbf{b} = 3 \tan am = 1{\cdot}420 = \sqrt{2}$, nearly.

Hence $\mathbf{a} : \mathbf{b} : \mathbf{c} = 1{\cdot}420 : 1 : 1{\cdot}024 = \sqrt{2} : 1 : 1$, nearly.

The latter ratios are the parameters of a cubic crystal, when the axes of reference are a tetrad axis and the pair of dyad axes perpendicular to it, and two of the octahedral faces become (110) and (011), respectively.

Now in a cubic substance the interfacial relations are exactly similar in azimuths about a dyad axis which differ by 180°, and in azimuths about a tetrad axis which differ by 90°. The arrangement of the particles about any point must have the same symmetry, although like groups of particles may only be interchanged by screw-rotation about an axis of symmetry. We may argue that similar relations hold for the possible arrangements of the particles in a pseudo-cubic substance; and that the crystals can grow when the particles are in positions or in orientations differing by rotations of 180° or 90° about the pseudo-axes of symmetry from those positions or orientations which hold for the growth of the simple crystal; the difference of azimuth admissible being 90° when the line of rotation is a pseudo-tetrad axis, and 180° when it is a pseudo-dyad axis. This hypothesis offers an explanation of the three twin-laws. The rectangular twin, Fig. 474, resembles the interpenetrating twins of pyrites described in Art. 15. The twin-axis should be regarded as OX, a pseudo-tetrad axis, and the angle of rotation 90°. The angle $c\underline{c}$ should be then 90°, and not 91° 22′. The inclined twin, Fig. 476, has for twin-axis a pseudo-dyad axis, the normal ζ, and an angle of rotation of 180°. The third twin, Fig. 477, has for twin-axis a pseudo-triad axis, if ξ (130) is taken as the twin-axis; and it then conforms with the law of twinning common in cubic crystals.

The method of drawing Fig. 476 will be now explained. In Fig. 475 the poles of the fixed crystal are represented by dots, a (100) being placed

at the centre and $c\,(001)$ at the top: the poles of the rotated crystal are indicated by crosses and barred letters. From the table of angles given it will be seen that a semi-revolution about ζ brings b to $\underline{b}$, which is nearly coincident with ζ', and c to $\underline{c}$ nearly coincident with $\zeta_{\prime\prime}$; and that the poles a and x nearly change places.

Assuming that the rotation produces exact coincidence, we can find indices of the rotated prism-faces $\underline{m}$ as if they were, in their new positions, faces of the fixed simple crystal. For the faces lie in the zone $[\underline{a}\underline{b}]$, which is now the same as $[x\zeta']=[62\bar{3}]$. This zone intersects $[ac]$ in $\chi\,(102)$ and $[ab]$ in $\xi_{\prime}\,(1\bar{3}0)$. The angles $x\chi$ and $x\xi_{\prime}$ can be calculated from the right-angled triangles $cx\chi$ and $b_{\prime}x_{\prime}\xi_{\prime}$: they are, $\wedge x\chi=55°\,29'$, $\wedge x_{\prime}\xi_{\prime}=54°\,11{\cdot}7'$. In the A.R. $\{x\underline{m}\chi\zeta'\}$ the symbols of three poles and all the angles are known, for $x\underline{m}=am=25°\,20'$. Hence $\underline{m}$ is (134). Similarly, from the A.R. $\{x_{\prime}\underline{m}'\xi_{\prime}\zeta'\}$, the pole $\underline{m}'$ is $(1\bar{6}\bar{2})$. The poles $\underline{m}$ and $\underline{m}'$ are not exactly in $[x\zeta']$, but the displacement of $\underline{m}$ from (134) is only about half-a-degree. When $\underline{m}$ and $\underline{m}'$ are taken to be (134) and $(1\bar{6}\bar{2})$, it is easy by the rule of Chap. V to find the directions in which any pairs of the faces intersect, and therefore to draw the twin.

Fig. 478 shows the relation of the principal zones which has led to the ideal drawing, Fig. 476. The zone $[\chi_{\prime}m\zeta\underline{m}]$ of Fig. 475 is placed in the primitive. Now $\chi_{\prime}$ is at 90° from $b\,(010)$ and at 88° 56′ from $r\,(101)$: it is therefore nearly the pole of the zone-circle $[b\zeta r]$, which is in Fig. 478 projected in a diameter perpendicular to $\chi_{\prime}\chi$. Further, the angles ζb, $\zeta\underline{b}$ are nearly 60°; and the zones $[\chi_{\prime}m\underline{m}]$, $[\chi_{\prime}m_{\prime}\underline{b}]$, $[\chi_{\prime}\underline{m}'b_{\prime}]$ are nearly at 60° to one another. The pairs of faces m and $\underline{m}$, $m_{\prime}$ and $\underline{b}$, $\underline{m}'$ and $b_{\prime}$, will therefore intersect in a regular hexagon in a plane perpendicular to $\chi\chi_{\prime}$: this plane is the base of Figs. 476 and 479. Further, the axial poles c and a, and the axes OZ and OX, of each crystal lie in a zone-circle through $\chi_{\prime}$ bisecting the angle $\zeta\zeta'$ (Fig. 475) $=\zeta\underline{b}$ (Fig. 478), where $\chi_{\prime}c_{\prime}=35°\,48'$ and $\chi_{\prime}a=54°\,12'$.

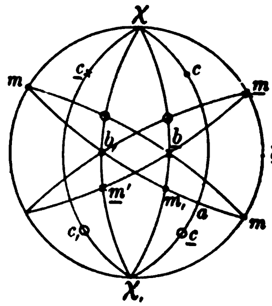


FIG. 478.

A hexagon, Fig. 479, is now projected in the way described in Chap. VI, Art. 19, having the back-and-fore cubic axis for the diagonal $nn_{\prime}$. The right-and-left cubic axis is the twin-axis $\zeta\zeta$, and the vertical axis is the normal $\chi\chi_{\prime}$. From Fig. 478 it is seen that $OB_{\prime}$ and $O\underline{B}$ are in the plane of the hexagon at 60° to $\zeta\zeta$. They are the bisectors of the other pairs of opposite sides of the hexagon.

Again OA'' and On are unit lengths on the vertical axis and on a horizontal axis. But the semi-diagonal of a hexagon is equal to each of its sides. The sides of the hexagons therefore give unit length along any line parallel to them. Further, since the angles at $B_{\prime}$, $\underline{B}$ and ζ are right angles,

$$OB_{\prime}=O\underline{B}=O\zeta=On\cos 30°:$$

the projected lines are therefore the unit length in their directions multiplied by $\cos 30°$, *i.e.* by $\sqrt{3}\div 2$.

The axes of Z and X of each individual are in the vertical plane perpendicular to the axis of Y: hence those of one crystal lie in the plane LOd perpendicular to $OB_,$, and those of the other in LOd, perpendicular to $O\underline{B}$. In the plane LOd construct a parallelogram $OSXe$ having for sides $OS = OA'' \sin 35° 48'$; and

$$Oe = Od \cos 35° 48'.$$

The diagonal OX is the direction of, and unit length on, the axis of X. This construction is identical with that given for the axis OX of an oblique crystal (Chap. VI, Art. 15). Similarly, the parallelogram $LOfZ$, with sides $OL = OA'' \cos 35° 48'$ and

$$Of = Od \sin 35° 48',$$

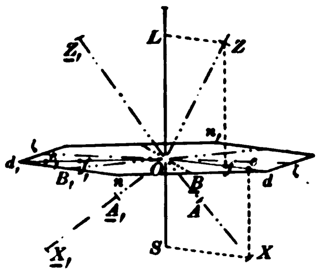


FIG. 479.

has for diagonal the unit-length on OZ. The unit-lengths on $O\underline{X}_,$ and $O\underline{Z}_,$ are found by constructing the parallelograms $OS\underline{X}_,e_,$ and $OL\underline{Z}_,f_,$, where the points $e_,$ and $f_,$ are got by drawing lines through e and f parallel to $\zeta\zeta$. $O\underline{B}$ and $OB_,$ being left unchanged, the lengths OX and $O\underline{X}_,$ are multiplied by $a \cos 30°$ giving OA and $O\underline{A}_,$, the lengths OZ and $O\underline{Z}_,$ by $c \cos 30°$. The axes of the two crystals have been now projected, and forms present on either crystal can be drawn.

Lines through the corners of the hexagon parallel to OZ and $O\underline{Z}_,$ give the edges $[mm_,]$ and $[bm]$ of the two individuals. Their lengths are immaterial, but, in any particular case, they should be made to correspond approximately with the lengths of the edges. The edges $[cm]$ and $[bc]$ of the crystals are easily found; for $[cm_,]$ is parallel to $AB_,$, $[cb]$ to OX, and so on.

Again, the faces in Fig. 476 intersect in a second hexagon gnp.... From Fig. 478 it is seen that $\underline{b}$ and $b_,$ are adjacent faces, and since their poles are in a plane perpendicular to $\chi\chi_,$ (the vertical line in the drawing), their line of intersection is parallel to $\chi\chi_,$. The re-entrant edge ng is therefore vertical. Again, it is clear that $\underline{m}$ and the face $\bar{1}\bar{1}0$ parallel to m must intersect in a horizontal line parallel to $d\zeta$ of Fig. 479. The remaining edge np in which $m_,$ and $\underline{m}'$ intersect is the zone-axis $[\zeta m_,]$, and is inclined to the vertical at an angle of nearly 34° 45′.

iv. TWINS OF THE RHOMBOHEDRAL SYSTEM.

42. As in cubic crystals, the triad axis is commonly the twin-axis in rhombohedral crystals, and it is immaterial whether the angle of rotation is taken to be 60° or 180°; for azimuths about the axis which differ by 120° are interchangeable by a rotation about it of 120°. For the sake of uniformity in the statement of the twin-orientation, it is usual to take 180° to be the angle of rotation. The combination-plane is often the base (111), when the law may be given by stating that the base is the twin-face. The crystals are sometimes united along a face of a hexagonal prism; and they

sometimes interpenetrate one another, as is the case in quartz and chabazite.

Twins with inclined triad axes occur, and in them the twin-axis is generally the normal to a face of a rhombohedron or of a trigonal pyramid; and the twins are most frequently combined along a plane perpendicular to the twin-axis.

We shall discuss the twins of calcite and quartz, and describe briefly some interesting twins of a few other substances.

Calcite.

43. *1. Twin-axis* [111]. Fig. 480 represents a twinned fundamental rhombohedron with (111) for twin-face. This is sometimes seen in fragments of calcite obtained by cleavage; but in complete crystals, it is more common in dolomite, $(Ca,Mg)CO_3$. The lower half has been rotated, and occupies the position of the inverse rhombohedron $\{\bar{1}22\}$: its polar edges are therefore the lines joining $V_{,}$ to $\bar{M}$, $M_{,}$, $M_{,,}$ of Fig. 335, p. 376; and the median coigns on these edges are the points of trisection nearest to the equatorial plane, which coincides with the combination-plane. The median edges are bisected in this plane at the points δ on the dyad axes. The salient and re-entrant edges join adjacent pairs of these points; and the angles $r\underline{r}$ are 90° 47.

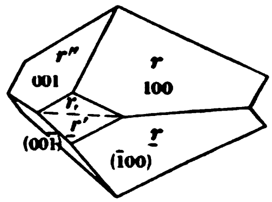


FIG. 480.

Fig. 481 represents a scalenohedron $\{2\bar{1}0\}$ twinned according to the same law. This twin is readily drawn with the aid of the twin shown in Fig. 480. Each of the rhombohedra in the latter is completed in fine pencil, and the median coigns are then joined to apices thrice as distant from the combination-plane as the apices of the auxiliary rhombohedra. The obtuse polar edges join the apex to coigns on the further side of the combination-plane, and these obtuse edges intersect in pairs in the points H. The median edges cross the combination-plane at the points δ, &c.; and these points united to the adjacent

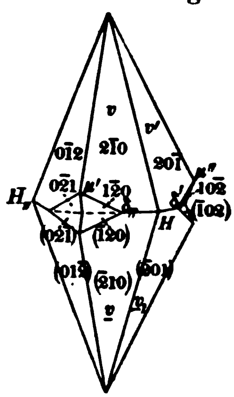


FIG. 481.

points H give the horizontal edges of the twin: these latter edges are in pairs which are alternately salient and re-entrant edges, the angles over them being in all cases 41° 56′.

The twinning is frequently repeated. If repeated only once, the portions at the opposite apices are in like orientation with a twin-lamella interposed; and if this lamella is very thin, it appears only as a fine line running horizontally across each face of the crystal.

Twinning according to this law is sometimes perceived in crystals in which the prism $\{2\bar{1}\bar{1}\}$ is the conspicuous form. Since the faces are parallel to the twin-axis and pairs of them to one another, twinning does not cause any interruption on them; for a portion of a rotated face is co-planar with one of the fixed crystal, as in the twinned rhombic dodecahedron of blende, Art. 9. The twin, if terminated by rhombohedral faces $e\,\{110\}$ at both ends, will differ from the simple crystal in Fig. 371, inasmuch as alternate prism-faces will be met in a like manner at both ends by faces e, and the equatorial plane will seem to be a plane of symmetry. If terminated by the pinakoid, the twin will be indistinguishable geometrically from a simple crystal, Fig. 328, but the cleavages at the two ends will be symmetrical to the combination-plane and not parallel.

A variety of the law is described by Haidinger as sometimes occurring in which a face of the prism $\{211\}$ is the combination-plane; and another variety occurs in which the individuals penetrate one another to a greater or less extent: the latter is common in dolomite.

44. *2. Twin-face* $e\,(110)$. This is the most common twin-law in calcite, frequently giving rise to polysynthetic twin-lamellæ, which are generally, though not always, parallel to one face of the form $\{110\}$. The lamellæ can be recognised by the striæ which they produce on some of the cleavage-faces, and by the internal reflexion caused at the combination-plane. In a cleavage-rhomb of Iceland spar traversed by such lamellæ, the four cleavages tautozonal with the twin-face are smooth and even; the remaining pair of cleavage-faces meet the twin-face at angles of 70° 52′ and 109° 8′. These latter faces are therefore traversed by alternate salient and re-entrant edges parallel to the horizontal diagonal of the rhombus, the normal-angle over each edge being $38°\,16' = 180° - 2 \times 70°\,52'$. If a thin plate is prepared parallel to this rhombic diagonal, and if the

plate is inserted in parallel light between crossed Nicols, the light is extinguished simultaneously in all the lamellæ, for the principal planes are all perpendicular to the diagonal. If a plate is prepared parallel to one of the smooth faces and similarly tested, the light in alternate lamellæ is extinguished simultaneously, but not that traversing adjacent lamellæ; the extinctions in the latter make a minimum angle of 11° 55′ with one another, for the principal planes perpendicular to the plate are inclined to one another at 101° 55′. The twin-lamellæ are often of secondary origin; *i.e.* produced after the formation of the crystal. They can, for instance, be produced in the laboratory by squeezing a cleavage-rhomb between the jaws of a vice, parallel polar edges being placed in contact with the jaws; or they can be produced by pressing a knife-edge placed across a polar edge into the substance.

A doublet of rare occurrence with (011) for twin-face is shown in Fig. 482. The upright individual is drawn in one of the ways given in Chap. XVI, Art. 57. The edges of the section in the combination-plane, and the directions of the twin-axis and of the triad-axis of the rotated crystal are then determined. The prism-edges through the corners of the section and lines through the middle points of its alternate sides are now drawn parallel to the rotated triad-axis. Lines parallel to the twin-axis are also drawn through the coigns in the faces *e* of the fixed crystal: they meet the edges and lines of construction already drawn in corresponding coigns of the rotated crystal, which can be then completed.

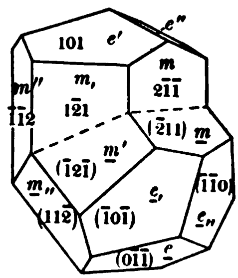


FIG. 482.

45. *3. Twin-face r* (100). Doublets of calcite according to this twin-law are fairly common in Cumberland. The twins vary much in aspect, the habit depending on the forms present and on the relative size of the faces either of the same or of different forms. Thus the twins sometimes consist of hexagonal prisms $\{2\bar{1}\bar{1}\}$ terminated either by {111}, or by the rhombohedron {110}, Fig. 486, sometimes of scalenohedra $\{20\bar{1}\}$, Fig. 485. The principal axes are inclined to one another at angles of 89° 13′ and 90° 47′.

Fig. 483 represents a doublet with (001) for twin-face. It can be imitated by placing two cleavage-rhombs side by side in twin-orientation. The drawing is made as follows.

The rhombohedron {100} is drawn in the way described in Chap. XVI, Art. 31. It is then bisected by the combination-plane $g\,\delta_{,,}e$ drawn through $\delta_{,,}$ parallel to (001), $\delta_{,,}$ being the middle point of the median edge $\mu_{,}\mu$. The normal Or'' to (001) meets the polar face-diagonal $V\mu''$ at r'', where $Vr'' : V\mu''=3 : 8$, nearly. For, from Fig. 484, $Vr''=OV\sin D$, and $V\mu''\sin D = Vt_{,}=4OV\div 3$.

$$\therefore\ Vr'' : V\mu''=3\sin^2 D\div 4=3 : 8, \text{ nearly.}$$

For, D being 44° 36·6′, $\sin^2 D=1\div 2$ very nearly. The point r'' is now found

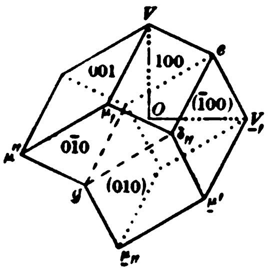


FIG. 483.

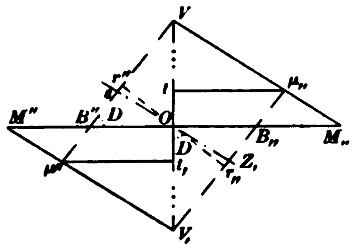


FIG. 484.

by proportional compasses, and Or'' is produced beyond (001) to Ω, where $O\Omega=2Or''$. ΩV is the rotated triad axis $O\underline{V}_{,}$ in direction and magnitude. The figure can be now completed, corresponding coigns lying on lines parallel to $V\underline{V}_{,}$.

The twin represented in Fig. 485 can be drawn as follows The triad axes in Fig. 483 are produced both ways to points V^3, V_3, $\underline{V}_3$, and $\underline{V}^3$. Each half-rhombohedron in Fig. 483 is also completed in faint pencil beyond the combination-plane; and the median coigns of each rhombohedron are then joined to the apices of the corresponding scalenohedron. The corresponding polar edges intersect in the combination-plane; and the adjacent pairs of these points give the alternate salient and re-entrant edges. The angles between corresponding faces are,

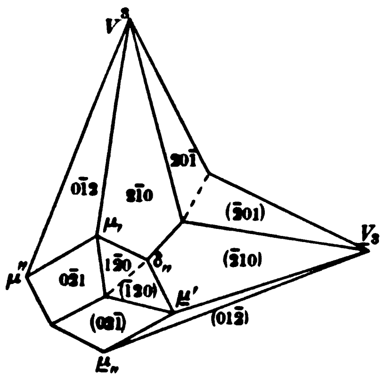


FIG. 485.

$$210 \wedge (\bar{2}10) = -1\bar{2}0 \wedge (120) = 180° - 2 \times 82° 29' = 15° 2';$$

$$20\bar{1} \wedge (\bar{2}01) = 180° - 2 \times 103° 56{\cdot}5' = -27° 53' = -0\bar{2}1 \wedge (02\bar{1}).$$

To construct Fig. 486 the two rhombohedra in Fig. 483 are completely drawn as required for the previous twin, and the middle points δ of their median edges are found. The lines through these points parallel to the respective triad axes are the prism-edges, corresponding pairs of which intersect in the combination-plane. The salient and re-entrant edges in this plane can then be drawn. A convenient length, corresponding fairly to that of the edge in the twin, is now cut off on one of the prism-edges of the fixed crystal, and the edges $[me']$, $[m_{,}e']$, &c., are drawn in the manner described in Chap. XVI, p. 407, for the combination $\{2\bar{1}\bar{1}\}$ and $\{110\}$. The coigns on the rotated crystal are found by drawing lines parallel to $V\underline{V}_{,}$ of Fig. 483 through each of the coigns of the fixed crystal to meet the triad axis $O\underline{V}_{,}$, the corresponding prism-edges, and lines through the middle points of $[m\underline{m}]$, &c., parallel to $O\underline{V}_{,}$. The coigns of the rotated crystal having been determined, the figure can be completed. The polar edges of the rotated crystal can also be found in the same way as those of the fixed crystal.

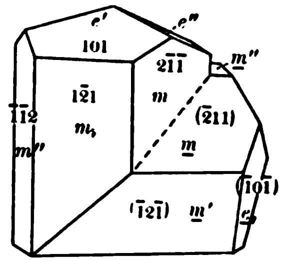


FIG. 486.

46. *4. Twin-face* $f\,(11\bar{1})$. Fig. 487 represents a scalenohedron v $\{2\bar{1}0\}$ twinned according to this law. In Chap. XVI, Art. 52, it was shown that the faces f of $\{11\bar{1}\}$ truncate the acute polar edges of the scalenohedron $\{2\bar{1}0\}$, and in Art. 37 that the apices of $\{11\bar{1}\}$ are at distances $2c$ from the origin, the equivalent symbol being $-2R$. The inclination of the normal f'' $(11\bar{1})$ to the triad axis is therefore $63°\,7'$; and in the twin the two triad axes include an angle of $180° - 2 \times 63°\,7' = 53°\,46'$. Further, the pair of acute polar edges in the plane of the triad axes are parallel, and the opposite pairs of obtuse polar edges in the same plane include a salient and re-entrant angle of $97°\,54'$. The faces which meet in the combination-plane form salient and re-entrant angles: these are

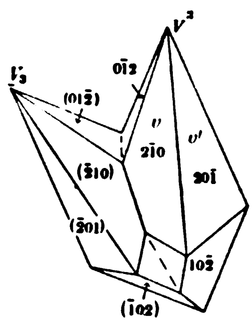


FIG. 487.

$$2\bar{1}0 \wedge (\bar{2}10) = -\,1\bar{2}0 \wedge (120) = 180° - 2 \times 71°\,23' = 37°\,14',$$
$$10\bar{2} \wedge (102) = -\,0\bar{1}2 \wedge (01\bar{2}) = 180° - 2 \times 51°\,18' = 77°\,24'.$$

This twin is rarer than the others. The several twins can be readily distinguished from one another by the positions of the cleavages which can be generally detected by means of flaws traversing the crystals.

The position of the rotated triad axis is easily found when the rhombohedron $\{11\bar{1}\}$ is drawn as in Fig. 370, p. 406; for the foot of the normal f'' lies in $V\lambda''$ at a distance given by $Vf'' : V\lambda'' = 3\sin^2 VOf'' \div 4 = 3\sin^2 63^\circ 7' \div 4 = \cdot 669 = 2 \div 3$ nearly. If a section in the plane $V\lambda''V_2$ of Fig. 370 is made similar to Fig. 484, we shall obtain similar relations to those established for the latter; the angle VOf'' replacing the angle VOr'', and O being at the middle point of VV_2. The normal Of'' is prolonged to Ω where $O\Omega = 2Of''$. ΩV is then the direction of the rotated axis $O\underline{V}_2$ and gives a length $2c$ upon it. The rest of the drawing presents no special difficulty.

Quartz.

47. ***Twin-axis*** [111]. *1. **Dextrogyral twins.*** Let Fig. 488 (a) represent a simple dextrogyral crystal of normal habit, having the faces of the forms, m {211}, r {100}, z $\{\bar{1}22\}$, and $x = a$ $\{41\bar{2}\}$; and let Fig. 488 (c) represent an exactly similar crystal, the position of

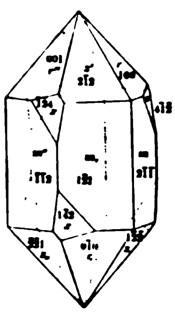


Fig. 488 (a).

Fig. 488 (b).

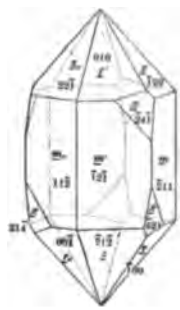


Fig. 488 (c).

which is such that a semi-revolution about the triad axis will bring it into the same orientation as the crystal (a). Let us now suppose three equal wedge-shaped segments to be cut out of each of the two crystals by planes inclined at 60° to one another and passing through the triad axis; and suppose the wedges to include similarly placed alternate edges of both crystals. Let the segments of crystal (c) include the prism-edges having faces x at both ends, then the similarly placed wedges of (a) include no x faces. Now interchange the wedges and insert those from (c) in the vacant spaces in (a) and vice versâ.

When the three wedges of (*c*) are inserted in (*a*), a twin, Fig. 488 (*b*), is obtained in which the faces *r* of the one individual are co-planar with the faces *z* of the other, and the trapezohedral faces *x* occur on all the prism edges. These faces *x* all lie in zones which, proceeding from a face of the apparently hexagonal pyramid towards a prism-face, slant in the direction of a right-handed screw; and the twin is, like the simple crystal, throughout dextrogyral. The necessity for this is obvious, for it has already been shown that a dextrogyral crystal cannot be brought into the same orientation as a lævogyral crystal by rotation about any axis.

By the insertion of the three wedges of (*a*) in (*c*), a twin is obtained which has no *x* faces, and is geometrically indistinguishable from a simple crystal having the faces *m*, *r* and *z*. Even in this latter case the twin character can often be detected by the difference in lustre and markings of the *r* and *z* portions of the composite-faces. Further, the prism-edges bearing faces *x* are at the analogous poles; consequently with falling temperature, the prism-edges in Fig. 488 (*b*) are all negatively electrified, and those of the apparently simple crystal are all positively electrified.

Other similar twins can be formed by the interchange of a single segment cut from (*a*) and (*c*), so as to include a similarly placed prism-edge. Two twins are then produced, one of which has faces *x* on four prism-edges, the other only on two. When two wedges are interchanged, twins having five prism-edges or only one modified by *x* faces are obtained. We are thus able to explain the irregularity of distribution of the *x* faces observed on crystals of quartz.

So far the wedge has been supposed to extend from apex to apex, but there is no necessity for this; and it often happens that a portion at one corner, or in the middle of a face, has been, as it were, excised from a crystal and replaced by an equal amount of matter in twin-orientation. It is indeed frequently observed that a crystal has several patches of such interpolated matter; the presence of such patches in the *r* and *z* faces being recognized by the difference in lustre and corrodibility, and on the prism-faces by the different orientation of the figures produced by corrosion.

Occasionally these twins and the twins given under *2*—*4* are in juxtaposition with a face of the prism $\{2\bar{1}\bar{1}\}$ for combination-plane. In these cases the individuals are sometimes sharply separated by the combination-plane, sometimes there is a greater or less amount of overlapping and interlocking.

2. *Lævogyral twins.* Figs. 489 (*a*), (*b*) and (*c*) serve to illustrate the production by an exactly similar process of a twin of a lævogyral crystal having faces $x_{,}$ of $a\{4\bar{2}\bar{1}\}$, arranged in a similar

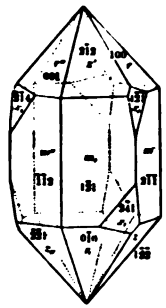


FIG. 489 (*a*).

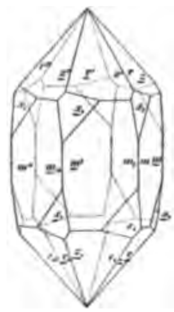


FIG. 489 (*b*).

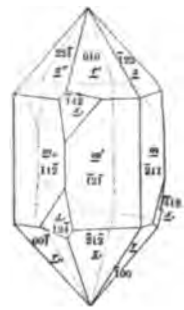


FIG. 489 (*c*).

manner on all the prism-edges. The preceding discussion holds for all the possible variations of the twins of lævogyral crystals, but in these crystals the tautozonal faces s', $x_{,}$, m follow one another along a left-handed screw when we proceed from either of the extreme faces. Thus we may have lævogyral twin-crystals showing no faces $x_{,}$, or faces $x_{,}$ only on one, two, &c. prism-edges; and we may have the crystals in juxtaposition along a face of $\{2\bar{1}\bar{1}\}$.

48. 3. *Lævo-dextrogyral twins* a. *Brazil-law.* Composite crystals of quartz with coincident triad axes occur in which the different portions have opposite rotations. Thus the twin, shown in Fig. 490 (*b*), which was first described by G. Rose, can be obtained by the interpenetration of crystals of opposite rotations.

Suppose wedges including similarly placed prism-edges to be cut from a dextro- and a lævo-gyral crystal, placed as in Figs. 490 (*a*) and (*c*) with their faces *r* parallel to one another; and suppose the wedges including the faces $x_{,}$ of Fig. 490 (*c*) to be inserted in equal spaces in (*a*). A composite crystal represented by Fig. 490 (*b*) is obtained. Now Figs. 490 (*a*) and (*c*) are both in normal orientation, and no rotation of either individual has been assumed. The crystal falls under the class of symmetric twins; and it is symmetrical with respect to each of three planes which are parallel to the prism-faces truncating alternate edges of the prism $\{2\bar{1}\bar{1}\}$. Had the wedges from (*a*) been inserted in the spaces in (*c*) a similar twin would be produced which would show no trapezohedral faces, and would be

indistinguishable, except by optical examination, from a simple crystal. Twins of this kind showing trapezohedral faces are very

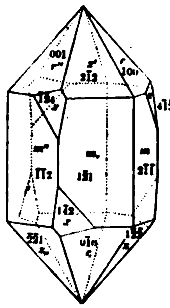


FIG. 490 (*a*).

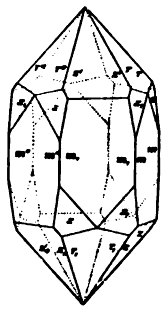


FIG. 490 (*b*).

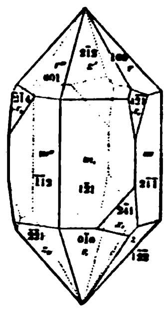


FIG. 490 (*c*).

rare. The boundaries of the individuals and the number of faces x actually present on any crystal seem to be as variable as in the case of dextrogyral or lævogyral twins. The rhombohedral faces r and z forming the hexagonal pyramid are in this twin each of one kind, and a face r of one individual is not co-planar with a face z of the other. The law of twinning is known as the *Brazil-law.*

4. Lævo-dextrogyral twins β. Another possible composite crystal of quartz into which both dextro- and lævo-gyral portions enter is shown in Fig. 491 (*b*). Suppose a lævogyral crystal, Fig. 491 (*a*), and a dextrogyral crystal to be placed side by side with faces r parallel; and let the latter crystal be then turned through 180° about the triad axis, when its position is given by Fig. 491 (*c*). Suppose both crystals to be cut into similar wedges of 30° by planes passing through the middle point of each crystal; one plane being horizontal, the others vertical; of the latter six pass through the prism-edges and six are perpendicular to the prism-faces. Replace twelve alternate wedges in Fig. 491 (*a*) by twelve similarly placed wedges from Fig. 491 (*c*). A twin is then obtained represented by Fig. 491 (*b*). This is also a symmetric twin; and it is geometrically and physically symmetrical to the equatorial plane and to each of the planes passing through opposite prism-edges, *i.e.* to planes parallel to the faces of $\{2\bar{1}\bar{1}\}$. The faces of the hexagonal pyramid are composite as in the twins of single rotation; the prism-faces are composed of four portions, those placed diagonally belonging to a crystal of the same rotation.

The composite nature of twins of laws *3* and *4* can be proved by

the examination in plane-polarised light of plates cut perpendicularly to the optic axis. Such examination has shown that the purple

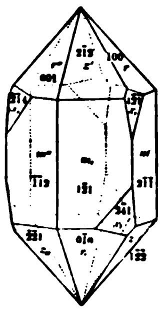


FIG. 491 (a).

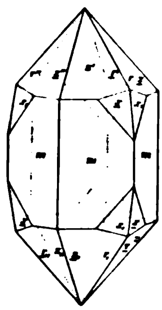


FIG. 491 (b).

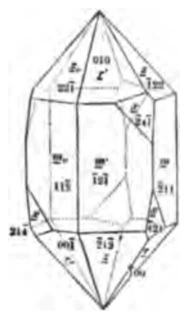


FIG. 491 (c).

variety (amethyst) consists of a number of thin layers alternately of opposite rotations and different colour; the layers being parallel to the faces *r*, and sometimes both to the faces *r* and to two or three of the faces *s* of $\alpha\{41\bar{2}\}$. The sections are therefore divided into three or six segments which are usually bounded by very definite lines, and they are sometimes separated by triangular spaces near those prism-edges on which the faces *z* stand. These triangular spaces are often again divided into two distinct triangles in which the rotations are opposite.

49. *5.* In these twins the triad axes are inclined to one another at angles of 84° 34′ and 95° 26′; and, irrespective of the rotatory character of the individuals, we have two varieties. i. In the first, Fig. 493, the one individual may be supposed to have been turned through 180° about the normal $O\xi$ to a pyramid-face which truncates the edge $[r''z]$. This face belongs to a trigonal bipyramid $\alpha\{\bar{1}25\}$. ii. In the second, Fig. 494, the twin-axis is the zone-axis $OT\,[210]$ parallel to the edge $[r''z]$. The twins of both kinds are united along a surface which is very nearly, if not exactly, parallel to the face $(\bar{1}25)$.

i. *Twin-face* $\xi\,(\bar{1}25)$. Fig. 492 is a plan of a simple crystal of quartz projected on a plane parallel to $(1\bar{2}1)$. Suppose the crystal to be bisected by a plane through OT perpendicular to $(1\bar{2}1)$, and the lower half to be turned through 180° about $O\xi$. The two halves, being then joined in the plane of section, give the twin shown in Fig. 493. The rhombohedral faces *r* and *z* of the one are

symmetrical to similar *r* and *z* faces of the other with respect to the combination-plane; but, if the crystals have the same rotatory

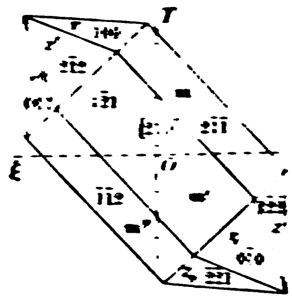


FIG. 492.

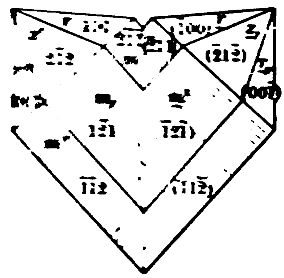


FIG. 493.

character, the trapezohedral faces *x* would, if present, not be symmetrical to this plane. The twin is neither geometrically nor optically symmetrical to the combination-plane; for a right-handed screw is reflected in a left-handed one. The prism-faces $1\bar{2}1$ and $(\bar{1}2\bar{1})$ are co-planar; and the boundary traversing this face is jagged and irregular. The other prism-faces meet in definite salient and re-entrant edges; the angles over them being

$$\bar{1}\bar{1}2 \wedge (11\bar{2}) = -2\bar{1}\bar{1} \wedge (\bar{2}11) = 79°\,41'.$$

If the individuals are of opposite directions of rotation, they are optically symmetrical to the combination-plane; and so are the faces *r*, *z*, *m* and the trapezohedral faces *x*: the twin is symmetric.

Each individual is often again twinned according to one of the preceding laws; for the faces *r* and *z* generally show patches of different character which indicate twinning. Whether the individual is in these cases dextrogyral or lævogyral or is lævo-dextrogyral can rarely be determined without sacrificing the specimen.

ii. *Twin-axis* $OT = [210]$. If now two crystals placed side by side in parallel orientation are bisected by the plane through OT of Fig. 492 perpendicular to $1\bar{2}1$, and the two upper halves are united together in the planes of section after one of them has been turned through 180° about OT, a twin represented by Fig. 494 is produced. In this twin the faces *r*, *z* and *x* are not symmetrical to the combination-plane, and the twin is asymmetric whether the crystals are of like or of opposite rotations. Fig. 494 represents a twin from Japan

in the Cambridge Museum, which, judging from the pyramidal faces of $a\{4\bar{2}1\}$ and the trapezohedral faces x, shown, consists of two lævogyral crystals: the relative dimensions of these faces to the others are exaggerated. The faces r and z seem to be composite, and the individuals composing the twin are probably far from simple. As in twins of the variety described under i, the individuals united according to this law may be of like rotation, or twins of the kind described under *3* and *4*.

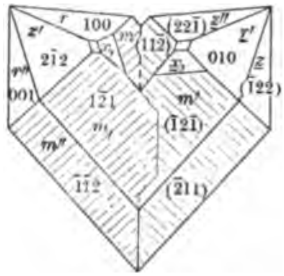


FIG. 494.

50. *Cinnabar.* Fig. 495 represents an interpenetrating twin, in which the forms of the individual in normal position are $r\{100\}$, $n\{5\bar{1}\bar{1}\}$, $\chi = a\{13, 1, \bar{5}\}$. The individual with barred letters is after a semi-revolution about the triad axis brought into a position in which the rhombohedral faces become coincident with the corresponding faces of the first crystal, but the trapezohedral faces $\underline{\chi}$ come into the position of the enantiomorphous form $a\{13, \bar{5}, 1\}$. Professor Tschermak (*Min. u. Petr. Mitth.* VII, p. 361, 1886) found a simple crystal having the forms r, n and χ to be dextrogyral. The twin is therefore a symmetric twin of a dextro- and a lævo-gyral crystal, the latter being in an azimuth differing by 180° from that given to the simple crystal. The individuals are reciprocally symmetrical with respect to the equatorial plane, and to three vertical planes parallel to the faces of the prism $\{2\bar{1}\bar{1}\}$. This twin is similar to the twin of quartz according to the fourth law.

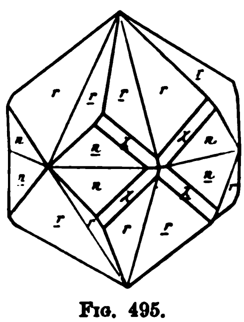


FIG. 495.

Symmetric twins were also observed in which the faces of the rhombohedron are coincident, but all the edges $[rn]$ are modified by faces of the enantiomorphous trapezohedra χ and $\underline{\chi}$. Such crystals therefore have the geometrical development characteristic of crystals of class III of the rhombohedral system, and fall into the same group of twins as those of quartz twinned according to the Brazil-law.

51. *Hematite.* Fig. 496 represents a twin in which the triad axis is the twin-axis, and the combination-plane is parallel to a face of the hexagonal prism $\{2\bar{1}\bar{1}\}$. As shown in the figure the faces of the pinakoids are co-planar, and likewise

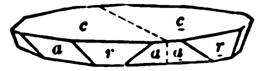


FIG. 496.

a pair of opposite faces *a* of the prism $\{10\bar{1}\}$. The faces shown are *c* $\{001\}$, *r* $\{100\}$ and *a* $\{10\bar{1}\}$. The composite character of the face *c* is often revealed by barbed striæ, which indicate precisely the position of the combination-plane.

52. *Chabazite.* The crystals are usually rhombohedra $\{100\}$ which resemble cubes, for the angle $rr' = 85° 14'$. The twins are interpenetrant twins with the triad axis for twin-axis. The twins consist usually, as shown in Fig. 497, of a large individual from the faces of which small portions of others protrude in twin-orientation. The appearance is similar to that observed in many cubes of fluor.

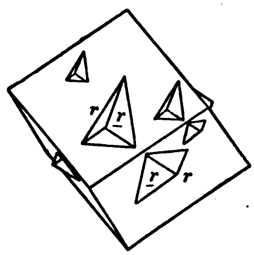


FIG. 497.

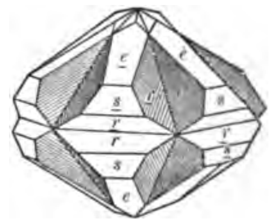


FIG. 498.

In Fig. 498 a twin of another habit is shown, which is common in the variety called phacolite (seebachite). This may be regarded as an interpenetrant twin of two rhombohedral crystals, crossing one another with fair regularity in the same way as the rhombohedra of cinnabar, Fig. 495. The forms present are $r\{100\}$, $e\{110\}$, $s\{\bar{1}11\}$. The faces *r* and *e* are striated parallel to the edges $[re]$; the striæ on some of the faces *r* are however exaggerated in the figure to show the deep re-entrant angles more conspicuously.

Plates cut perpendicularly to the triad axis have been found to consist of six segments similar to those of potassium sulphate, and the segments are often, if not generally, biaxal, the acute bisectrix being perpendicular to the plate. But the angle of the optic axes is not the same in all the segments of the same plate, and the double refraction also varies in strength. The substance loses some of the water of crystallization very easily, and the anomalous optical characters are probably due to the strains caused by the loss of water. Since different segments may be unequally affected, we have an explanation of the variation from segment to segment; and this interpretation of the phenomena is supported by the fact that a further expulsion of water by heat strengthens the double refraction where it is already manifest, and brings it into prominence in segments in which it was at first inconspicuous.

53. *Pyrargyrite.* The crystals of this mineral belong to the acleistous ditrigonal class, and afford an instance of complementary twins having their triad axes and also their planes of symmetry coincident in direction. The twin shown in Fig. 499 may be supposed to consist of two similar halves of separate crystals united along the equatorial plane (the plane of section) after one has been turned through 180° about the normal to one of the faces a of the prism $\{10\bar{1}\}$. The forms present are $a\{10\bar{1}\}$, $m = \mu\{21\bar{1}\}$, $v = \mu\{20\bar{1}\}$ and $t = \mu\{310\}$. A section of the simple crystal in the equatorial plane is a hexagon, having its alternate corners truncated by the traces of the trigonal prism $\mu\{2\bar{1}\bar{1}\}$. A semi-revolution of such a hexagon about a line in its plane which bisects a pair of its opposite sides, interchanges the sides of the hexagon, but exchanges a modified with an unmodified corner. Consequently the truncated edges of the rotated hexagonal prism stand below the unmodified edge of the fixed crystal, and vice versâ. The pyramid-faces on the two portions are parallel in pairs, and the like faces together build up each a scalenohedron characteristic of class III of the rhombohedral system. The small triangular faces, which appear at the points where the edge [aa] overlaps the face of $(2\bar{1}\bar{1})$, belong to the trigonal pyramid $\mu\{\bar{1}00\}$.

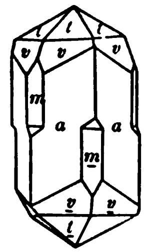


FIG. 499.

V. TWINS OF THE HEXAGONAL SYSTEM.

54. Twins of crystals with their hexad axes coincident in direction have been mentioned in Art. 19 of Chap. XVII. Such twins are complementary twins and are only possible in classes I and III. Their geometry offers no difficulty, and we need not dwell further on them.

Arguing from the fact established in Chap. IX, Art. 21, that only a single hexad axis is possible in a crystal, we should expect that twins with inclined hexad axes would be inconsistent with the possibility of crystal-growth. A single twin of apatite with inclined axes is described in the *Am. Journ. of Sci.* [iii], XXXIII, p. 503, 1887: the axis of rotation is given as the normal to a face of $\{11\bar{2}1\}$. A similar composite crystal with the same twin-face is recorded as observed in a rock-section (*Jour. Geol.* III, p. 25, 1895). But in a

mineral so common and abundant as apatite, these instances seem hardly sufficient to establish a twin-law.

vi. Twins of the oblique system.

55. Crystals of this system are most commonly twinned about a normal or zone-axis lying in the axial plane XOZ; this plane being a plane of symmetry in classes II and III, and perpendicular to a dyad axis in classes I and III of the system. The plane of symmetry and the dyad axis have therefore the same directions in the several portions of the twin. When the portions are in juxtaposition, and the twin-axis is a possible normal, the combination-plane is usually perpendicular to the axis; the twin-law may then be shortly described by specifying the position of the twin-face.

The orientation of these hemitropes may in many cases be also given by taking for twin-axis a line in XOZ perpendicular to that usually adopted. Thus, the twin of hornblende, Fig. 503, is usually said to have (100) for twin-face: it may also be described as having the vertical axis [001] for twin-axis, and (100) for combination-plane. In the Carlsbad-twin of orthoclase, we adopt the vertical axis for twin-axis: we might also refer it to the normal to (100) as twin-axis, but by doing so important relations of the twin are lost sight of. When there is no special reason for choosing the zone-axis rather than the normal, the latter is generally taken as the twin-axis.

The directions of the axes of X and Z have usually been selected arbitrarily, so as to give simple indices to the most conspicuous faces. They are therefore dependent on the habit of the crystals first described, and it may prove that these directions have no apparent connection with those of the twin-axes. It follows that the twin-face may in some cases be (100), in others (001) and in others ($h0l$).

Twins having for twin-axis a normal or zone-axis inclined to the dyad axis or plane of symmetry at angles other than 90° or 0°, are rare except in the case of orthoclase and the harmotome group. In such twins, the twin-axis may be a normal ($hk0$), ($0kl$) or (hkl), according to the edges which have been selected to give the axes of X and Z. As a rule, however, the twin-axes have symbols with very low indices.

56. *Gypsum. 1. Twin-face* (100). Fig. 500 represents a doublet of gypsum, which is common in many localities, or can be produced in the laboratory. The drawing has been made by bisecting the crystal represented in Fig. 140, p. 188, in the plane YOZ, and then turning the front half through 180° about the normal to this plane. The position of the twin-axis is the back-and-fore cubic axis in the projection which serves as basis for the construction of the oblique axes (Chap. VI, Arts. 15 and 16). Similar coigns of the fixed and rotated portions lie on lines parallel to the twin-axis at equal distances from the combination-plane. The faces l of {111} meet in pairs in two re-entrant and in two salient edges, the angle over each of them being 70° 51′. The angles between the edges [bl] in the common face b are 75° 9′ and 104° 51′.

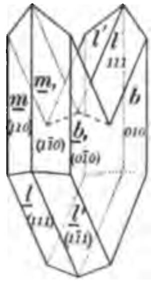


Fig. 500.

The portions occasionally intercross, forming a twin which has common (010) faces and shows sometimes re-entrant angles at both ends, and sometimes salient angles.

2. Twin-face (101). Another twin of gypsum having (101) for twin-face and the face (010) common to both portions is shown in Fig. 501 (after Hessenberg). In this twin the re-entrant angles are formed by faces β having the symbol {509}, and the faces l {111} of the two portions are tautozonal.

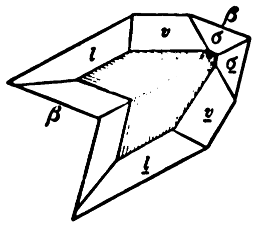


Fig. 501.

In well-developed twins the distinction between the laws *1* and *2* is easily perceived; for the faces m are brighter and more even than the faces l, and the angle $mm_{,} = 68°\ 30'$, whilst $ll_{,}$ is 36° 12′. If sufficiently translucent for optical examination between crossed Nicols, the twins can be distinguished by the minimum angle between the directions of extinction in the plane of symmetry; and plates suitable for the purpose are easily obtained by cleavage.

We have seen in p. 188 that Bx_a is inclined to OZ at an angle of + 52° 27′. Hence in twin (*1*) the angle between the acute bisectrices is 104° 54′, and the least angle between the directions of extinction in the two portions is 14° 54′. In twin (*2*) the vertical axis is inclined to the combination-plane at an angle of 52° 25′, and

Bx_a therefore at an angle of 52° 25′ + 52° 27′ = 104° 52′ (or its supplement 75° 8′). The two directions of extinction are then inclined at 150° 16′, and the least angle between them is 180° − 150° 16′ = 29° 44′, which is almost identically double that in the first twin.

Again in the first twin the interruptions in the perfect cleavage caused by the conchoidal cleavage 100 are parallel to the combination-plane, those due to the fibrous cleavage n $\{\bar{1}11\}$ are inclined to this plane at angles of 66° 10′. In the second twin both sets of interruptions are inclined to the combination-plane, those due to {100} making angles with it of 52° 25′ and those due to n $\{\bar{1}11\}$ angles of 61° 25′.

57. *Hornblende.* *Twin-face* (100). The simple black crystal, Fig. 502, of this substance resembles a rhombohedral crystal, for the angles bm = 62° 15′, $mm_{,}$ = 55° 30′, cm = 76° 48′ and br = 74° 14′. If such a crystal is divided in the plane YOZ, and the front half turned through 180° about the normal to the plane of section, a twin is obtained, which would only differ from Fig. 503, inasmuch as portions of two faces c would meet in a re-entrant furrow modifying the highest coign made by four faces r, and at the other end there would be small indentations at the coigns where faces b meet

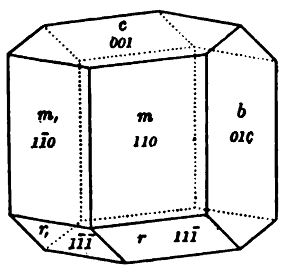


Fig. 502.

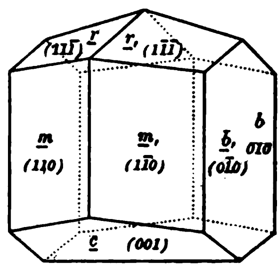


Fig. 503.

the two c faces. The top would then resemble the end of Fig. 504 which is directed to the front. Such re-entrant edges are very rare in the twins, which, as shown in Fig. 503, consequently simulate hemimorphic crystals of the prismatic system in which an acleistous pyramid is developed at one end and a gonioid (or hemidome) of two faces at the other.

A thin section cut parallel to the common face b (010) would between crossed Nicols act like a cleavage-plate of a twin of gypsum,

and would extinguish the light in directions including a minimum angle of about 40°.

Orthoclase.

58. This mineral is stated to form twins according to eleven different laws, most of them rare and some doubtful. We shall describe some of the twins according to the three common and well established laws. These laws are (*1*) twin-face *c* (001), the law being often known as the *Manebach-law*, (*2*) twin-face *n* (021), the *Baveno-law*, and (*3*) twin-axis *OZ*, the *Carlsbad-law*; these last twins occasionally have (010) for combination-plane, but more commonly interpenetrate to some slight extent. The Carlsbad and Baveno twins are the most common.

59. *1. Manebach-law. Twin-face* *c* (001). The appearance of these twins, Figs. 504 and 505, differs much according to the faces which predominate. We shall describe them separately as (α) and (β).

α. Good specimens of these twins have been found at Manebach in Thuringia, Four-la-Brouque in Auvergne, the Mourne Mts. in Ireland, and Pike's Peak, U.S.A. The faces *b* (010) and *c* (001) are largely developed and form a rectangular prism, in which the parallel faces *c* belong each to a separate portion, the faces *b* are composite and are often traversed by striæ or markings parallel to the edges [*bm*] of the respective portions. At one end four faces *m* of {110} are developed, and form, as do the faces *r* in the twin of hornblende, an apparent pyramid with an angle $m\underline{m} = 44° 26'$, and an angle $mm_{,} = 61° 13'$. Occasionally, as shown in Fig. 504, the apex of this apparent pyramid is replaced by a small cavity bounded by faces *y* ($20\bar{1}$), $y \wedge \underline{y}'$ being $-19° 24'$. The other end is formed by two faces *y*, making with one another a salient angle of 19° 24'. In specimens from the Mourne Mts. the coigns at which these faces *y* meet the common faces *b* are modified

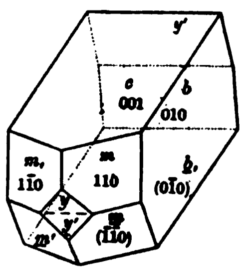


Fig. 504.

by pairs of *m* faces parallel to those forming salient angles to the front in Fig. 504: they therefore form re-entrant angles of $-44° 26'$; and in one of the Irish specimens at Cambridge the faces *m* forming

re-entrant angles are developed so as to obliterate the faces **y**: this end of the twin has the same appearance as that end of Fig. 505 which is shown to the front; the faces *b* and *c* are large, and the crystal is attached to the rock at the end at which the salient pyramid of faces *m* would be developed.

β. Fig. 505 represents a twin of adularia from the St Gothard with (001) for twin-face. In these twins the prism-faces *m* are largely developed, and form, on opposite sides of the twin, salient and re-entrant angles of 44° 26′. The crystals are terminated by small triangular faces *c* (001) and *x* (10$\bar{1}$), the face *c* being easily distinguished by its pearly lustre. The twins of this habit sometimes intercross; and the portions forming the twin are often also combined with portions of other individuals twinned to them according to the Baveno-law.

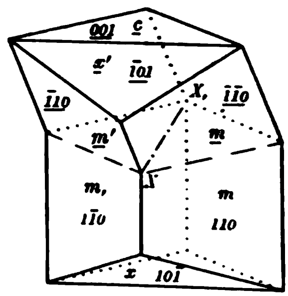


Fig. 505.

The doublet has been drawn by finding the direction and length of the normal on the face *c* (001). The normal lies in the plane *XOZ* at an angle of 26° 3′ to *OZ*, and its direction is therefore found by the same construction as that given in Chap. VI, Art. 15 for finding *OX*. By drawing through *C* on *OZ* a line parallel to *XX*, to meet the normal, its length is obtained. Doubling this length, the origin Ω of the rotated axes is determined. The prism-edges [$\underline{mm}'$] are then drawn through the points in which the prism-edges of the fixed portion meet the combination-plane; corresponding coigns on the rotated and fixed prism-edges lie on lines parallel to the twin-axis. Twin *α*, Fig. 504, is drawn from the same axes, and only differs from *β*, Fig. 505, in the relative size of the faces present.

The zone-axis $XX_{,}$, Fig. 505, may be taken as twin-axis; for a semi-revolution about it gives the twin-orientation correctly. We have given the twin-law by means of a twin-face for the sake of brevity in its enunciation.

60. *2. Baveno-law. Twin-face* *n* (021). Fig. 506 represents the habit of these twins, as commonly observed on specimens from Baveno and elsewhere; the faces *b* and *c* being largely developed. Since $b \wedge n = 45° \; 3{\cdot}5'$, the pair of adjacent faces *b* include an angle of 90° 7′, and the pair of faces *c*, which complete the almost rectangular

prism, are at 89° 53′ to one another. Opposite faces are of dissimilar character, and are not strictly parallel. At one end pairs of faces m and y meet in small salient angles, which can be computed from the angles given on p. 190. Faces $o\{\bar{1}11\}$, $x\{\bar{1}01\}$, and $z\{130\}$ are also often present. The angles are

$$m \wedge \underline{m} = 180° - 2n'm = 10° \, 34',$$

$$y \wedge \underline{y} = 180° - 2n'y = 13° \, 42',$$

and $$o \wedge \underline{o} = 180° - 2n'o = 92° \, 34'.$$

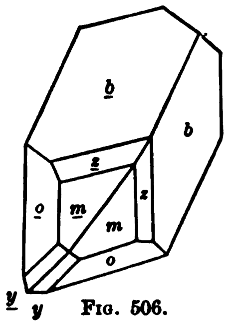


FIG. 506.

The crystals are usually broken, having been attached to the rock at the end opposite to that shown in the figure, but occasionally specimens showing both ends are preserved. In the perfect specimens in the Cambridge Museum, the faces at the end opposite to that shown to the front in Fig. 506 make re-entrant angles.

The individuals are rarely united along a plane passing, as in Fig. 506, through the edges $[b\underline{b}]$, $[c\underline{c}]$, but parts of a face b of one individual overlap, and are practically co-planar with, a face c of the other, and vice versâ; and the boundaries of the two individuals are often very irregular. When portions of b and c faces are coplanar or nearly so, the distinction in their character is plainly seen, and much assists the student in tracing the boundary of the twin; for the faces b are often marked in directions parallel to the edges $[bm]$, which are inclined to the edge $[bc]$ at angles of 63° 57′; the faces c have a pearly lustre, and are sometimes corroded in lines parallel to the edges $[cxy]$ which are at right angles to the edge $[bc]$.

Adularia from the St Gothard twinned according to this law is frequently found in triplets and occasionally in quartets, which Figs. 507—510 serve to illustrate. In these plans the edge $[bc]$ is perpendicular to the paper, the edges $[cx]$ are parallel to it, and the edges $[bm]$ are inclined to it at angles of 26° 3′.

The plans are easily constructed; for a square being drawn, it is only necessary to find the angle which the projected zone-axis $[xm]$ makes with the trace c to be able to complete each of the figures. Now any face in the zone $[bc] = [100]$ meets the paper in a straight line, which makes with the trace $c_{,}$ the same angle as the face makes with the face $c_{,}$. Also the face which is common to the zones $[xm] = [1\bar{1}1]$ and $[bc] = [100]$ has the symbol (011). Knowing the angle cn to be 44° 56·5′, the angle $c \wedge 011$ is found to be 26° 31′. The lines $[mx]$, $[m_{,}x]$, &c., are then drawn each at 26° 31′ to

the traces $c_{,}$, $\underline{c}$, &c., through the corners of the square. The edges [$mm_{,}$] are parallel to the traces b.

Fig. 507 represents two crystals in contact along an edge [bc], that labelled I being in the usual position of an oblique crystal, the second in an orientation differing from that of I by a semi-revolution about the normal to the face n (021). Suppose now the two crystals to be bisected by planes through the corners $n_{,}$ and $\underline{n}'$ parallel to the twin-face n, and the more remote halves to be united in the planes of section. We thus get the doublet given in the plan, Fig. 508. In this doublet the faces $m_{,}$ and $\underline{m}'$ meet in a salient angle of 10° 34′,

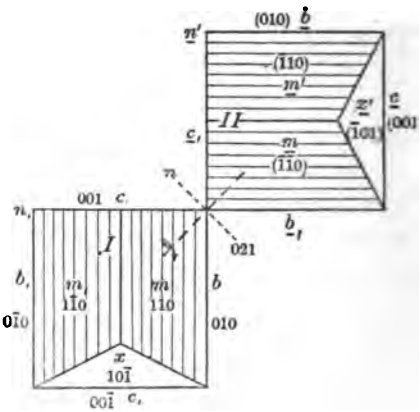


FIG. 507.

the small portions of m and $\underline{m}$ in a re-entrant angle of $-77°\ 42'$, and the faces x make a salient angle $x\underline{x}' = 53°\ 47{\cdot}5'$. The faces m are usually striated parallel to the edge [mb].

If now at the corner $\underline{n}'$, Fig. 507, a third crystal is introduced in twin-orientation to, and touching individual II with (02$\bar{1}$) for twin-face, and the three crystals are cut down so as to form a rect-

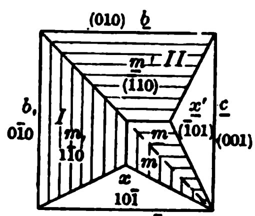


FIG. 508.

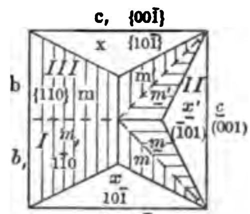


FIG. 509.

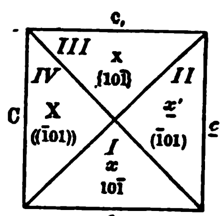


FIG. 510.

angular prism, we obtain the triplet shown in Fig. 509. In this case, three of the faces bounding the rectangular prism are c faces, and the fourth consists of two b portions inclined to one another at a salient angle of 14′: for the angles $b_{,}c_{,} = bc_{,} = 90°$, and $c_{,}\underline{c} = \underline{c}c_{,} = 89°\ 53'$. The individuals I and III are very nearly, but not absolutely in twin-orientation according to the Manebach-law; for the plane through

$[m_, m]$ in which they meet is very nearly parallel to the faces $c_,$ and $c_,$, but the faces $m_,$ and m meet in a re-entrant angle of $-44° 16'$. The edges $[c_, x]$ of the triplet are all parallel to the paper, the edges $[b_, m_,]$ and [bm] are both inclined to it at angles of 26° 3'. Triplets of this kind are fairly common, but the re-entrant angles $m \wedge \underline{m}$ and $\underline{m}' \wedge m'$, are almost always obliterated as shown in Fig. 510 by the development of the faces x, and the filling up of the troughs.

Fig. 510 represents a quartet of the same kind which may be supposed to be produced by associating a fourth individual in twin-orientation at $n_,$ in Fig. 507. Individuals III and IV are not then in strict twin-orientation, and the faces of the prism are not quite parallel. The twin approximates in habit and development to a tetragonal prism surmounted by a tetragonal pyramid. The twins are therefore of interest in showing how a crystal may be developed as a mimetic twin out of portions of crystals of much inferior symmetry. Specimens corresponding to each of the drawings of the ideal twins given above, are far from showing the regularity of development assumed in the description. The crystals are generally much corroded, and the faces are sometimes encrusted with chlorite. The faces c seem to be most easily corroded, and those of b to be most readily encrusted.

61. *Manebach-Baveno twins.* In the collection of minerals recently presented to Cambridge by the Rev. T. Wiltshire, M.A., Hon. Sc.D., of Trinity College, is a remarkable twin of adularia from the St Gothard shown in Fig. 511. It forms a nearly rectangular prism bounded by composite faces b (010), and is strictly speaking an octet; each segment, such as that having its faces labelled b_1, m_1, x_1, being twinned to an adjacent segment (that carrying suffixes 2) on one side according to the Baveno-law, and to the adjacent one on the other side (that lettered b'_3, m_3, x'_3) according to the Manebach-law. The combination according to the two laws is not as regular as that seen in ordinary doublets, and there is a certain amount

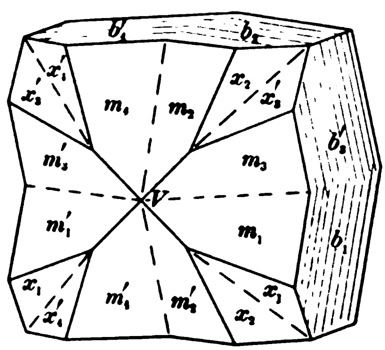


Fig. 511.

of interlocking of the matter of adjacent portions across the combination-planes. Thus, as is slightly indicated in the composite faces *b*, the markings transgress the trace of the ideal combination-plane of the Manebach-twin. A similar irregular interlocking is observed at the back of the crystal, where two faces *m* meet at a small salient angle over an edge which should lie in the twin-plane (021) of the Baveno-law. The edges of the prism formed by the composite *b* faces are also modified, and are replaced by very narrow rectangular grooves having the faces *c* for sides. These grooves much resemble the well-marked grooves on twins of harmotome, Fig. 516, p. 540. The twin is not entirely free from attached matter, but sufficient of the back can be made out to show that the edges separating faces of different portions are all salient.

The faces in Fig. 511 have been labelled to correspond to an interpretation of the twin as consisting of two intercrossing Manebach-twins; the one having even suffixes, the other odd ones. Again adjacent wedges consisting of portions of odd and even labelled individuals are twinned according to the Baveno-law. Similarly labelled faces *x* give almost simultaneous reflexions of a bright signal. Hessenberg (*Min. Not.* II, p. 4) has described a similar twin.

62. *3. Carlsbad-law. Twin-axis* [001]. The twins shown in Figs. 512 and 513 have the vertical axis for twin-axis with an angle of rotation of 180°. Consequently in each twin the faces *b* {010} and *m* {110} of one individual are parallel respectively to faces *b* and *m* of the other. The twins are often in juxtaposition, the

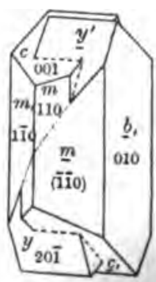


FIG. 512.

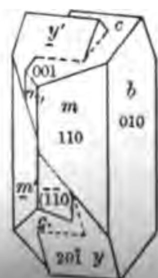


FIG. 513.

combination-plane being parall[illegible] (010): in oth[illegible] cases there is, as shown in the figures, a [illegible] interlocking of the individuals. This int[illegible]ved when the crystals have the faces [illegible]oped.

Since the portions of an oblique crystal on opposite sides of the plane of symmetry are antistrophic, two twins according to this law are possible which, depending on the relative positions of the components, are sometimes described as right- and left-handed. Suppose two exactly similar crystals, having the faces in Figs 512 and 513, to be placed side by side in the usual position of an oblique crystal, and suppose them to be bisected in their planes of symmetry. Suppose now the two portions to the left of the plane of symmetry to be united in this plane after one of them has been turned through 180° about the vertical axis [*bm*]: a left-handed twin identical, save for interlocking portions, with Fig. 512 is produced. Fig. 513 is obtained by the similar union of the two portions to the right of the plane of symmetry, and may be called a right-handed twin. If the two twins are placed side by side with the faces *b* and the edges [*bm*] parallel, the pair are reciprocal reflexions in a mirror parallel to their planes of symmetry, and cannot therefore be placed in similar positions. Good instances of right- and left-handed twins have been found in many localities, and have been recently obtained in great abundance at Bob Tail Gulch, Colorado.

When the faces *c* and *x* $\{10\bar{1}\}$ are the only faces associated with the prism {110} and pinakoid {010}, the juxtaposed twins resemble a simple prismatic crystal, for a face *c* of the one individual is as shown in Fig. 514 practically co-planar with a face *x* of the other. The inclinations of the two faces to the vertical axis are nearly the same, for, Fig. 142, p. 191, $C \wedge Z = 26° 3'$ and $x \wedge Z = 24° 13{\cdot}5'$. G. Rose, who carefully examined the twins, was however unable to detect any divergence of the two portions of the composite face, corresponding to the difference of 1° 50′, between their inclinations. Owing to the difference in physical character and corrodibility of the faces *c* and *x*, the composite character of the faces in the twin is well seen. This has been indicated in the figure by marking the faces *x* with dots. A repetition of the twinning according to this law is sometimes indicated by narrow interruptions traversing the faces *c* and *x* parallel to the edges [*bc*] and [*bx*] of an otherwise simple crystal.

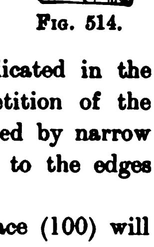


Fig. 514.

A rotation of 180° about the normal to the face (100) will satisfy the relations of this twin. The prism-faces of the rotated

individual would in that case have different indices, for after a semi-revolution about the normal each of them is brought into a position coincident with that of a different homologous face of the prism. But such an interpretation of the twin will not satisfy the relations of the similar twins of the closely allied anorthic felspars. For in these latter substances the angle $100 \wedge 010$ differs from 90°, and the faces (010) of the two portions would not be parallel, as has been found to be the case in the twins.

63. J. D. Dana pointed out in the 5th edition of his *Mineralogy* a remarkable approximation in the angles of orthoclase and the anorthic felspars to those between faces of the more common forms of a cubic crystal. Thus $bm = b_{,}m_{,} = 59° 23{\cdot}5'$ and $mm_{,} = 61° 13'$ are all comparable with 60°, the angles between the faces of the rhombic dodecahedron parallel to a triad axis. Hence the axis OZ of orthoclase corresponds to a triad axis of a cubic crystal, and may be regarded as a pseudo-triad axis. The faces z in the zone $[bm]$ make 29° 24′ with b and 29° 59·5′ with adjacent faces m: they correspond to faces of {211} which truncate the edges of the rhombic dodecahedron.

Again, n being (021), the angles $bn = 45° 3{\cdot}5'$ and $cn = 44° 56{\cdot}5'$ are very nearly 45°. The zone $[bnc]$ is therefore comparable with a tetragonal zone, and the zone-axis $[bc]$ (the axis OX) is a pseudo-tetrad axis. But from the zone $[bmm_{,}]$, the face b is seen to correspond to a face of the rhombic dodecahedron, and its normal is a dyad axis. Hence the face c also corresponds to a face of the rhombic dodecahedron and its normal to a pseudo-dyad axis. The faces n then correspond to the tautozonal faces of the cube, and their normals to pseudo-tetrad axes; the relation in this zone being similar to that described in the zone $[bc]$ of staurolite (p. 511).

The third face of the cube has to be sought in the zone $[cxy]$ at right angles to c and n. The nearest important face to this position is y $(\bar{2}01)$, for $cy = 80° 18'$, $ny = 83° 9'$. Knowing the angles cx and cy, we find that the indices of the face $(h0l)$ at 90° from c must satisfy the equation $1{\cdot}43h = 3{\cdot}86l$. Hence $(\bar{2}01)$, $(\bar{3}01)$, $(\bar{8}03)$ and $(\bar{27}, 0, 10)$ are successive approximations to the position: they correspond to angles

$$001 \wedge \bar{3}01 = 93° 50{\cdot}5',\ 001 \wedge \bar{8}03 = 89° 39{\cdot}5' \text{ and } 001 \wedge \bar{27}, 0, 10 = 90° 0'.$$

By the aid of the relations just given, many of the twins of orthoclase can be explained from considerations as to the probable arrangements of the particles in the different classes of crystals. Amongst the regular arrangements possible, there are some in which groups of the particles about one point can be brought into the position of similar groups by translation through some finite interval. When the crystals have also axes of symmetry, pairs or triads or tetrads of like groups can be interchanged by rotations of 180°, 120° and 90° about the respective axis. Such arrangements of the groups of particles may be said to be *conformable*. A

rotation through the respective angle about an axis which is only approximately one of symmetry will not bring a group into coincidence with a similar group; but the relations of the rotated group to the matter in its neighbourhood may so nearly conform to those required for the attachment of the group as to allow growth to proceed, and the more nearly the axis approximates to a true axis of symmetry the greater the probability of twinning and the more stable is the twin likely to be.

From this point of view, the Manebach-twin is due to the normal to the face $c\,(001)$ being a pseudo-dyad axis; the Carlsbad-twin to the axis OZ being a pseudo-triad axis, and to the structure about this axis having much the character which is connected with the twinning about a triad axis in cubic crystals. The Baveno-law may, as in the twins of pyrites and the rectangular twins of staurolite, be given in two ways: (α) a rotation of 180° about the normal (021) which is a pseudo-tetrad axis, or (β) a rotation of 90° about the zone-axis OX (the same as $[bnc]$) which is also a pseudo-tetrad axis.

Further, this hypothesis suggests that the normals (110), $(20\bar{1})$ and $(\bar{1}11)$ should also be twin-axes; for each of them is nearly in the direction of a pseudo-axis of symmetry of even degree. Twins with each of these normals for twin-axis—the angle of rotation being 180°—have been described, but they are very rare. Since these lines are not so nearly in the directions of axes of symmetry as the axes OX, OZ and the normals (001), (021), the conditions which would give rise to twinning about them are not likely to be as frequently met with as in the case of the three important laws.

Harmotome Group.

64. The twins of orthoclase and their relations to the pseudo-cubic symmetry of the crystals throw light on the remarkable twins of harmotome, phillipsite and wellsite, in none of which has the simple crystal been observed. Fig. 515 represents a crystal of harmotome from Scotland, which was long believed to be a simple prismatic crystal. Des Cloizeaux showed, however, by optical examination between crossed Nicols, that sections cut parallel to the face marked b and $\underline{b},$ are composed of four quadrants, such that the directions of the extinctions in opposite quadrants (those marked by similar letters b) are the same, whilst those in adjacent quadrants (marked b and $\underline{b},$ respectively) are inclined to one another. One of the directions of extinction in each quadrant is inclined, as shown in Fig. 515, at an angle of nearly 60° to the edge $[bc]$. This crystal resembles a Manebach twin of orthoclase, in which, however, the two individuals intercross so that the two ends show a similar pyramid of four like faces. Occasionally the

apex of this pyramid is modified by a small uneven face f, the inclination of which to c cannot be distinguished from 90°.

Des Cloizeaux took the upper pair of faces m to be those of the prism {110} of an oblique crystal, in which b (010) is parallel to the plane of symmetry, c is the base (001) and f is (10$\bar{1}$). The faces are very uneven and striated, so that accurate measurements are not possible. He adopted the following values, viz. $mm_{,} = 59° 59'$, $cm = 60° 21'$, and $cf = 90° 0'$. From these angles we can compute the elements of the crystal, and the angles which a face e (011) truncating the edge [bc] makes with the faces on the crystal. The elements of the crystals of the other minerals can be found from similar data. We thus have for:

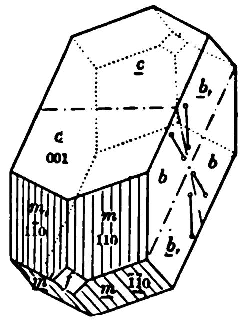


Fig. 515.

	β	$a:b:c$	be	me	$m_{,}e$	$m'm'''$
Harmotome	55° 10′	·703 : 1 : 1·231	44° 42′	45° 18·5′	90° 25′	0° 50′.
Phillipsite	55 37	·709 : 1 : 1·256	43 58	45 25	91 27	2 54.
Wellsite	53 27	·768 : 1 : 1·245	45 0	43 8	90 45	1 30.

65. The crystals are, however, for the most part complex twins similar to Figs. 516 and 517. The former is common in harmotome, the latter in phillipsite and wellsite. The striated faces in Fig. 516 correspond to those marked b in Fig. 515, and are, as far as measurements can be made, at right angles to one another; so also are the faces c which form the sides of the grooves running the whole length of the edges of the apparently rectangular prism. Similar crystals are often observed in phillipsite, but the faces c are

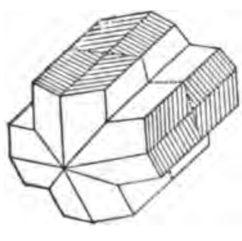


Fig. 516.

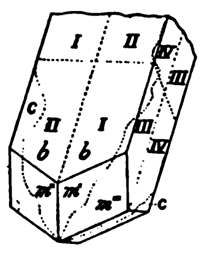


Fig. 517.

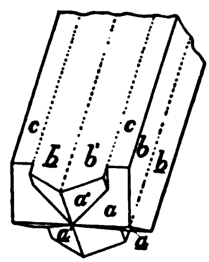


Fig. 518.

very often those forming the prism, and the faces *b* form the sides of the groove. Regarding the intercrossing doublet, Fig. 515, as if it were a simple crystal, the complex twin is formed of two such individuals having *e* (011) for twin-face; or it may be treated as consisting of four wedges cut from the doublets and, like the wedges of orthoclase in Fig. 510, united in succession along a twin-face (011). Or, again, as was pointed out in the case of orthoclase, we may suppose the wedges to be turned through 90° about the zone-axis [*bc*] (a pseudo-tetrad axis), and each wedge to be united to the adjacent ones more or less regularly along a face (011). The twin agrees most closely, however, with the octet of adularia, Fig. 511, in the Wiltshire collection, save that in the twins of the harmotome group of minerals the ends showing salient angles are alone developed.

Since $e\underline{m} = 89° \, 35'$, the faces *m* separated by a line joining the apex to the end of the groove are almost co-planar. On the assumption of a semi-revolution about *e* (011), they make in harmotome a salient angle of 50′; and on the assumption of a quarter-revolution about [*bc*], one of 25′. The composite character of the *m* faces is however most clearly demonstrated by the barbed striæ on them, the striæ being parallel to the edges [*bm*] of the respective individuals.

These complex twins are very regularly developed in harmotome, and the grooves along the edges of the prism are well marked. In phillipsite the faces composing the rectangular prism are free from striæ and belong to *c* {001}; the grooves are bounded by faces *b*, which sometimes show patches of striæ similar to those on the prism-faces of Fig. 516. But in phillipsite and wellsite the grooves are for the most part obliterated, and the twins closely resemble a tetragonal prism {100} terminated by a tetragonal bipyramid {111}. Examination between crossed Nicols of plates cut parallel to the faces of the pseudo-tetragonal prism shows that combination does not take place regularly along the plane (011), but portions of a face *c* are co-planar with a face *b* of the adjacent individual. This is indicated in Fig. 517, where the irregular lines of dots show the boundaries between the individuals, labelled I—IV, in a twin of wellsite described by Messrs Pratt and Foote (*Am. Jour. of Sci.* [iv], III, p. 443, 1897). They observed that the boundary between the portions *b* forming part of the doublet is a definite line corresponding to combination parallel to *c* (001).

Wellsite[1] was found by the authors to form other octets (or intercrossing quartets), Fig. 518, in which the faces *a* (100) are largely developed. The faces *a* of the doublet, having (001) for twin-face, make with one another salient angles of 73° 6′ (an angle used in the computation of the crystal-elements); and adjacent faces *a* of different doublets, such as *a* and *a′*, make re-entrant angles of 49° 48′. Similar twins of harmotome from Kongsberg, St. Andreasberg, and Scotland, and of phillipsite from Asbach have been described.

66. Applying to these twins the views as to pseudo-cubic symmetry already propounded in the discussion of the twins of staurolite and orthoclase, we see that the zone [*bc*] is one of pseudo-tetragonal symmetry, the normals *b* and *c* being the one a dyad, and the other a pseudo-dyad, axis. The doublet, Fig. 515, is then due to crystal-growth being possible when the particles or groups of particles are deposited in orientations differing from that of conformability by a semi-revolution about a pseudo-dyad axis (the normal to (001)). The same orientation is obtained by a semi-revolution about the zone-axis [*bc*]. Again, the complex twin is due to [*bc*] being a pseudo-tetrad axis; a quarter-revolution about this line bringing the groups of particles sufficiently near to a position of conformability for growth to be possible.

Further, the edge [$bmm_{,}$], Fig. 515, is a pseudo-triad axis; and a rotation of 120° about it should leave the particles in approximate conformability with their first orientation. This axis is inclined to [*bc*] at the angle β, which is in all the crystals very nearly equal to 54° 44′—the angle between a triad and tetrad axis of a cubic crystal. Still more complex twins, such as that of phillipsite shown in Fig. 519 (after Köhler, *Pogg. Ann.* XXXVII, p. 561, 1836), are therefore possible, which may be explained as due to this pseudo-triad axis. The same twin is arrived at by regarding it as the result of twinning about the pair of pseudo-tetrad axes which are the normals to the faces *e* of {011}, and are at right angles to [*bc*]; the angle of rotation about each of these normals being 90°.

[1] Wellsite is shown by Messrs. Pratt and Foote to be a member of the group of zeolites formed by phillipsite, harmotome and stilbite; the composition being:

Wellsite ....$(Ba, Ca, K_2)\,Al_2Si_3O_{10}\,.\,3H_2O$.
Phillipsite ......$(Ca, K_2)\,Al_2Si_4O_{12}\,.\,4\tfrac{1}{2}H_2O$.
Harmotome......$(Ba, K_2)\,Al_2Si_5O_{14}\,.\,5H_2O$.
Stilbite............$(Ca, Na_2)\,Al_2Si_6O_{16}\,.\,6H_2O$.

Seeing that the determination of the true state of hydration is very difficult, and that the analyses of phillipsite vary considerably, it is probable that the number of molecules of water of crystallization should be 4 and not 4½. The authors point out also that a lower member of the series, having two molecules of water, may exist. Such a member would be a dimorphous form of natrolite.

The upright individual in Fig. 519 is placed as if the largely developed faces were {110} of a tetragonal crystal and the normals to the faces *e* were the axes of *X* and *Y*. By a rotation of 90° about the back-and-fore normal (*OX*), the vertical axis is brought into the position of the axis of *Y*. By joining two quartets related to one another in this manner, an intercrossing rectangular twin is obtained, such as is sometimes found in phillipsite: it may be represented by the two individuals of Fig. 519 which have their axes [*bc*] parallel to the paper. If now a third quartet is joined to the two, after it has been turned through 90° from the upright position about the axis *OY*, the twin, Fig. 519, is produced. The composite faces *m* of Fig. 516 are in Fig. 519 arranged in sets of two which are very nearly co-planar, and are parallel respectively to faces of the rhombic dodecahedron. If, further, the matter is deposited so as to fill up the re-entrant spaces between the prisms, and if the faces *m* are developed so as to obliterate all the other faces, a twelve-faced figure is produced, which is very nearly a rhombic dodecahedron, each of the faces being composed of segments which belong to four separate individuals. A mimetic dodecahedron of phillipsite of this kind has been described by Streng (*N. Jahrb. f. Min.* 1875, p. 585).

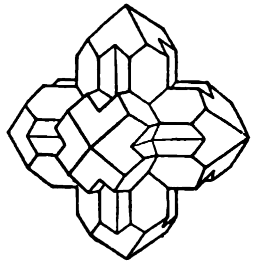


FIG. 519.

Langemann (*N. Jahrb. f. Min.* 1886, II, p. 83) has shown that plates cut parallel to the face *cc* of the doublet, Fig. 515, consist of four segments, such that, between crossed Nicols, opposite segments extinguish the light simultaneously, but adjacent segments have different directions of extinction. The face *b* is not therefore parallel to a plane of symmetry, or its normal a dyad axis; and the simple crystal of harmotome and phillipsite must be regarded as anorthic. The intercrossing doublet, Fig. 515, is then a mimetic octet or a mimetic intercrossing quartet, the individuals being separated along planes parallel to *b* and *c* passing through the edges $mm_{,}$ and $m\underline{m}$. The relations to pseudo-cubic symmetry apply to the crystals as before, but the normal to the face *b* is now only a pseudo-dyad axis.

vii. TWINS OF THE ANORTHIC SYSTEM.

67. We shall limit our discussion to a few of the twins of the felspars, microcline, $KAlSi_3O_8$, albite, $NaAlSi_3O_8$, and anorthite, $CaAl_2Si_2O_8$, mentioning in the intermediate members of the group a few cases which are pertinent to the discussion. The plagioclastic felspars are all characterised by having two perfect cleavages inclined to one another at angles of nearly 90° and by the prominence of faces parallel to these cleavages: they also have faces

of interlocking of the matter of adjacent portions across the combination-planes. Thus, as is slightly indicated in the composite faces *b*, the markings transgress the trace of the ideal combination-plane of the Manebach-twin. A similar irregular interlocking is observed at the back of the crystal, where two faces *m* meet at a small salient angle over an edge which should lie in the twin-plane (021) of the Baveno-law. The edges of the prism formed by the composite *b* faces are also modified, and are replaced by very narrow rectangular grooves having the faces *c* for sides. These grooves much resemble the well-marked grooves on twins of harmotome, Fig. 516, p. 540. The twin is not entirely free from attached matter, but sufficient of the back can be made out to show that the edges separating faces of different portions are all salient.

The faces in Fig. 511 have been labelled to correspond to an interpretation of the twin as consisting of two intercrossing Manebach-twins; the one having even suffixes, the other odd ones. Again adjacent wedges consisting of portions of odd and even labelled individuals are twinned according to the Baveno-law. Similarly labelled faces *x* give almost simultaneous reflexions of a bright signal. Hessenberg (*Min. Not.* II, p. 4) has described a similar twin.

62. *3. Carlsbad-law. Twin-axis* [001]. The twins shown in Figs. 512 and 513 have the vertical axis for twin-axis with an angle of rotation of 180°. Consequently in each twin the faces *b* {010} and *m* {110} of one individual are parallel respectively to faces *b* and *m* of the other. The twins are often in juxtaposition, the

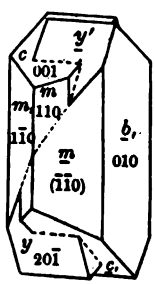


Fig. 512.

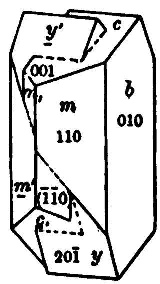


Fig. 513.

combination-plane being parallel to (010): in other cases there is, as shown in the figures, a greater or less amount of interlocking of the individuals. This interlocking is generally perceived when the crystals have the faces *c* {001} and *y* {201̄} well developed.

lines, so that the figure is self-congruent, *i.e.* exactly fits, after it has been turned through 180° about the normal to its plane. Fig. 521 gives a good representation of a doublet of oligoclase; but in albite the thickness of the twin perpendicular to the faces M is much less in proportion to the other dimensions, and the twinning is often repeated. The pairs of faces P, x and y (when present) which meet at the upper end of the vertical axis include re-entrant angles, the opposite pairs equal salient angles. The angles are $P \wedge \underline{P} = 7^\circ 12'$, $x \wedge \underline{x} = 7^\circ 20'$ and $y \wedge y = 4^\circ 41'$. Notwithstanding the small difference in the angles, the pearliness of the faces P renders it easy to distinguish them from the others; and they can often be recognised by the directions of the cleavage-flaws, which traverse the crystals parallel to P.

Fig. 522 represents an intercrossing doublet of albite from Roc Tourné in Savoy which was first described by Rose (*Pogg. Ann.* cxxv, p. 456, 1865). The faces indicated by letters only are $f\{130\}$, $l\{1\bar{1}0\}$, $p'\{1\bar{1}\bar{1}\}$, $o'\{11\bar{1}\}$. In these twins the faces P make re-entrant angles on opposite sides of the centre, and the faces y salient angles; further, the largely developed faces M are traversed by narrow troughs bounded by faces f. The twin is composed of four portions separated by the combination-plane and by the plane through the troughs perpendicular to M, adjacent portions being in twin-orientation.

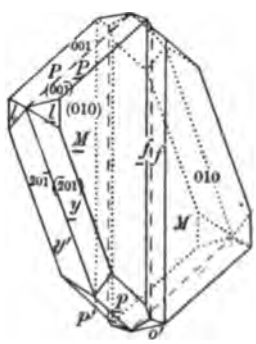


FIG. 522.

By breaking the crystal along the pearly cleavage P, a fragment shown in Fig. 523 was obtained. Rose found that the angle between the cleavages changed at the troughs from a re-entrant to a salient angle. The cleavage portions labelled P are parallel, and so are those labelled $\underline{P}$.

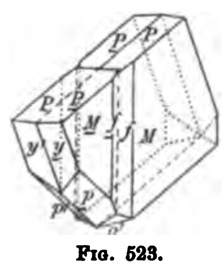


FIG. 523.

Twinning according to the albite-law is often repeated, giving rise to polysynthetic twin-lamellæ of varying thickness. A second semi-revolution of 180° about the same line brings any face to its original position. Hence the faces P, y and x

of alternate lamellæ will be parallel, and faces of the same form will in equably developed twin-crystals always meet in the combination-planes. The presence of such twin-lamellæ can be often detected in cleavage-fragments of the plagioclastic felspars by the presence of a series of parallel striæ due to the alternating salient and re-entrant angles at which the pearly cleavages of the several portions meet. Such striæ, when present, serve as a means of distinction between these minerals and orthoclase; for in the latter the normal to (010) is a dyad axis, and the faces $010 \wedge 001$ are at 90° to one another.

Even when the above test fails, polysynthetic twinning according to this law can be often recognised by examining a thin cleavage-flake parallel to P between crossed Nicols; for, except in the case of some specimens of oligoclase and andesine, the directions of extinction in the twin-lamellæ of the cleavage-flake are inclined to the combination-plane at very appreciable angles.

69. *Twin-face* P (001). Fig. 524 is a sketch, which serves to show the unequal development of the faces and individuals in a twin with twin-face (001) of the green variety of microcline called Amazon-stone from Pike's Peak, U.S.A. The figure is a plan with $[P_{,}M]$ perpendicular to the paper, and is made in the same way as the plans of the Baveno-twins of orthoclase, Figs. 507—510. Owing to the close approximation to 90° of the angle $P \wedge M$—the value adopted by Des Cloizeaux being 89° 45′—the faces M and $\underline{M}_{,}$ are practically co-planar, the theoretical angle being 30′. The edge $[y'\underline{y}]$ is interrupted by a dimple having for sides faces parallel to l and T. A portion of the rotated individual projects on one side beyond the plane $M_{,}$ of the other; and at the irregular line the crystal is attached to a fragment of another.

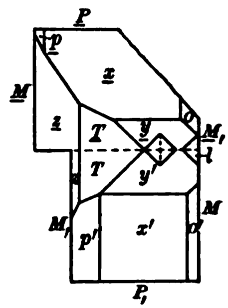


FIG. 524.

Crystals of pericline twinned according to this law have been observed, the faces M on the different portions of the twin being inclined to one another at angles of 7° 12′.

70. *Twin-axis* $XX_{,} = [100]$. The twin of microcline, Fig. 524, may like the similar twin of orthoclase, Fig. 504, be referred to $XX_{,}$ as twin-axis. In this case it should be regarded as composed

of two like portions of separate crystals, and the faces {110} and {1$\bar{1}$0} meeting in the combination-plane belong to different forms. Thus l would meet a face T, and a face z of {1$\bar{3}$0} would meet a face f of {130}. The faces M and $\underline{M}_{,}$ would be exactly co-planar; but the traces of l and T, z and f in the plane XOY would not be accurately congruent. The crystals are too irregular and the faces too uneven for such minute differences to be determined.

A twin of andesine having $XX_{,}$ for twin-axis has been described by vom Rath. His drawing of the twin has a fairly close resemblance to that of orthoclase shown in Fig. 504, but several additional forms are given. The composite faces M are co-planar and the faces P parallel; whereas, if the twin-axis were the normal (001), the faces M would meet in the combination-plane at an angle of 7° 32′. The angle between the axes of X and Y is 89° 59′, so that the traces of l{110} and T{1$\bar{1}$0} in this plane make practically a rhombus, and after a semi-revolution about $XX_{,}$ a face l meets a face T in a line lying in the plane XOY as combination-plane.

A twin of pericline having $XX_{,}$ for twin-axis and P (001) for combination-plane is given by Des Cloizeaux, but it generally occurs associated with the albite-law in complex twins. In the simple twin the faces M are co-planar, but the traces l and T in the plane XOY will, after a semi-revolution about $XX_{,}$, be inclined to one another at angles of 2° 59′: there is therefore a certain amount of overlapping. If this overlapping is avoided, then combination must take place along a plane inclined to the base in a manner similar to that described under the pericline-law.

71. *Twin-axis* $ZZ_{,}$ [001]; *combination-plane* (010). This twin occurs in several of the felspars, *e.g.* in anorthite and andesine from Japan, and it is fairly common in albite, although, from the continual occurrence of twinning according to the albite-law, its recognition in this last mineral is often difficult. The doublet according to this law of the ideal crystal, Fig. 520, will so nearly resemble Fig. 521 that the difference is not perceptible in a drawing; for, taking the elements of albite adopted by Dana, the angle $ZOP = 26° 42′$ and $ZOx = 24° 10′$. But the re-entrant angles at one end and the equal salient angles at the other are each made by a face P and a face x of different individuals. When the faces are wide enough to be distinctly seen, their difference of lustre makes it easy to discriminate between them, and therefore to determine the twin-law.

Another easy test for distinguishing between twins according to this and the albite-law is afforded by the cleavage-cracks which so frequently traverse the crystals of albite. When these cracks occur and the crystal is sufficiently translucent for us to get the reflexions from them of light traversing the crystal, the cleavages are perceived to cross one another at angles of 52° 58′ in this twin, whilst they are parallel in the albite-twin. This latter test often proves an albite-twin to consist of several lamellæ united together successively according to the albite and the present laws. When faces *y* are also present, portions of the individuals overlap in a manner similar to that observed in the Carlsbad-twins of orthoclase.

Pericline-law. *Twin-axis* $YY_{,} = [010]$.

72. This twin—called after the variety of albite in which the faces $P\{001\}$ and $x\{\bar{1}01\}$ are largely developed—will be best understood if the corresponding twins of anorthite represented in Figs. 526, 528, and 529 are first described. The Vesuvian crystals of anorthite, having smooth and bright faces, admit of accurate measurement; and, though variable in habit, generally show the forms in Fig. 525. The crystals and twins of anorthite are described by vom Rath, in two classical memoirs (*Pogg. Ann.* CXXXVIII, p. 449, 1869; CXLVII, p. 22, 1872), from which the figures have been copied with slight modifications. The forms shown in Fig. 525 are given on p. 159, and the positions of their poles and the axial points in Fig. 121, p. 158. We need only to know that P is $\{001\}$, $M\{010\}$, $l\{110\}$, $T\{1\bar{1}0\}$, $h\{100\}$; and that the elements and angles are:

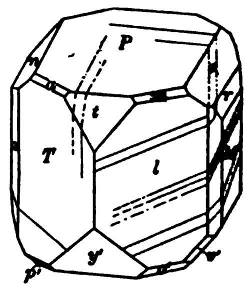


FIG. 525.

$$\alpha = YOZ = 93^\circ\ 13{\cdot}3', \quad \beta = ZOX = 115^\circ\ 55{\cdot}5', \quad \gamma = XOY = 91^\circ\ 11{\cdot}6'.$$

$$a : b : c = {\cdot}63473 : 1 : {\cdot}55007.$$

$$MP = 85^\circ\ 50', \quad Ph = 63^\circ\ 57', \quad hM = 87^\circ\ 6', \quad lT = 59^\circ\ 29', \quad TM_{,} = 62^\circ\ 26{\cdot}5'.$$

Suppose two crystals like Fig. 525 to be placed in similar positions and to be bisected by planes parallel to the base. If now the upper halves are united in their planes of section, after one of them has been turned through 180° about $YY_{,}$, with this axis in common, a twin is produced in which the faces M to the right make

a salient angle with one another, and those to the left a re-entrant angle. An actual twin of this kind is shown in Fig. 526. If the lower halves are united in a similar way, the twin produced has the re-entrant angle between the faces M on the right and the salient angle on the left. The two modifications resemble the similar modifications of the Carlsbad-twin of orthoclase; and, like them, the pairs of twins composed from the same two crystals of anorthite can be placed so that they are reciprocal reflexions in a vertical plane ZOX: this relation can be fairly well seen by comparing the twins shown in Figs. 528 and 529.

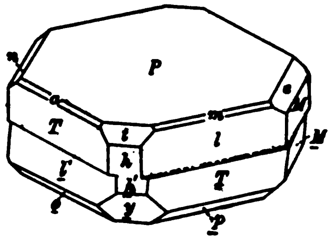


FIG. 526.

73. *The rhombic section.* Any plane through $YY_,$ of Fig. 530 meets the two pairs of faces l and T, produced beyond M to meet in vertical lines, in a parallelogram having $YY_,$ for one diagonal and for the other a line OR in the plane ZOX which contains the vertical edges $[lT]$. If the plane of section is XOY, the two diagonals are $XX_,$ and $YY_,$, the angle between them being the crystal-element γ; and the parallelogram is $BXB_,X_,$ marked by the continuous lines l and T in Fig. 527. When this parallelogram is turned through 180° about $BB_,$, it is brought into the position shown by the interrupted lines. The traces M parallel to OX cross one another at an angle $\underline{X}OX_, = 2\gamma - 180°$; and the pairs of traces $\underline{l}$ and $T_,$, &c., do not exactly fit. An actual twin, Fig. 526, having XOY for combination-plane, has been observed in anorthite. In Figs. 526 and 527 the overlapping of the parallelograms and of the individuals in the twin has been considerably exaggerated so as to show with distinctness the lack of congruence. But, generally, the individuals meet without any overlapping, as is seen in Figs. 528 and 529 showing actual twins of anorthite. The trace of the plane of section on M and the parallel diagonal OR in the plane XOZ must then be at right angles to $YY_,$, for such a line

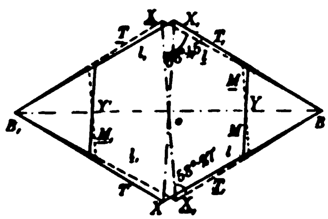


FIG. 527.

is the only one which retains the same direction after a semi-revolution about $YY_,$. Now the parallelogram having its diagonals

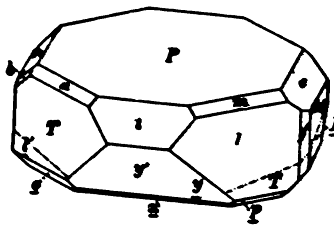


FIG. 528.

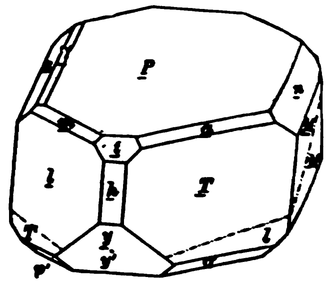


FIG. 529.

at 90° is a rhombus, and the plane of section through $YY_,$ in which the traces l and T are all equal is known as the plane of the *rhombic section* of the plagioclastic felspars. The pericline-law is therefore that the twin-axis is $YY_,$, and that combination takes place in the plane of the rhombic section. We proceed to determine the position of this plane.

Let OR, in Fig. 530, be perpendicular to OY, and the plane ROY be that of the rhombic section. About O as centre describe a sphere, meeting OR and the axes at R, X and Y. Then from the spherical triangle XRY,

$$\cos RY = \cos RX \cos XY + \sin RX \sin XY \cos RXY.$$

But $\qquad \wedge RXY = \wedge PM$, and, RY being 90°, $\cos RY = 0$;

$$\therefore \cot RX = -\tan XY \cos RXY = -\tan \gamma \cos PM.$$

Also $\qquad \sin XYR = \sin RX \sin RXY = \sin RX \sin PM,$

gives the inclination of the plane to the base (001). Introducing the angles γ and PM given for anorthite, we find

$$\wedge RX = 16^\circ\, 1', \text{ and } \wedge XYR = 15^\circ\, 58'.$$

The angles of albite are very variable and the values of γ and PM to be used for any particular crystal are very uncertain. If the values adopted by vom Rath for those from Schmirn are taken, viz. $\gamma = 87^\circ\, 51{\cdot}5'$ and $PM = 86^\circ\, 30'$; then $RX = 31^\circ\, 29{\cdot}4'$, and $XYR = 31^\circ\, 25{\cdot}5'$. If Dana's values, $\gamma = 88^\circ\, 8{\cdot}6'$ and $PM = 86^\circ\, 24'$, are taken; then $RX = 27^\circ\, 17{\cdot}5'$ and $XYR = 27^\circ\, 14'$. If Breithaupt's angles for pericline, viz.

$$\gamma = 89^\circ\, 13{\cdot}3' \text{ and } PM = 86^\circ\, 41';$$

then $RX = 13^\circ\, 12{\cdot}3'$ and $XYR = 13^\circ\, 11'$.

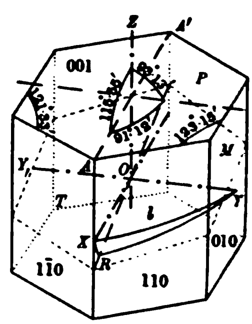


FIG. 530.

From the cases given above it is seen that,

since the tangent of an angle of nearly 90° changes very rapidly for small increments, the angles RX and XYR vary much more widely than the elements of the crystal.

Whilst, moreover, the angle $\alpha = YOZ$ is in all the felspars $> 90°$ (varying from 93° 13′ to 94° 5·3′) the angle γ is less than 90° in albite and greater than 90° in anorthite. Hence, if a plane, passing through $YY_{,}$ in Fig. 530, is turned about $YY_{,}$ from the position ZOY to XOY and then to $Z_{,}OY$, the angle which the front half of the line of intersection with the plane ZOX makes with OY diminishes continuously from $\alpha > 90°$ to its supplement. The angle becomes 90° only when the plane coincides with that of the rhombic section. But in albite $\gamma = XOY$ is less than 90°; and the line OR, where $\wedge ROY = 90°$, lies between OZ and OX: the plane of the rhombic section is then inclined to $P(001)$ on the same side as $x(\bar{1}01)$. In anorthite γ is greater than 90°, and the line OR at 90° to OY must lie between OX and $OZ_{,}$: the plane of the rhombic section is inclined to P on the same side as $t(201)$. Since the combination-edges in most of the twins of anorthite, and, as far as can be judged, in all those of pericline, are congruent, the edge $[M\underline{M}]$ in which the faces M meet is parallel to OR: the edge is therefore in albite inclined to the edge $[PM]$ in a direction lying between $[PM]$ and $[xM]$, and in anorthite it is inclined the other way so as to lie between $[PM]$ and $[tM]$. The former direction is sometimes indicated by affixing a plus sign to the angle, and the latter by affixing a minus sign. Thus in albite the angle between the edges $[M\underline{M}]$ and $[PM]$, computed respectively from the values of γ and PM given in the preceding paragraph, is $+31° 29·4′$, $+27° 17·5′$ and $+13° 12·3′$; in anorthite it is $-16° 1′$.

In albite γ is less than 90°, in anorthite it is greater than 90°, and it has been shown that the position of OR, and therefore of the parallel edge $[M\underline{M}]$, depends mainly on the value of γ. Hence, if for the intermediate felspars the angle γ changes more or less regularly with the chemical composition, the angle $[PM] \wedge [M\underline{M}]$ will serve as a means of discriminating between them. Thus vom Rath was led, by observation of striæ on the face M of a crystal of esmarkite, to think that an error had been made in placing it near anorthite. The striæ on M, due to polysynthetic twinning according to this law, are inclined to $[PM]$ at an angle of about $+4°$, and such an angle requires γ to be less than 90° and that the substance should be an oligoclase, not far removed from

albite. Analysis confirmed this surmise, for the mineral was found to contain 61·9 of silica and only 4·45 of calcium oxide.

Knowing the angles hP, Px of the felspars, it is easy, from the anharmonic ratio of four tautozonal planes, to calculate the indices of possible faces which are approximately parallel to the plane of the rhombic section. In albite such a face will require a different symbol for values of γ differing by small increments. The plane can therefore have no place amongst possible crystal-faces of the substance. In anorthite the face (307) is inclined to P at an angle of 15° 59·5′; and it may be that this approximation to the position of a possible crystal-face with fairly low indices accounts for the great uniformity of the trace on M of the rhombic section in crystals of this substance.

74. The twins of anorthite sometimes intercross as shown in Fig. 531, so that the faces M make re-entrant angles at both ends. This figure shows clearly the enantiomorphous character of the pair of doublets which, being formed from two simple crystals, have, like the doublets in Figs. 528 and 529, re-entrant angles $M \wedge \underline{M}$ only at one end. The twin, Fig. 531, can be composed by bisecting in the plane XOZ two such doublets, and uniting in the plane of section the portions to the extreme right and left; *i.e.* the portions on the left may be composed of upper parts on the left of two like crystals, and the portions on the right of lower parts on the right of the same crystals. A similar intercrossing twin with salient angles $M \wedge \underline{M}$ at both ends, though possible, has not been observed.

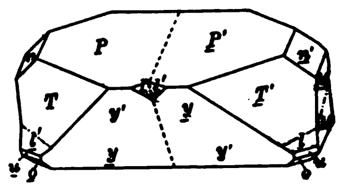


FIG. 531.

75. *Pericline.* The twins of pericline, when both ends can be perceived, are seen to be always intercrossing doublets similar to that of anorthite, Fig. 531, and the angles $M \wedge \underline{M}$ are invariably re-entrant. The general habit is fairly well given by Fig. 531, but the trace of the combination-plane on M is inclined the other way to the edge $[MP_{\prime}]$. Further, the combination along the rhombic section does not seem to be very regular, as is shown in the sections Figs. 532 and 533. The first shows the natural face, in which the exact division between the twinned individuals is partly

masked by a deposit of albite substance on the pericline-twin. Fig. 533 shows the appearance of a section of the same crystal obtained by cleavage parallel to M.

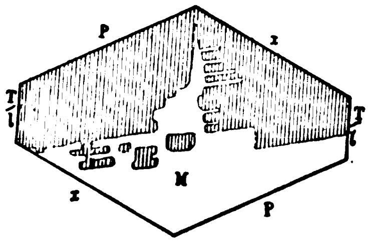


FIG. 532.

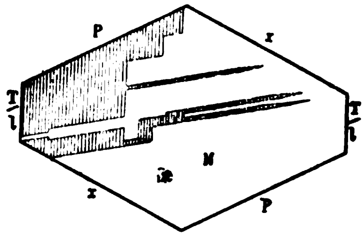


FIG. 533.

76. The twinning is often repeated, giving rise to striæ on the faces M, l and T. Such is the origin of the striæ shown traversing the faces M, f, l in Fig. 525. The lines of interrupted strokes indicate twin-lamellæ according to the albite-law, and vom Rath observed that the lamellæ of different kinds never intersected one another in crystals of anorthite.

The crystals of microcline are traversed by numbers of extremely thin lamellæ which cross one another nearly at right angles. The lamellæ of one series are parallel to $[PM]$ and are combined according to the albite-law. The others are due to twinning parallel to $YY_{,}$, combination taking place parallel to a plane making a very considerable angle with the base. The direction of extinction on the pearly cleavage P is $+ 15° 30'$. Hence a cleavage-flake parallel to this face shows between crossed Nicols a characteristic division resembling a grating which is due to the fine lamellæ extinguishing the light in different azimuths.

77. The views as to pseudo-cubic symmetry propounded to explain the common twins of orthoclase receive considerable confirmation from the laws of twinning observed in the plagioclastic felspars; for the angles being nearly the same in all the felspars, we have a similar approximation to a tetragonal zone in the cleavage-zone $[PM]$, and its axis $XX_{,}$ is one of pseudo-tetrad symmetry. Thus, e being (021) and $n\,(0\bar{2}1)$, we have in albite $eP = 43° 10'$ and $nP = 46° 46'$; in anorthite $eP = 42° 38{\cdot}5'$ and $nP = 46° 46'$. The relations of the particles about the normal to $M\,(010)$ and about the zone-axis $YY_{,}$ may both be considered to be nearly the same as those about a true dyad axis; and suggest a reason for the occurrence of twinning about both these axes, *i.e.* according to the albite- and pericline-laws. Several crystallographers indeed regard orthoclase and microcline

to be merely varieties of the same anorthic mineral, and the former to be a mimetic twin composed of ultra-microscopic lamellæ which are combined according to the albite- and pericline-laws. Again, the normal to (001) is a pseudo-dyad axis, and is the twin-axis adopted in microcline. We have also twins of microcline, and very rarely of albite, which may be referred to the same law as the Baveno-twin of orthoclase, *i.e.* twin-face (021) or ($0\bar{2}1$): in microcline it makes little difference whether the twin-face is *e* or *n*, or whether the twin-axis is $XX_,$ with a rotation of 90°; but in albite and the other felspars there would be considerable differences according to the twin-axis and rotation adopted. In fact from the theoretical views under discussion three twins are possible, (i) a twin in which *e* is the twin-face, (ii) a twin with *n* for twin-face—both with a rotation of 180°—and (iii) a twin in which $XX_,$ is the twin-axis with an angle of rotation of 90°. Further, a second rotation of 90° about the pseudo-tetrad axis $XX_,$ gives the twin-law in andesine, &c. Again, the faces *M*, *l* and *T* are nearly at 60° to one another, and the vertical axis $ZZ_,$ is a pseudo-triad axis: hence we have an explanation of the twins with this line for twin-axis.

Twin-axis a line in a face perpendicular to one of its edges.

78. A twin-axis of this kind has been proposed as the twin-axis of the pericline-law, and for certain complex twins of albite and anorthite; and single instances of its occurrence in labradorite and cyanite have also been given. The line has no crystallographic significance for it is neither a zone-axis nor the normal of a possible face; and it is difficult to understand how such a line can have so important a relation to the crystalline structure.

Kayser (*Pogg. Ann.* XXXIV, p. 109, 1835) proposed the line in the base (001) perpendicular to the edge [*PM*] for the axis of semi-revolution in the pericline-twin; he adopted this line as the twin-axis rather than $[Px] = YY_,$, for he perceived that the combination-edge $[M\underline{M}]$ must with the latter twin-axis be inclined to the edge [*PM*] at an angle ϕ, computed from Breithaupt's measurements to be 13° 12·3′ (see Art. 73), in order that the individuals should meet in congruent edges. He points out that, in spite of the imperfections of the faces *M* on the crystals of pericline, the combination-edge is on the whole parallel to [*PM*], and that this is the more apparent the more perfect the face. Consequently, since the faces *M* meet in congruent edges parallel to the edges [*MP*] of the two individuals, and the faces *P* are parallel, the twin-axis must be a line in *P* perpendicular to [*PM*]. Rose (*Pogg. Ann.*, CXXIX, p. 1, 1866) adopts Kayser's twin-axis, but points out that the combination-edge is not often parallel to [*PM*] and that it is always in a direction intermediate between [*PM*] and [*Mx*]: his careful drawings serve indeed to strengthen the argument in favour of [*Px*] as twin-axis. Kayser supports his view by the fact that, in the twins of oligoclase from Arendal in Norway, the combination-edge is parallel to

[PM]. We have already shown in Art. 73 that the angle ϕ changes rapidly for very small increments of γ, and it follows that, since γ is very nearly 90° in the crystals of oligoclase and andesine, the combination-edge is, in their twins, substantially parallel to [PM]. Further, the twins of anorthite according to the pericline-law were unknown, and the evidence supplied by the well-marked inclination of the combination-edge to [PM] was wanting. Seeing then that, except when the faces M meet in a jagged line indicative of combination along a very irregular surface, the direction of the combination-edge is generally in conformity with that required for congruence with [Px] as twin-axis, and, further, that Kayser himself insists on the identity of the law of twinning in the several felspars, we must reject his interpretation of the law, and adopt $YY_{,}$ as the twin-axis.

In the complex twins of albite and anorthite we may possibly have to deal with lamellæ twinned according to different laws; and the presence of an extremely thin lamella twinned on one side according to the albite-law and on the other side according to the Carlsbad-law (extending this term to embrace the plagioclase-twins described in Art. 71 which have [001] for twin-axis) will give rise to a composite crystal, one portion of which seems to be turned through 180° about a line at right angles to the two axes of the separate laws. For, by Euler's theorem, successive rotations of 180° about two lines at right angles to one another are equivalent to a single semi-revolution about a line perpendicular to the two first axes. In a complicated group and in an ill-developed crystal, the presence of a thin lamella can easily escape detection.

Too little is known of the crystals of labradorite, to justify the acceptance of a twin-axis which is neither a normal nor a zone-axis; and more especially as the specimen itself to which such a twin-axis has been ascribed did not admit of accurate determination.

In cyanite the crystal-elements are not capable of determination with great precision, and possibly differ slightly from one crystal to another. In this mineral the edge [ac], Fig. 118, p. 155, is an accepted twin-axis, combination taking place parallel to (100). The angle between the edges [ac] and [ab] is very nearly 90°; the value computed by vom Rath varying from 90° 0′ to 90° 5′, that by Professor Bauer being 90° 23′. Hence, with the latter angle and [ac] as twin-axis, the prism-faces of the rotated individual are slightly out of the zone formed by the fixed individual, and this was observed to be the case. But in a single instance (*Zeitsch. d. Deutsch. geol. Ges.* xxx, p. 306, 1878) the faces of the prisms of both individuals were found to be exactly in one zone. Professor Bauer therefore inferred that rotation had taken place about the line at 90° to [ab] of Fig. 118; this axis of rotation making with the edge [ac] an angle of 23′. Considering the untrustworthiness of the angles on twin-crystals, as indicated, for instance, by the apparent coincidence of the faces c and x in the twin of orthoclase, Fig. 514, the evidence supplied by this observation is insufficient for the acceptance of the twin-axis.

On the method of determining the position of the twin-axis.

79. The method employed to determine the position of the twin-axis and, when the portions are juxtaposed, of the combination-plane will depend on the development of the crystal; and in many cases the law is perceived without difficulty. Thus, after the student has examined a few twinned octahedra of spinel, he will easily recognise the same twin-law in the case of octahedral crystals of other substances. The general method to be followed may be given as follows: (i) Determine the crystal-forms present on each portion; and by measurement the elements of the crystal and the symbols of the faces. (ii) The positions of the like faces on the two portions having been determined, the student must consider round what line one portion must be turned through 180° to bring like faces into coincidence or parallelism. The perception of the direction of the twin-axis is much aided when certain faces or edges on the two portions are seen to be parallel to one another, or to be coincident in direction. (iii) A semi-revolution of a normal about any origin-line brings it again into the plane containing the normal and axis of rotation, and the latter bisects the angle between the initial and final positions of the normal. Thus, if TT' in Fig. 534 is a twin-axis with an angle of rotation of 180° and P and P' are the poles of corresponding faces, then a semi-revolution about TT' brings P to p', the opposite extremity of the diameter through P', and $\wedge TP = \wedge Tp'$. Also M being the point midway between P and P', the arc $MT = 90°$. So far as these two faces are concerned the axis may equally well be the diameter through T or M. If, however, the angle between any other two corresponding poles Q and Q', not in a zone with P and P', is also measured, the position of TT' is fixed. For $T'Q = T'q' = TQ'$. If the twin is combined along a plane perpendicular to the twin-axis, then P and Q' are known poles belonging to one individual. These being projected in the way usual for a simple crystal of its system, the position of T can be found by the construction given in Chap. VII, Art. 19; and its relation to the axes of the crystal can be then determined.

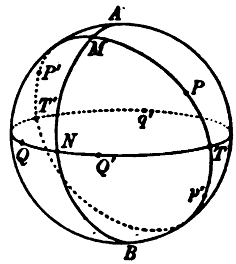


FIG. 534.

If P and Q belong to one individual, then the triangle TPQ is known, and can be projected in the way given for the previous case, and the point T determined.

In a juxtaposed twin the relations of the individuals to the combination-plane must be recognised in order to determine the twin-axis; and, except in the anorthic system as already illustrated in the discussion of the pericline-law, seldom present any serious difficulty.

CHAPTER XIX.

ANALYTICAL METHODS AND DIVERS NOTATIONS.

1. SEVERAL of the problems examined in the preceding Chapters, and a few which were then omitted, are capable of simple treatment by the methods of co-ordinate geometry.

2. To find the equation of the origin-plane (see p. 35) parallel to a face (hkl).

Let the co-ordinate axes OX, OY, OZ, Fig. 535, be parallel to any three edges of the crystal which are not parallel to one plane. Let, also, a, b, c be three finite lengths giving the distances from the origin at which OX, OY, OZ are respectively met by a definite face of the crystal. Let any other face (hkl) meet the axes at points H, K, L, so that $OH = a \div h$, $OK = b \div k$, $OL = c \div l$; h, k, l being rational numbers.

Then the equation of the face is

$$\frac{x}{OH} + \frac{y}{OK} + \frac{z}{OZ} = 1.$$

Replacing the intercepts by their values in terms of the parameters and indices, we have

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 1 \text{............} (1).$$

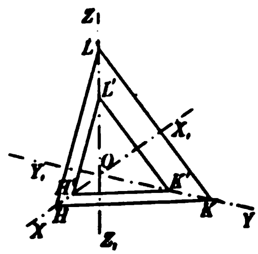


FIG. 535.

Now a parallel face will meet the axes at points H', K', L', where
$OH' = p \,.\, OH = pa \div h$; $OK' = p \,.\, OK = pb \div k$;
$OL' = p \,.\, OL = pc \div l$.

The equation of the plane $H'K'L'$ is

$$\frac{x}{OH'} + \frac{y}{OK'} + \frac{z}{OL'} = 1;$$

which, after the introduction of the values of OH', &c., becomes

$$h\frac{x}{a}+k\frac{y}{b}+l\frac{z}{c}=p \qquad \text{(2)};$$

an equation only differing from (1) in the constant term.

When, however, the plane passes through the origin, equation (2) must be satisfied by the values $x=y=z=0$; $\therefore p=0$.

Hence, a plane through the origin parallel to the face (hkl) is given by the equation

$$h\frac{x}{a}+k\frac{y}{b}+l\frac{z}{c}=0 \qquad \text{(3)}.$$

This plane will be called the *origin-plane* (hkl).

Since the angles measured by the goniometer give only the relative directions of the faces, and the computations into which these angles enter are not concerned with the actual distances of the faces from the origin, it follows that all the angular relations between the faces of a crystal will be the same as those of the origin-planes parallel to them.

3. To find the equations of the straight line through the origin parallel to two non-parallel faces ($h_1k_1l_1$), ($h_2k_2l_2$); *i.e.* of their zone-axis.

The equations of the two origin-planes are:

$$h_1\frac{x}{a}+k_1\frac{y}{b}+l_1\frac{z}{c}=0,$$

$$h_2\frac{x}{a}+k_2\frac{y}{b}+l_2\frac{z}{c}=0.$$

But for the line of intersection the two equations must be satisfied by the same values of x, y and z; hence the equations of the zone-axis are

$$\frac{x}{a(k_1l_2-l_1k_2)}=\frac{y}{b(l_1h_2-h_1l_2)}=\frac{z}{c(h_1k_2-k_1h_2)} \qquad \text{(4)}.$$

If, as in Chap. v, the differences of the products of the indices occurring in the denominators of (4) are called the zone-indices, and expressed by u_{12}, v_{12}, w_{12}; then equations (4) can be written

$$\frac{x}{au_{12}}=\frac{y}{bv_{12}}=\frac{z}{cw_{12}} \qquad \text{(5)}.$$

This line is drawn by constructing a parallelepiped having for edges in the axes of X, Y and Z lengths equal to au_{12}, bv_{12}, cw_{12},

respectively. The diagonal OT through the origin is the zone-axis $[u_{12}\ v_{12}\ w_{12}]$; and any three numbers in the ratios $au_{12} : bv_{12} : cw_{12}$ may be called its *direction-ratios*.

4. To find the condition that any face (hkl) shall be in a zone with the two faces $(h_1k_1l_1)$, $(h_2k_2l_2)$.

From the definition of a zone (p. 1) the line of intersection of the faces $(h_1k_1l_1)$, $(h_2k_2l_2)$ must be parallel to the face (hkl). Hence the origin-plane (hkl) must contain the zone-axis $[h_1k_1l_1, h_2k_2l_2]$, the equations of which are given in (4) and (5).

Hence the equations

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 0,$$

$$\frac{x}{au_{12}} = \frac{y}{bv_{12}} = \frac{z}{cw_{12}} \dots,$$

must both be satisfied by the same values of x, y and z.

Introducing the ratios for x, y and z given in the latter equations into that of the plane, we obtain the required condition; which is

$$hu_{12} + kv_{12} + lw_{12} = 0 \text{ ....(6).}$$

This is the relation which in Chap. v, Art. 8, we designated as Weiss's zone-law. It was there pointed out that it can be written as a determinant, involving the indices of the three faces in an exactly similar manner.

5. To show that the plane containing two zone-axes is parallel to a possible face.

Let $[u_1v_1w_1]$, $[u_2v_2w_2]$ be the zone-axes, and let

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 0$$

be the origin-plane containing them.

The zone-axes have the equations

$$\frac{x}{au_1} = \frac{y}{bv_1} = \frac{z}{cw_1},$$

and

$$\frac{x}{au_2} = \frac{y}{bv_2} = \frac{z}{cw_2}.$$

If the origin-plane (hkl) passes through the zone-axes, the values of x, y, z which satisfy each of the equations of the zone-axes also satisfy that of the plane.

Hence replacing $x:y:z$ in the equation of the plane by their values given in the equations of the zone-axes, we have

$$hu_1 + kv_1 + lw_1 = 0,$$

$$hu_2 + kv_2 + lw_2 = 0.$$

$$\therefore \frac{h}{v_1w_2 - w_1v_2} = \frac{k}{w_1u_2 - u_1w_2} = \frac{l}{u_1v_2 - v_1u_2} \text{............(7).}$$

Therefore h, k and l are all commensurable numbers; for u_1, v_1, &c., being zone-indices, are commensurable. The plane is therefore parallel to a possible face, for the equation is of the form given in (3) and the indices h, k and l are rational.

The origin-plane may be written

$$(v_1w_2 - w_1v_2)\frac{x}{a} + (w_1u_2 - u_1w_2)\frac{y}{b} + (u_1v_2 - v_1u_2)\frac{z}{c} = 0 \text{......(8).}$$

As stated in Chap. v, the indices of a face common to two zones $[u_1v_1w_1]$, $[u_2v_2w_2]$ are obtained from the zone-indices by the same rule as that by which zone-indices are deduced from the indices of two non-parallel faces.

6. To change the axes of reference to another set of axes, so that $[u_1v_1w_1]$ becomes OX', $[u_2v_2w_2]$ becomes OY', and $[u_3v_3w_3]$ becomes OZ'; supposing the parametral face (111) to retain the same indices.

Let the parametral face (111) meet the new axes at the points A', B', C' respectively. Now the equations of the parametral face and of the axis $[u_1v_1w_1]$, referred to the original axes, are

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \text{............................(9);}$$

$$\frac{x}{au_1} = \frac{y}{bv_1} = \frac{z}{cw_1} \text{.........................(10).}$$

The co-ordinates of the point A', Fig. 536, are obtained by introducing the ratios of $x:y:z$ from the latter equations into that of the plane. Let them be x', y', z': then

$$\frac{x'}{au_1} = \frac{y'}{bv_1} = \frac{z'}{cw_1} = \frac{\frac{x'}{a} + \frac{y'}{b} + \frac{z'}{c}}{u_1 + v_1 + w_1} = \frac{1}{u_1 + v_1 + w_1} \text{.........(11).}$$

The new parameter OA' on OX' is the diagonal of the parallelepiped $OYA'Z$, having for edges along the original axes the co-ordinates $x' = NY$, $y' = OY$, $z' = NA'$.

The co-ordinates of B' and C' are obtained in an exactly similar manner by combining equation (9) with those of each of the axes $[u_2v_2w_2]$, $[u_3v_3w_3]$: the co-ordinates may be represented by $(x''y''z'')$ and $(x'''y'''z''')$ respectively. The parameters OB' and OC' are the diagonals of parallelepipeds having these co-ordinates for edges.

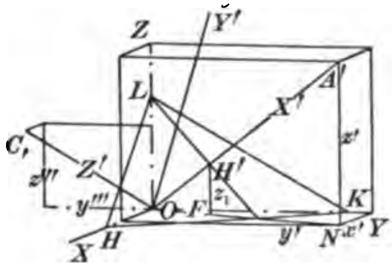


Fig. 536.

Let a face (hkl) meet the original axes at H, K, L; and the new ones at H', K', L'; and let its new symbol be $(h'k'l')$.

When referred to the original axes, the equation of the face is

$$h\frac{x}{a} + k\frac{y}{b} + l\frac{z}{c} = 1 \text{........................}(12).$$

To find the co-ordinates of the point H' in which the face meets the new axis OX', we must combine equation (12) with (10). Let the co-ordinates of H' be x_1, y_1, z_1, then

$$\frac{x_1}{au_1} = \frac{y_1}{bv_1} = \frac{z_1}{cw_1} = \frac{h\frac{x_1}{a} + k\frac{y_1}{b} + l\frac{z_1}{c}}{hu_1 + kv_1 + lw_1} = \frac{1}{hu_1 + kv_1 + lw_1} \text{....}(13).$$

By combining (12) with the equations of the axes OY', OZ', similar expressions are obtained for the co-ordinates of K', L' in which the face meets these axes. But the new indices are:

$$h' = OA' \div OH', \quad k' = OB' \div OK', \quad l' = OC' \div OL'.$$

And by similar triangles $OA'N$, $OH'F$, we have

$$OA' : OH' = A'N : H'F = z' : z_1 = y' : y_1 = x' : x_1,$$

$$\left.\begin{aligned} \therefore \text{ from (11) and (13) } h' &= \frac{z'}{z_1} = \frac{hu_1 + kv_1 + lw_1}{u_1 + v_1 + w_1} \\ \text{Similarly,} \qquad k' &= \frac{hu_2 + kv_2 + lw_2}{u_2 + v_2 + w_2} \\ l' &= \frac{hu_3 + kv_3 + lw_3}{u_3 + v_3 + w_3} \end{aligned}\right\} \text{....}(14).$$

In the above transformation the axes OX', OY', OZ' are not necessarily zone-axes; for any lines through the origin may be represented by equations of the form employed in (10), and the lines will only be possible zone-axes when the indices u_1, v_1, &c., are all commensurable.

In a similar manner equations of transformation can be obtained in which the symbol of the parametral plane is also changed. The equations have been found in Chap. VIII, Art. 20, on the assumption that the new axes are zone-axes, and have the same form whether this is the case or not.

7. To show that the face-indices are rational only when the axes of reference are zone-axes.

In the transformation given in Chap. VIII, Art. 20, the new axes were zone-axes, for the formulæ were obtained by means of anharmonic ratios which involved face- and zone-indices; these being all necessarily commensurable numbers. The new indices were therefore also commensurable. But in the analysis given in the last Article, the new axes may be any lines whatever; and they will only be possible zone-axes when the ratios $u_1 : v_1 : w_1$, &c., are all commensurable. For other lines these ratios will be irrational.

From the first of equations (14) we have

$$(h'-h)\,u_1+(h'-k)\,v_1+(h'-l)\,w_1=0 \quad\text{...........(15).}$$

The original axes are supposed to be zone-axes, and for a possible face the indices h, k, l are then rational (Chap. IV, Art. 9). If in (15) h' is rational, the coefficients of u_1, v_1, w_1 are all rational, and the ratios $u_1 : w_1$ and $v_1 : w_1$ must also be rational. For take any other possible face (pqr), and suppose it to become $(p'q'r')$ when referred to the new axes; p', q', r' being also supposed to be rational. Replacing the face-indices in (15) by those of the new face; we have

$$(p'-p)\,u_1+(p'-q)\,v_1+(p'-r)\,w_1=0 \quad\text{...........(16).}$$

Solving equations (15) and (16), we find

$$\frac{u_1}{(h'-k)(p'-r)-(h'-l)(p'-q)}=\frac{v_1}{(h'-l)(p'-p)-(h'-h)(p'-r)}$$

$$=\frac{w_1}{(h'-h)(p'-q)-(h'-k)(p'-p)} \quad\text{......(17).}$$

The numbers in the denominators of (17) are all rational: therefore u_1, v_1, w_1 are also rational, and the axis OX' is a possible zone-axis.

Hence if the face-indices referred to the new axes are to be rational, the new axes must be possible zone-axes.

8. To find the value of the anharmonic ratio of four tautozonal faces in terms of their indices.

Let OT, Fig. 537, be the line of intersection of four origin-planes TOP_1, TOP_2, TOP_3, TOP_4, parallel respectively to the four tautozonal faces $(h_1k_1l_1)$, $(h_2k_2l_2)$, $(h_3k_3l_3)$, $(h_4k_4l_4)$, no two of which are parallel. Let the origin-planes meet the plane through O perpendicular to OT in the lines OP_1, OP_2, OP_3, OP_4; and the plane XOY in the lines OL_1, OL_2, OL_3, OL_4. Through any point T on the zone-axis draw a plane TP_1P_4 parallel to OX; and let it meet OY in K and the origin planes in the straight lines TL_1P_1, TL_2P_2, TL_3P_3, TL_4P_4, respectively. Two planes necessarily intersect in a straight line, so that the points P_1, P_2, P_3, P_4 are co-linear; and so are the points K, L_1, L_2, L_3, L_4. Then using the symbolical notation for the anharmonic ratio given in Chap. VIII, Art. 9, and $O\{L_1L_2L_3L_4\}$ to represent that of four lines meeting at a point O we have

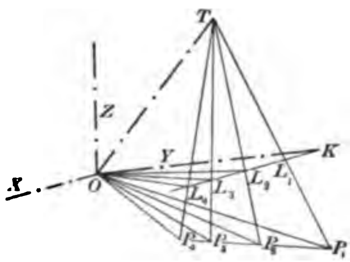


FIG. 537.

$$\{P_1P_2P_3P_4\} = \frac{\sin P_1OP_2}{\sin P_1OP_3} \div \frac{\sin P_4OP_2}{\sin P_4OP_3} = \frac{P_1P_2}{P_1P_3} \div \frac{P_4P_2}{P_4P_3};$$

where P_1P_2, &c. are the lengths on the line P_1P_4 intercepted between the several planes. The relation between the sines of the angles and the corresponding lengths on P_1P_4 can be easily proved from the known properties of plane triangles, in a manner similar to that adopted in proving a corresponding relation in spherical triangles in Chap. VIII, Art. 19.

In a similar manner we can show that the A.R. of the intercepts on a line is equal to that of the pencil of four lines drawn to any point whatever lying outside the line. Hence, we have

$$\{P_1P_2P_3P_4\} = T\{P_1P_2P_3P_4\} = T\{L_1L_2L_3L_4\}.$$

But $$T\{L_1L_2L_3L_4\} = \{L_1L_2L_3L_4\} = \frac{L_1L_2}{L_1L_3} \div \frac{L_4L_2}{L_4L_3}.$$

And, $$L_1L_2 = KL_2 - KL_1;\ L_4L_2 = KL_2 - KL_4;\ \&c.$$

The lines OL_1, OL_2, &c. are given by the following equations:

$$\left.\begin{array}{l} \text{for } OL_1,\ h_1\dfrac{x}{a}+k_1\dfrac{y}{b}=0,\ z=0 \\ \ \ ,,\ \ OL_2,\ h_2\dfrac{x}{a}+k_2\dfrac{y}{b}=0,\ z=0 \\ \ \ ,,\ \ OL_3,\ h_3\dfrac{x}{a}+k_3\dfrac{y}{b}=0,\ z=0 \\ \ \ ,,\ \ OL_4,\ h_4\dfrac{x}{a}+k_4\dfrac{y}{b}=0,\ z=0 \end{array}\right\} \ldots\ldots\ldots\ (18).$$

Moreover the line $KL_1 \ldots L_4$ is parallel to OX, and the lengths KL_1, KL_2, &c. are the ordinates x in equations (18) when y is made equal to OK.

$$\therefore\ KL_1 = -\frac{a}{b}\frac{k_1}{h_1}OK;\ KL_2 = -\frac{a}{b}\frac{k_2}{h_2}OK, \text{ \&c.}$$

$$\therefore\ \{P_1P_2P_3P_4\} = \frac{L_1L_2}{L_1L_3} \div \frac{L_4L_2}{L_4L_3} = \frac{KL_2 - KL_1}{KL_3 - KL_1} \div \frac{KL_2 - KL_4}{KL_3 - KL_4}$$

$$= \frac{-\frac{a}{b}OK\left(\frac{k_2}{h_2}-\frac{k_1}{h_1}\right)}{-\frac{a}{b}OK\left(\frac{k_3}{h_3}-\frac{k_1}{h_1}\right)} \div \frac{-\frac{a}{b}OK\left(\frac{k_2}{h_2}-\frac{k_4}{h_4}\right)}{-\frac{a}{b}OK\left(\frac{k_3}{h_3}-\frac{k_4}{h_4}\right)}$$

$$= \frac{h_1k_2 - k_1h_2}{h_1k_3 - k_1h_3} \div \frac{h_4k_2 - k_4h_2}{h_4k_3 - k_4h_3} \ldots\ldots\ldots\ldots\ (19).$$

9. To find the angles which a line PQ makes with the axes in terms of its direction-ratios e, f, g; also the length intercepted between any two points the co-ordinates of which are given.

Let α, β, γ be the angles YOZ, ZOX, XOY, respectively; and let λ, μ, ν be the angles which PQ makes with the axes of X, Y and Z. Construct the parallelepiped $PNLQM$, Fig. 538, having for edges PL, PM, PN parallel to the axes and PQ for diagonal. Then

$$PL = e\,.\,PQ,\quad PM = LF = f\,.\,PQ,$$
$$PN = FQ = g\,.\,PQ \ \ldots\ldots\ldots\ (20).$$

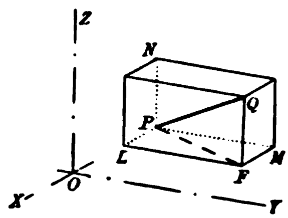


Fig. 538.

Again, if P is supposed to be a fixed point $(x_1y_1z_1)$ and Q the moveable one (xyz), the equations of PQ are

$$\frac{x-x_1}{e} = \frac{y-y_1}{f} = \frac{z-z_1}{g} \ldots\ldots\ldots\ldots\ (21).$$

Now the orthogonal projection of PQ on the axis of X is equal to the sum of the orthogonal projections on the same axis of the lines PL, LF and FQ.

Hence, $PQ \cos\lambda = PL + LF\cos\gamma + FQ\cos\beta$; similarly, by taking the orthogonal projections of the same lines on the axes of Y and Z, we have

$$\left.\begin{aligned} PQ\cos\lambda &= PL + LF\cos\gamma + FQ\cos\beta;\\ PQ\cos\mu &= PL\cos\gamma + LF + FQ\cos\alpha\\ PQ\cos\nu &= PL\cos\beta + LF\cos\alpha + FQ \end{aligned}\right\}\ \ldots\ldots\ldots (22).$$

Introducing into (22) the values of PL, LF and FQ from (20), we have

$$\left.\begin{aligned} \cos\lambda &= e + f\cos\gamma + g\cos\beta;\\ \cos\mu &= e\cos\gamma + f + g\cos\alpha;\\ \cos\nu &= e\cos\beta + f\cos\alpha + g. \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23).$$

The angles λ, μ, ν are therefore known in terms of the direction-ratios e, f, g and the angles between the axes.

Again, since the sum of the orthogonal projections of the lines PL, LF and FQ on PQ is equal to PQ, we have

$$PQ = PL\cos\lambda + LF\cos\mu + FQ\cos\nu\ \ldots\ldots\ldots\ldots(24).$$

Introducing the values given in (20) and (23), we have

$$\begin{aligned} 1 &= e\,(e + f\cos\gamma + g\cos\beta) + f(e\cos\gamma + f + g\cos\alpha)\\ &\qquad + g\,(e\cos\beta + f\cos\alpha + g)\\ &= e^2 + f^2 + g^2 + 2fg\cos\alpha + 2ge\cos\beta + 2ef\cos\gamma\ \ldots\ldots\ldots(25). \end{aligned}$$

If the three equations in (22) are multiplied respectively by PL, LF and FQ, and the equations are then added, we have

$$\begin{aligned} &PQ\,(PL\cos\lambda + LF\cos\mu + FQ\cos\nu) = (\text{from } (24))\ PQ^2\\ &= PL^2 + LF^2 + FQ^2 + 2LF\,.\,FQ\cos\alpha + 2FQ\,.\,PL\cos\beta\\ &\qquad\qquad + 2PL\,.\,LF\cos\gamma. \end{aligned}$$

But, $PL = x - x_1,\ \ LF = y - y_1,\ \ FQ = z - z_1$;

$$\begin{aligned} \therefore\ PQ^2 &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(y - y_1)(z - z_1)\cos\alpha\\ &\quad + 2\,(z - z_1)(x - x_1)\cos\beta + 2\,(x - x_1)(y - y_1)\cos\gamma\ \ldots\ldots (26). \end{aligned}$$

10. To find the direction-ratios e, f, g in terms of the angles which the line makes with the axes.

Multiply the second equation of (23) by $\cos\alpha\cos\beta$, and the third by $\cos\gamma\cos\alpha$, and add the products to the first equation; then

$$\begin{aligned} &\cos\lambda + \cos\mu\cos\alpha\cos\beta + \cos\nu\cos\gamma\cos\alpha = e\,(1 + 2\cos\alpha\cos\beta\cos\gamma)\\ &\quad + f(\cos\gamma + \cos\alpha\cos\beta + \cos\gamma\cos^2\alpha) + g\,(\cos\beta + \cos^2\alpha\cos\beta + \cos\gamma\cos\alpha). \end{aligned}$$

Again, multiply the first equation of (23) by $\cos^2 a$, the second by $\cos\gamma$ and the third by $\cos\beta$, and add the resulting equations; then

$$\cos\lambda\cos^2 a+\cos\mu\cos\gamma+\cos\nu\cos\beta=e(\cos^2 a+\cos^2\beta+\cos^2\gamma)$$
$$+f(\cos^2 a\cos\gamma+\cos\gamma+\cos a\cos\beta)+g(\cos^2 a\cos\beta+\cos a\cos\gamma+\cos\beta).$$

Subtract the second equation from the first, and we have

$$e(1-\cos^2 a-\cos^2\beta-\cos^2\gamma+2\cos a\cos\beta\cos\gamma)=eN^2\text{ (say)}$$
$$=\cos\lambda\sin^2 a+\cos\mu(\cos a\cos\beta-\cos\gamma)+\cos\nu(\cos\gamma\cos a-\cos\beta)\ldots(27).$$

This equation gives the direction-ratio e in terms of the angles which the line makes with the axes.

By a similar process the direction-ratios f and g can be shown to be

$$\left.\begin{aligned} fN^2&=\cos\mu\sin^2\beta+\cos\nu(\cos\beta\cos\gamma-\cos a)\\ &\qquad\qquad+\cos\lambda(\cos a\cos\beta-\cos\gamma)\\ gN^2&=\cos\nu\sin^2\gamma+\cos\lambda(\cos\gamma\cos a-\cos\beta)\\ &\qquad\qquad+\cos\mu(\cos\beta\cos\gamma-\cos a)\end{aligned}\right\}\ldots(27^*).$$

where N^2 is written for $1-\cos^2 a-\cos^2\beta-\cos^2\gamma+2\cos a\cos\beta\cos\gamma$.

But from (25), $\qquad e\cos\lambda+f\cos\mu+g\cos\nu=1$;

$\therefore$ from (27) and (27*)

$$\cos\lambda\{\cos\lambda\sin^2 a+\cos\mu(\cos a\cos\beta-\cos\gamma)+\cos\nu(\cos\gamma\cos a-\cos\beta)\}$$
$$+\cos\mu\{\cos\mu\sin^2\beta+\&c.\}+\cos\nu\{\cos\nu\sin^2\gamma+\&c.\}=N^2$$

collecting the terms, we have

$$\cos^2\lambda\sin^2 a+\cos^2\mu\sin^2\beta+\cos^2\nu\sin^2\gamma+2\cos\mu\cos\nu(\cos\beta\cos\gamma-\cos a)$$
$$+2\cos\nu\cos\lambda(\cos\gamma\cos a-\cos\beta)+2\cos\lambda\cos\mu(\cos a\cos\beta-\cos\gamma)$$
$$=1-\cos^2 a-\cos^2\beta-\cos^2\gamma+2\cos a\cos\beta\cos\gamma\ldots\ldots(28).$$

Equation (28) gives a relation connecting the angles which any line makes with the axes.

When the axes are rectangular $\cos a=\cos\beta=\cos\gamma=0$, and equation (28) becomes $\qquad \cos^2\lambda+\cos^2\mu+\cos^2\nu=1\ldots\ldots\ldots\ldots(29).$

11. To find the angle θ between two lines the direction-ratios (efg) and $(e'f'g')$ of which are known.

Let PQ, Fig. 538, be one of the lines having the direction-ratios e, f, g, and making the angles λ, μ, ν with the axes; and let $P'Q'$ be the second line having the direction-ratios e', f', g' and making with the axes the angles λ', μ', ν'.

Take on PQ any definite length PQ, and with PQ as diagonal construct, as in Art. 9, a parallelepiped having PL, PM, PN for edges parallel to the axes, their lengths are given by equations (20). Let them all be now projected orthogonally on the second line $P'Q'$. The projected lengths are:

$$PQ\cos\theta,\ PL\cos\lambda',\ PM\cos\mu' \text{ and } PN\cos\nu'.$$

Now the orthogonal projection of PQ is equal to the sum of those of the three others. Therefore

$$PQ \cos\theta = PL \cos\lambda' + PM \cos\mu' + PN \cos\nu' \quad\text{.........(30).}$$

And from (23), $\cos\lambda' = e' + f' \cos\gamma + g' \cos\beta$;

and similar expressions for $\cos\mu'$, $\cos\nu'$.

Introducing these values, and those of PL, &c., given in (20) into equation (30), we have

$$\begin{aligned}\cos\theta &= e\,(e' + f' \cos\gamma + g' \cos\beta) + f\,(e' \cos\gamma + f' + g' \cos\alpha) \\ &\qquad + g\,(e' \cos\beta + f' \cos\alpha + g') \\ &= ee' + ff' + gg' + (fg' + gf') \cos\alpha + (ge' + eg') \cos\beta \\ &\qquad + (ef' + fe') \cos\gamma \quad\text{............... (31).}\end{aligned}$$

When the axes are rectangular, equation (31) reduces to

$$\cos\theta = ee' + ff' + gg' \quad\text{.....................(32).}$$

Again, from (31) we can show that

$$\sin^2\theta = \Sigma\,(fg' - gf')^2 \sin^2\alpha + 2\Sigma\,(\cos\beta\cos\gamma - \cos\alpha)(ge' - eg')(ef' - fe') \text{...(33)};$$

where

$$\Sigma\,(fg' - gf')^2 \sin^2\alpha = (fg' - gf')^2 \sin^2\alpha + (ge' - eg')^2 \sin^2\beta + (ef' - fe')^2 \sin^2\gamma\,;$$

and

$$\begin{aligned}&\Sigma\,(\cos\beta\cos\gamma - \cos\alpha)(ge' - eg')(ef' - fe') \\ &\quad = (\cos\beta\cos\gamma - \cos\alpha)(ge' - eg')(ef' - fe') \\ &\qquad + (\cos\gamma\cos\alpha - \cos\beta)(ef' - fe')(fg' - gf') \\ &\qquad + (\cos\alpha\cos\beta - \cos\gamma)(fg' - gf')(ge' - eg').\end{aligned}$$

12. To find in terms of x, y, z the equations of the normal OP to a face (hkl).

Let the normal be inclined to the axes at angles λ, μ, ν. Then from equations (1) of Chap. IV,

$$OP = p = \frac{a\cos\lambda}{h} = \frac{b\cos\mu}{k} = \frac{c\cos\nu}{l} \quad\text{............(34).}$$

If a parallelepiped is constructed with edges in the axes and OP for diagonal, similar to that made in Art. 9 with PQ as diagonal, then the equations of the last Articles apply to the normal and to the angles λ, μ, ν. Hence, if P has the co-ordinates x, y, z,

$$\left.\begin{aligned} OP\cos\lambda &= x + y\cos\gamma + z\cos\beta, \\ OP\cos\mu &= x\cos\gamma + y + z\cos\alpha, \\ OP\cos\nu &= x\cos\beta + y\cos\alpha + z \end{aligned}\right\} \quad\text{............ (35).}$$

Introducing these values in (34), we have for the equations of the normal to (hkl)

$$\frac{a(x+y\cos\gamma+z\cos\beta)}{h}=\frac{b(x\cos\gamma+y+z\cos a)}{k}$$
$$=\frac{c(x\cos\beta+y\cos a+z)}{l}\;\ldots\ldots(36).$$

When the axes are rectangular, $\cos a=\cos\beta=\cos\gamma=0$, and the equations of the normal become

$$\frac{ax}{h}=\frac{by}{k}=\frac{cz}{l}=\frac{r}{\sqrt{\frac{h^2}{a^2}+\frac{k^2}{b^2}+\frac{l^2}{c^2}}}\;\ldots\ldots\ldots\ldots\ldots(37),$$

where r is the distance of any point $P(x, y, z)$ from the origin.

13. To find the angles λ, μ, ν which the normal to a face (hkl) makes with the axes of reference, and also its direction-ratios e, f, g in terms of the indices h, k, l and of the angles between the axes; and hence the angle θ between any two normals.

For the normal $OP=p$ (say) we have from equations (34) and (23),

$$\left.\begin{aligned}\cos\lambda&=p\frac{h}{a}=e+f\cos\gamma+g\cos\beta\\ \cos\mu&=p\frac{k}{b}=e\cos\gamma+f+g\cos a\\ \cos\nu&=p\frac{l}{c}=e\cos\beta+f\cos a+g\end{aligned}\right\}\;\ldots\ldots\ldots\ldots\ldots\ldots(38).$$

Introducing the first values of $\cos\lambda$, $\cos\mu$, $\cos\nu$ into (27) and (27*), we have

$$\left.\begin{aligned}N^2e&=p\left\{\frac{h}{a}\sin^2a+\frac{k}{b}(\cos a\cos\beta-\cos\gamma)+\frac{l}{c}(\cos\gamma\cos a-\cos\beta)\right\}\\ N^2f&=p\left\{\frac{h}{a}(\cos a\cos\beta-\cos\gamma)+\frac{k}{b}\sin^2\beta+\frac{l}{c}(\cos\beta\cos\gamma-\cos a)\right\}\\ N^2g&=p\left\{\frac{h}{a}(\cos\gamma\cos a-\cos\beta)+\frac{k}{b}(\cos\beta\cos\gamma-\cos a)+\frac{l}{c}\sin^2\gamma\right\}\end{aligned}\right\}\ldots(39).$$

From (25), (38) and (39) the value of p can now be determined: it is given by

$$N^2\div p^2=\frac{h}{a}\left\{\frac{h}{a}\sin^2a+\frac{k}{b}(\cos a\cos\beta-\cos\gamma)+\frac{l}{c}(\cos\gamma\cos a-\cos\beta)\right\}+\&c.$$
$$=\Sigma\frac{h^2}{a^2}\sin^2a+2\Sigma\frac{kl}{bc}(\cos\beta\cos\gamma-\cos a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40);$$

where $$\Sigma\frac{h^2}{a^2}\sin^2a=\frac{h^2}{a^2}\sin^2a+\frac{k^2}{b^2}\sin^2\beta+\frac{l^2}{c^2}\sin^2\gamma,$$

and $$\Sigma\frac{kl}{bc}(\cos\beta\cos\gamma-\cos\alpha)=\frac{kl}{bc}(\cos\beta\cos\gamma-\cos\alpha)$$
$$+\frac{lh}{ca}(\cos\gamma\cos\alpha-\cos\beta)+\frac{hk}{ab}(\cos\alpha\cos\beta-\cos\gamma).$$

Introducing the value of p into equations (38), we have

$$\left.\begin{aligned}\cos\lambda&=\frac{h}{a}\frac{N}{\sqrt{\Sigma\frac{h^2}{a^2}\sin^2\alpha+2\Sigma\frac{kl}{bc}(\cos\beta\cos\gamma-\cos\alpha)}}=\frac{h}{a}\frac{N}{\phi}\\ \cos\mu&=\frac{k}{b}\frac{N}{\sqrt{\Sigma\frac{h^2}{a^2}\sin^2\alpha+\&c.}}=\frac{k}{b}\frac{N}{\phi}\\ \cos\nu&=\frac{l}{c}\frac{N}{\sqrt{\Sigma\frac{h^2}{a^2}\sin^2\alpha+\&c.}}=\frac{l}{c}\frac{N}{\phi}\end{aligned}\right\}\ldots(41).$$

Let $p_{\prime}$ be the normal on any other face $(h_{\prime}k_{\prime}l_{\prime})$; then from (30) and (34)

$$\cos\theta=ep_{\prime}\frac{h_{\prime}}{a}+fp_{\prime}\frac{k_{\prime}}{b}+gp_{\prime}\frac{l_{\prime}}{c}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (42).$$

Hence from (39)

$$\frac{N^2\cos\theta}{pp_{\prime}}=\frac{h_{\prime}}{a}\left\{\frac{h}{a}\sin^2\alpha+\frac{k}{b}(\cos\alpha\cos\beta-\cos\gamma)+\frac{l}{c}(\cos\gamma\cos\alpha-\cos\beta)\right\}$$
$$+\frac{k_{\prime}}{b}\left\{\frac{h}{a}(\cos\alpha\cos\beta-\cos\gamma)+\frac{k}{b}\sin^2\beta+\frac{l}{c}(\cos\beta\cos\gamma-\cos\alpha)\right\}$$
$$+\frac{l_{\prime}}{c}\left\{\frac{h}{a}(\cos\gamma\cos\alpha-\cos\beta)+\frac{k}{b}(\cos\beta\cos\gamma-\cos\alpha)+\frac{l}{c}\sin^2\gamma\right\}$$
$$=\Sigma\frac{hh_{\prime}}{a^2}\sin^2\alpha+\Sigma\frac{kl_{\prime}+lk_{\prime}}{bc}(\cos\beta\cos\gamma-\cos\alpha)\ \ldots\ldots\ldots\ldots\ldots\ (43).$$

Therefore, introducing the values of p and $p_{\prime}$ from (40),

$$\cos\theta=\frac{\Sigma\frac{hh_{\prime}}{a^2}\sin^2\alpha+\Sigma\frac{kl_{\prime}+lk_{\prime}}{bc}(\cos\beta\cos\gamma-\cos\alpha)}{\sqrt{\left\{\Sigma\frac{h^2}{a^2}\sin^2\alpha+2\Sigma\frac{kl}{bc}(\cos\beta\cos\gamma-\cos\alpha)\right\}\left\{\Sigma\frac{h_{\prime}^2}{a^2}\sin^2\alpha+2\Sigma\frac{k_{\prime}l_{\prime}}{bc}(\cos\beta\cos\gamma-\cos\alpha)\right\}}}$$
$$\ldots\ldots\ldots\ldots(44);$$

where $$\Sigma\frac{hh_{\prime}}{a^2}\sin^2\alpha=\frac{hh_{\prime}}{a^2}\sin^2\alpha+\frac{kk_{\prime}}{b^2}\sin^2\beta+\frac{ll_{\prime}}{c^2}\sin^2\gamma,$$

and $$\Sigma\frac{kl_{\prime}+lk_{\prime}}{bc}(\cos\beta\cos\gamma-\cos\alpha)=\frac{kl_{\prime}+lk_{\prime}}{bc}(\cos\beta\cos\gamma-\cos\alpha)$$
$$+\frac{lh_{\prime}+hl_{\prime}}{ca}(\cos\gamma\cos\alpha-\cos\beta)+\frac{hk_{\prime}+kh_{\prime}}{ab}(\cos\alpha\cos\beta-\cos\gamma).$$

14. **To find $\cos\theta$, $\cos\lambda$, $\cos\mu$, $\cos\nu$ of the last Article in terms of the face-indices and the angles $a_{,}$, $\beta_{,}$, $\gamma_{,}$ between the poles A, B and C of the axial planes.**

To obtain the required expressions we have to replace, in the expressions of Art. 13, $\sin^2 a$, &c. by their equivalents in terms of $a_{,}$, $\beta_{,}$, $\gamma_{,}$.

From the polar triangles XYZ and ABC, Fig. 539, we have

$YZ=a=\pi-A$, $ZX=\beta=\pi-B$, $XY=\gamma=\pi-C$;

$BC=a_{,}=\pi-X$, $CA=\beta_{,}=\pi-Y$, $AB=\gamma_{,}=\pi-Z$.

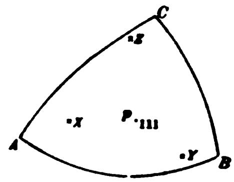


Fig. 539.

From the triangle ABC,

$$\cos a_{,}=\cos\beta_{,}\cos\gamma_{,}+\sin\beta_{,}\sin\gamma_{,}\cos(A=\pi-a).$$

$$\therefore\quad \cos a=\frac{\cos\beta_{,}\cos\gamma_{,}-\cos a_{,}}{\sin\beta_{,}\sin\gamma_{,}};$$

and
$$\sin^2 a=1-\cos^2 a=\frac{(1-\cos^2\beta_{,})(1-\cos^2\gamma_{,})-(\cos\beta_{,}\cos\gamma_{,}-\cos a_{,})^2}{\sin^2\beta_{,}\sin^2\gamma_{,}}$$

$$\left.\begin{aligned}
&=\frac{1-\cos^2 a_{,}-\cos^2\beta_{,}-\cos^2\gamma_{,}+2\cos a_{,}\cos\beta_{,}\cos\gamma_{,}}{\sin^2\beta_{,}\sin^2\gamma_{,}}=\frac{N_{,}^2\sin^2 a_{,}}{\sin^2 a_{,}\sin^2\beta_{,}\sin^2\gamma_{,}};\\
&\text{where}\quad N_{,}^2=1-\cos^2 a_{,}-\cos^2\beta_{,}-\cos^2\gamma_{,}+2\cos a_{,}\cos\beta_{,}\cos\gamma_{,}.\\
&\text{Similarly,}\qquad \sin^2\beta=\frac{N_{,}^2\sin^2\beta_{,}}{\sin^2 a_{,}\sin^2\beta_{,}\sin^2\gamma_{,}}\\
&\qquad\qquad\quad\ \sin^2\gamma=\frac{N_{,}^2\sin^2\gamma_{,}}{\sin^2 a_{,}\sin^2\beta_{,}\sin^2\gamma_{,}}
\end{aligned}\right\}\qquad\ldots\ldots\ldots(45).$$

Again, from the triangle XYZ, we have

$$\cos\beta\cos\gamma-\cos a=-\sin\beta\sin\gamma\cos X=\sin\beta\sin\gamma\cos a_{,}$$

$$\left.\begin{aligned}
&=(\text{from }(45))\ \frac{N_{,}^2\sin\beta_{,}\sin\gamma_{,}\cos a_{,}}{\sin^2 a_{,}\sin^2\beta_{,}\sin^2\gamma_{,}};\\
&\text{similarly,}\quad \cos\gamma\cos a-\cos\beta=\frac{N_{,}^2\sin\gamma_{,}\sin a_{,}\cos\beta_{,}}{\sin^2 a_{,}\sin^2\beta_{,}\sin^2\gamma_{,}}\\
&\qquad\qquad\ \ \cos a\cos\beta-\cos\gamma=\frac{N_{,}^2\sin a_{,}\sin\beta_{,}\cos\gamma_{,}}{\sin^2 a_{,}\sin^2\beta_{,}\sin^2\gamma_{,}}
\end{aligned}\right\}\ \ldots\ldots\ (46).$$

Introducing into (44) the values of $\sin^2 a$, $\cos\beta\cos\gamma-\cos a$, &c., given (45) and (46), we have

$$\cos\theta=\frac{\Sigma\dfrac{hh_{,}}{a^2}\sin^2 a_{,}+\Sigma\dfrac{kl_{,}+lk_{,}}{bc}\sin\beta_{,}\sin\gamma_{,}\cos a_{,}}{\sqrt{\left\{\Sigma\dfrac{h^2}{a^2}\sin^2 a_{,}+2\Sigma\dfrac{kl}{bc}\sin\beta_{,}\sin\gamma_{,}\cos a_{,}\right\}\left\{\Sigma\dfrac{h_{,}^2}{a^2}\sin^2 a_{,}+2\Sigma\dfrac{k_{,}l_{,}}{bc}\sin\beta_{,}\sin\gamma\cos a_{,}\right\}}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(47).$$

Again,

$$N^2 = 1 - \Sigma\cos^2 a + 2\cos a\cos\beta\cos\gamma = \sin^2\beta\sin^2\gamma - (\cos\beta\cos\gamma - \cos a)^2$$

$$= (\text{from (45) and (46)}) = \frac{N_,^4}{\sin^4 a_,\sin^2\beta_,\sin^2\gamma_,} - \frac{N_,^4\cos^2 a_,}{\sin^4 a_,\sin^2\beta_,\sin^2\gamma_,}$$

$$= \frac{N_,^4}{\sin^2 a_,\sin^2\beta_,\sin^2\gamma_,} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48).$$

Introducing the values of N, $\sin^2 a$, $\cos\beta\cos\gamma - \cos a$, and those of the similar factors into equations (41), we have

$$\left.\begin{aligned}
\cos\lambda &= \frac{h}{a}\frac{N_,}{\sqrt{\Sigma\frac{h^2}{a^2}\sin^2 a_, + 2\Sigma\frac{kl}{bc}\sin\beta_,\sin\gamma_,\cos a_,}} = \frac{h}{a}\frac{N_,}{\phi_,}\\
\cos\mu &= \frac{k}{b}\frac{N_,}{\sqrt{\Sigma\frac{h^2}{a^2}\sin^2 a_, + \&c.}} = \frac{k}{b}\frac{N_,}{\phi_,}\\
\cos\nu &= \frac{l}{c}\frac{N_,}{\sqrt{\Sigma\frac{h^2}{a^2}\sin^2 a_, + \&c.}} = \frac{l}{c}\frac{N_,}{\phi_,}
\end{aligned}\right\} \ldots (49).$$

15. To find the equation of an origin-plane perpendicular to a zone-axis OP having the direction-ratios au, bv, cw.

Let the zone-axis make angles λ, μ, ν with the axes, and let it meet the perpendicular plane at distance $OP = p$ at the point P, Fig. 540: in this plane take any point $T(x, y, z)$, and draw the co-ordinates TF, FM parallel respectively to OZ and OX.

Then, since TP is perpendicular to OP, the sum of the projections of OM, MF, FT on OP is equal to OP.

$$\therefore\ p = x\cos\lambda + y\cos\mu + z\cos\nu;$$

and the parallel origin-plane is

$$x\cos\lambda + y\cos\mu + z\cos\nu = 0.$$

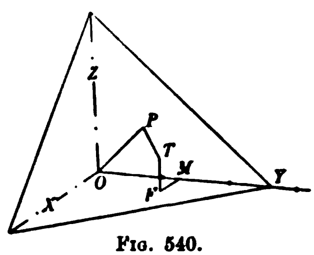


FIG. 540.

Introducing from (23) the values of $\cos\lambda$, &c., in terms of the direction-ratios, we have

$$x(au + bv\cos\gamma + cw\cos\beta) + y(au\cos\gamma + bv + cw\cos a)$$
$$+ z(au\cos\beta + bv\cos a + cw) = 0 \ldots (50).$$

16. To find the condition that a zone-axis should be perpendicular to a possible face.

If the plane given by (50) is parallel to a face (hkl), the coefficients of x, y, z in (3) and (50) must be proportional. Hence

$$\frac{a(au + bv\cos\gamma + cw\cos\beta)}{h} = \frac{b(au\cos\gamma + bv + cw\cos\alpha)}{k}$$
$$= \frac{c(au\cos\beta + bv\cos\alpha + cw)}{l} \quad \ldots\ldots\ldots (51).$$

The zone-indices u, v, w being integers, h, k, l can only be integers in very special cases; and a zone-axis is not, as a rule, perpendicular to a possible face. We shall give a few of the exceptional cases in which the zone-axis is a possible normal.

i. In the cubic system, every zone-axis is perpendicular to a possible face; for $a = b = c$, and $\cos\alpha = \cos\beta = \cos\gamma = 0$;

$$\therefore \frac{h}{u} = \frac{k}{v} = \frac{l}{w}.$$

Hence the zone-axis $[uvw]$ is perpendicular to a face (uvw).

ii. In the tetragonal, rhombohedral and hexagonal systems every zone-axis perpendicular to the principal axis (p. 112) is perpendicular to a possible face.

Thus, in the rhombohedral system $\alpha = \beta = \gamma$, and $a = b = c$. Equations (51) then become

$$\frac{h}{u + (v + w)\cos\alpha} = \frac{k}{v + (w + u)\cos\alpha} = \frac{l}{w + (u + v)\cos\alpha} \ldots (52).$$

For the principal axis, $u = v = w$, and $h = k = l$.

When the zone-axis is in the equatorial plane, it may be taken to be the intersection of a face (pqr) with (111). But the indices of this zone are: $u = q - r$, $v = r - p$, $w = p - q$.

Equations (52) then become

$$\frac{h}{q - r + (r - q)\cos\alpha} = \frac{k}{r - p + (p - r)\cos\alpha} = \frac{l}{p - q + (q - p)\cos\alpha};$$
$$\therefore \frac{h}{q - r} = \frac{k}{r - p} = \frac{l}{p - q}:$$

the plane (hkl) is therefore a possible face.

A zone-axis having any general position in these systems and in the three other systems is not perpendicular to a possible face.

iii. We have already proved that a plane of symmetry is perpendicular to a zone-axis, and that an axis of symmetry of even degree is perpendicular to a possible face. The student can easily verify this for (say) (010) in the oblique system.

Some propositions relating to the rhombohedral system.

17. To find the equations of the faces of the fundamental rhombohedron and of the pinakoid, taking $f = V_{,}M \div 3$ for parameter on the Millerian axes.

Let Fig. 541 represent a section of the rhombohedra R and mR by the plane Σ containing OX, $V_{,}M$ and the triad axis, and let us adopt the same letters and notation that were adopted in Chap. XVI in the discussion of class III. Let $V\mu$ the polar face-diagonal of (100) meet OX in s; and let OV be the linear element c measured on the principal axis.

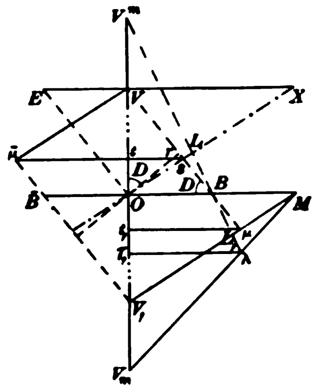


FIG. 541.

From the similar triangles VOs, $VV_{,}\mu$,

$$Os : V_{,}\mu = OV : V_{,}V = 1 : 2.$$

From the relations established in Chap. XVI, Art. 31, $V_{,}\mu = 2V_{,}M \div 3$.

$$\therefore\ Os = V_{,}M \div 3 = f \ldots (53).$$

But the face of the fundamental rhombohedron through $V\mu$ is parallel to the axes of Y and Z. Hence the equation of

$$\left.\begin{array}{lll}(100) & \text{is} & x = f\\ (010) & ,, & y = f\\ (001) & ,, & z = f\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

Again, taking the face (111) through V, its equation is

$$x + y + z = 3f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55);$$

for it meets the axis OX at X, where $OX = V_{,}M = 3f$, and similarly it meets the axes of Y and Z at distances $3f$ from the origin.

18. To find the equations of the faces of $\{hll\} = mR$ (p. 378).

We shall suppose the faces of all rhombohedra to be drawn through pairs of the points A, $A_{,}$, &c., of Fig. 542 in which the faces of the fundamental rhombohedron R $\{100\}$ meet the dyad axes. As shown in Chap. XVI, these points coincide with the points δ, $\delta_{,}$, &c., of the figures of rhombohedra and scalenohedra of class III.

The face (hll) of the rhombohedron mR has for its equation

$$hx + ly + lz = tf$$

where t is an arbitrary constant, which is determined when the face passes through a known point or line. Now this face meets the face (100) in the line $A'A_{\prime\prime}$ lying in the equatorial plane, *i.e.* in the origin-plane parallel to (111). But (100) is given by $x = f$, and the origin-plane (111) by

$$x + y + z = 0 \ldots\ldots(56).$$

Introducing the values

$$x = -(y + z) = f$$

which satisfy these two equations into that of the face (hll), we have

$$f(h - l) = tf.$$

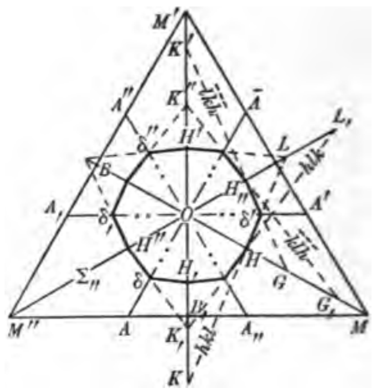


Fig. 542.

The equation of the face (hll) passing through $A'A_{\prime\prime}$ is therefore

$$\left.\begin{array}{lll} & & hx + l(y + z) = (h - l)f, \\ \text{similarly} & (lhl) \text{ is } & hy + l(x + z) = (h - l)f, \\ & (llh) \;\;,, & hz + l(x + y) = (h - l)f \end{array}\right\} \ldots\ldots\ldots\ldots (57).$$

To find the parallel faces, we have only to change the sign of the constant term. Thus

$$(\bar{h}\bar{l}\bar{l}) \text{ is } hx + l(y + z) = (l - h)f.$$

19. To find the relation between the equivalent symbols $\{hll\}$ and mR, and the length of an edge of the rhombohedron.

Let the face (hll) passing through $A'A_{\prime\prime}$ meet the triad axis OV at V^m, where $OV^m = mc$.

Now the co-ordinates of V at a distance c on the triad axis are seen from equations (54) and (55) to be (f, f, f); and the equations of the triad axis are

$$x = y = z = t_{\prime}f \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58);$$

$t_{\prime}$ being an arbitrary constant depending on the distance of the point from the origin. For the point V^m lying in (hll) the values of x, y and z given by (58) must satisfy each equation in (57);

$$\therefore\; t_{\prime}(h + 2l) = h - l \ldots\ldots\ldots\ldots\ldots\ldots\ldots (59).$$

Again, the apex V^m being at m times the distance OV, then by similar triangles the co-ordinates of V^m are (mf, mf, mf). Hence in (58) for the point V^m, $t_{\prime} = m$.

The two values of $t_{,}$ corresponding to V^m must be equal,

$$\therefore\ m = \frac{h-l}{h+2l} \dots\dots\dots\dots\dots\dots\dots\dots (60).$$

This is the same equation as (33) of Chap. XVI, Art. 35. When m is negative, the rhombohedron is said to be an *inverse* one, and is denoted by $-mR$.

Again, since the polar edges join V^m to the points M in the equatorial plane; and, Figs. 541 and 542,

$$OM = 2OB = 2a\cos 30^\circ = a\sqrt{3} \dots\dots\dots\dots (61).$$

Also $OV^m = OV_m = mc$.

$$\therefore\ V^m M^2 = V_m M^2 = 3a^2 + m^2c^2 \dots\dots\dots\dots (62).$$

But the orthogonal projections of the edges of the rhombohedron on the triad axis are all equal. The coigns λ lie therefore on horizontal lines parallel to OM, OM', &c. through the points of trisection of $V^m V_m$. By similar triangles $V_m \lambda \tau_{,}$, $V_m OM$, Fig. 541, we have

$$V_m\lambda : V_m M = V_m \tau_{,} : V_m O = 2 : 3.$$

Hence, Fig. 351,

$$\lambda\lambda_{,,} = V_m\lambda = 2V_m M \div 3 = \tfrac{2}{3}\sqrt{3a^2 + m^2c^2} = \tfrac{2}{3}\sqrt{3a^2 + \left(\frac{h-l}{h+2l}\right)^2 c^2} \dots (63).$$

20. To find the length of a line OP to any point (x, y, z), and expressions connecting the parameter f and the angle a between the axes with the linear elements c and a.

From equation (26) we have, by making $\beta = \gamma = a$, for the length of any line PQ

$$PQ^2 = (x-x_{,})^2 + (y-y_{,})^2 + (z-z_{,})^2 + 2\cos a\,\{(y-y_{,})(z-z_{,}) + (z-z_{,})(x-x_{,}) + (x-x_{,})(y-y_{,})\} \dots (64).$$

Hence, making $x_{,} = y_{,} = z_{,} = 0$, we have

$$OP^2 = x^2 + y^2 + z^2 + 2\cos a\,(yz + zx + xy) \dots\dots\dots (65).$$

Therefore, since V, where $OV = c$, has the co-ordinates (f, f, f), we have

$$OV^2 = c^2 = 3f^2(1 + 2\cos a) \dots\dots\dots\dots\dots\dots (66).$$

But from the right-angled triangle $OV_{,}M$, Fig. 541,

$$V_{,}M^2 = c^2 + 3a^2 = (\text{from } (53))\ 9f^2 \dots\dots\dots\dots (67).$$

$\therefore$ from (66) and (67), $3c^2 = (c^2 + 3a^2)(1 + 2\cos a)$;

and $$2\cos a = \frac{2c^2 - 3a^2}{c^2 + 3a^2} \dots\dots\dots\ \dots\dots\dots (68).$$

The scalenohedron $\{hkl\} = mRn$ (p. 387).

21. To find the equations of the faces of the scalenohedron $\{hkl\}$.

The equation of the face (hkl) is

$$hx + ky + lz = tf;$$

t depending on the distance of the face from the origin. Its value can be found when the face is taken to pass through A', Fig. 542.

Now A' is the point in which (100) meets an origin-line which lies in the equatorial plane (56) and is parallel to (010). Its co-ordinates are therefore given by $x = f$, $y = 0$, and $x + y + z = 0$; and they may be expressed shortly by $(f, 0, -f)$.

Introducing these co-ordinates into the equation of (hkl), we have

$$(h - l)f = tf; \text{ and } t = h - l.$$

Similarly the face (hlk) is given by

$$hx + ly + kz = t_{,}f:$$

where $t_{,}$ is found by drawing the face through $A_{,,}$, the co-ordinates of which are $(f, -f, 0)$.

$$\therefore (h - l)f = t_{,}f; \text{ and } t_{,} = h - l.$$

The equations of all the other faces can be shown to have the same form, the coefficients of x, y and z being taken in the two cyclical orders, the constant term remaining unchanged. We therefore have

$$\left.\begin{array}{lll} \text{for} & (hkl), & hx + ky + lz = (h - l)f \\ ,, & (hlk), & hx + ly + kz = (h - l)f \\ ,, & (klh), & kx + ly + hz = (h - l)f \\ & ,, & ,, \qquad ,, \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(69).$$

The parallel faces passing through $A_{,}$, A'', &c. are obtained by changing the sign of the constant term.

22. To find the equations and length of the median edge $\lambda\lambda_{,,}$.

This edge, Fig. 351, is the intersection of the faces (hkl) and $(\bar{l}\bar{k}\bar{h})$; and, by Art. 3, the zone-axis has the symbol $[\bar{k}, h + l, \bar{k}]$. Since the edge passes through $\delta'\,(f, 0, -f)$, δ' in Fig. 351 being the same point as A' in Fig. 542, its equations are

$$\frac{x - f}{-k} = \frac{y}{h + l} = \frac{z + f}{-k}$$

$$= \frac{\sqrt{(x-f)^2 + y^2 + (z+f)^2 + 2\cos\alpha\,\{y\,(z+f) + (z+f)\,(z-f) + (x-f)\,y\}}}{\sqrt{k^2 + (h+l)^2 + k^2 + 2\cos\alpha\,\{k^2 - 2k\,(h+l)\}}}$$

$$= \frac{p}{\phi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(70);$$

where p is the distance from δ' of any point (x, y, z) on the edge, and

$$\phi^2 = 2k^2 + (h + l)^2 + 2\cos a\,\{k^2 - 2k(h + l)\}\text{.........}(71).$$

Introducing into (71) the value of $2\cos a$ from (68), we have, since $c^2 + 3a^2 = 9f^2$,

$$\begin{aligned}9f^2\phi^2 &= (c^2 + 3a^2)\{2k^2 + (h + l)^2\} + (2c^2 - 3a^2)\{k^2 - 2k(h + l)\}\\ &= 3a^2\{k^2 + 2k(h + l) + (h + l)^2\} + c^2\{4k^2 - 4k(h + l) + (h + l)^2\}\\ &= (h + k + l)^2\left\{3a^2 + \left(\frac{h - 2k + l}{h + k + l}\right)^2 c^2\right\}\\ &= (h + k + l)^2\{3a^2 + m^2c^2\}\text{ ....} (72);\end{aligned}$$

where
$$m = \frac{h - 2k + l}{h + k + l}\text{..........................}(73).$$

Now the edge meets the face $(\bar{l}\bar{h}\bar{k})$ in the coign λ, and for λ the co-ordinates x, y, z in (70) must satisfy the equation of the face

$$lx + hy + kz = -(h - l)f\text{....................}(74).$$

The corresponding length of p is $\delta'\lambda$: it is most easily found by multiplying the numerator and denominator of the first term in (70) by l, those of the second by h and of the third by k, and then adding the numerators and also the denominators.

Hence for $p = \delta'\lambda$, we have

$$\frac{p = \delta'\lambda}{\phi} = \frac{l(x - f) + hy + k(z + f)}{-lk + h(h + l) - k^2} = \frac{lx + hy + kz + f(k - l)}{(h - k)(h + k + l)}$$

$$= (\text{from } (74))\ \frac{(l - h)f + f(k - l)}{(h - k)(h + k + l)} = -\frac{f}{h + k + l}\text{.........}(75).$$

$$\therefore\ \delta'\lambda = \frac{f\phi}{h + k + l} = (\text{from } (72))\ \frac{\sqrt{3a^2 + m^2c^2}}{3}\text{ ......}(76).$$

The minus sign in (75) arises from the fact that λ is the lower coign of the edge: a plus sign is obtained when $\lambda_{,,}$ in (khl) is taken.

In (60) it was seen that the edge of the rhombohedron mR is given by $\frac{2}{3}\sqrt{3a^2 + m^2c^2}$. Hence, equation (73) gives the value of m for the auxiliary rhombohedron mR. It is the same as that given in (60) of Chap. XVI.

23. To find n of mRn in terms of h, k, l.

By the method of construction given in Chap. XVI, Art. 40, the apex V^n is at distance $nOV^m = mnc$ from the origin, m having the

value given in (73); and, the apex being on the triad axis, its coordinates are found by similar triangles to be (mnf, mnf, mnf).

Introducing these into the equation of one of the faces (69), we have

$$(h+k+l)\,mnf = (h-l)f\,;$$

$$\therefore\ n = \frac{h-l}{h+k+l}\,\frac{1}{m} = \text{(from (73))}\ \frac{h-l}{h-2k+l} \quad\text{.........(77).}$$

Equations (73) and (77) are those already found in Chap. XVI, for the conversion of the Millerian symbol $\{hkl\}$ of the scalenohedron to the corresponding Naumannian symbol mRn. By a slight transformation, they suffice also to determine h, k, and l, when the Naumannian symbol is given. When m is negative, both the scalenohedron and the auxiliary rhombohedron mR are said to be inverse.

24. *The trapezohedron*, $\alpha\{hkl\}$. To find the length of the median edge $2\delta'\gamma$.

Geometrically the trapezohedron $\alpha\{hkl\}$ consists of the three pairs of faces of the scalenohedron $\{hkl\}$ which pass through the alternate median edges interchangeable by rotation about the triad axis. The faces are therefore

$hkl \quad lhk \quad klh \quad \bar{l}\bar{k}\bar{h} \quad \bar{h}\bar{l}\bar{k} \quad \bar{k}\bar{h}\bar{l}.$

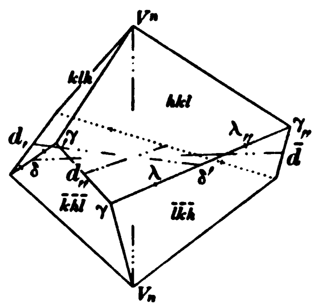


FIG. 543.

One pair of the faces which are common to the trapezohedron and scalenohedron being supposed to pass through δ', V^n and V_n, the equations of the median edge $\delta'\gamma$ are the same as those given in (70) for $\delta'\lambda$. This edge has in the trapezohedron to be extended to meet $(\bar{k}\bar{h}\bar{l})$, Fig. 543, at γ; and the length is found, in the same way as that of $\delta'\lambda$, by multiplying the numerators and denominators of the first, second and third terms of equations (70) by $\bar{k}$, $\bar{h}$ and $\bar{l}$ respectively.

Therefore

$$\frac{p=\delta'\gamma}{\phi} = \frac{-k(x-f)-hy-l(z+f)}{k^2-h(h+l)+lk} = \frac{(h-l)f+f(k-l)}{(k-h)(h+k+l)},$$

$$\therefore\ \delta'\gamma = -\frac{h+k-2l}{h-k}\,\frac{f\phi}{h+k+l} = \text{(from (75))}\ \frac{h+k-2l}{h-k}\,\delta'\lambda \quad\text{....(78).}$$

If 2ξ and 2η are the angles over the obtuse and acute polar edges of the scalenohedron $\{hkl\}$, then from equation (74) of Chap. XVI,

$$\frac{k-l}{h-k}=\frac{\sin\xi}{\sin\eta};$$

$$\therefore\ \delta'\gamma=\left(1+2\frac{k-l}{h-k}\right)\delta'\lambda=\left(1+2\frac{\sin\xi}{\sin\eta}\right)\delta'\lambda\text{......(79).}$$

This is the relation (90) used in p. 413 for drawing the trapezohedron.

Hexagonal axes.

25. The correlation of the four axes to which hexagonal and rhombohedral crystals are often referred is fully given in Chap. XVII, Arts. 5—14. We shall therefore assume Miller's (100) to be $(01\bar{1}1)$ in hexagonal symbols. Then a and c being the parameters on the equatorial and principal axes, let (hkil) be the symbol of a face, which in Millerian notation is (hkl).

As already stated in Chap. XVII, the fourth axis is superfluous when we require to determine the zone-indices, the anharmonic ratio of four tautozonal faces or other relations between the faces and edges. The faces are therefore referred to three axes OX, OY, and OZ; where the first two axes are inclined to one another at an angle of 120° and are perpendicular to the principal axis OZ. Hence the equation of the face (hkil) is

$$\mathrm{h}\frac{x}{a}+\mathrm{k}\frac{y}{a}+\mathrm{l}\frac{z}{c}=t\text{..............................(80);}$$

the constant t depending on the distance from the origin. The equation of the parallel origin-plane is

$$\mathrm{h}\frac{x}{a}+\mathrm{k}\frac{y}{a}+\mathrm{l}\frac{z}{c}=0\text{..........................(81).}$$

26. To find the formulæ of transformation from Miller's to hexagonal notation.

Miller's (hkl) becoming (hkil), we have to find the intercepts made by (hkl) on the hexagonal axes OV, $O\delta$, $O\delta'$, $O\delta''$. Referred to Millerian axes, the equations of these lines are

$$\left.\begin{array}{lll}\text{for} & OV & \dfrac{x}{f}=\dfrac{y}{f}=\dfrac{z}{f}=\dfrac{p}{\sqrt{3f^2(1+2\cos a)}}=(\text{from (66)})\ \dfrac{p}{c}\\ \text{,,} & O\delta & \dfrac{x}{0}=\dfrac{y}{-f}=\dfrac{z}{f}=\dfrac{p_{,}}{\sqrt{2f^2(1-\cos a)}}=\dfrac{p_{,}}{a}\\ \text{,,} & O\delta' & \dfrac{x}{f}=\dfrac{y}{0}=\dfrac{z}{-f}=\dfrac{p_{,,}}{\sqrt{2f^2(1-\cos a)}}=\dfrac{p_{,,}}{a}\\ \text{,,} & O\delta'' & \dfrac{x}{-f}=\dfrac{y}{f}=\dfrac{z}{0}=\dfrac{p_{,,,}}{\sqrt{2f^2(1-\cos a)}}=\dfrac{p_{,,,}}{a}\end{array}\right\}\text{......(82);}$$

$p, p_{,}$, &c. being the distances of any points on the lines from the origin; and from (66) and (68)

$$2f^2(1-\cos a)=f^2\left(2-\frac{2c^2-3a^2}{c^2+3a^2}\right)=\frac{9f^2a^2}{c^2+3a^2}=a^2.$$

If now the values of x, y and z which satisfy each line are introduced into the Millerian equation of the face

$$hx+ky+lz=(h-l)f,$$

we find the intercept made on each of the hexagonal axes.

Hence multiplying the numerator and denominator of the first term in each of equations (82) by h, those of the second by k and of the third by l, we have

$$\left.\begin{array}{lll} \text{on} & OZ, & \dfrac{OV^n}{c}=\dfrac{hx+ky+lz}{(h+k+l)f}=\dfrac{h-l}{h+k+l} \\ \text{,,} & OX, & \dfrac{p_{,}}{a}=\dfrac{hx+ky+lz}{(l-k)f}=\dfrac{h-l}{l-k} \\ \text{,,} & OY, & \dfrac{p_{,,}}{a}=\dfrac{hx+ky+lz}{(h-l)f}=\dfrac{h-l}{h-l} \\ \text{,,} & OU, & \dfrac{p_{,,,}}{a}=\dfrac{hx+ky+lz}{(k-h)f}=\dfrac{h-l}{k-h} \end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots (83).$$

The intercepts $OV^n, p_{,}, p_{,,}, p_{,,,}$ are in the ratios

$$c\div\mathrm{l} : a\div\mathrm{h} : a_{,}\div\mathrm{k} : a_{,,}\div\mathrm{i}.$$

$$\therefore\ \frac{\mathrm{l}}{h+k+l}=\frac{\mathrm{h}}{l-k}=\frac{\mathrm{k}}{h-l}=\frac{\mathrm{i}}{k-h}\ \ldots\ldots\ldots\ldots\ldots\ldots (84).$$

These are the equations found in Chap. XVII, Art. 10.

Each term in (84) is equal to $\dfrac{\mathrm{h}+\mathrm{k}+\mathrm{i}}{l-k+h-l+k-l}$. Since the denominator of this last expression is zero, the numerator must be so too;

$$\therefore\ \mathrm{h}+\mathrm{k}+\mathrm{i}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (85).$$

27. To find the equations of the six faces of τ {hkil}.

Let the pole P of $(hkl)=$(hkil) be so situated that $Pr<Pr'<Pr''$; r, r', r'' being the poles (100), (010), (001) respectively; and let the face pass through γ' at distance a on OY, then its equation referred to X, Y and Z is

$$\mathrm{h}\frac{x}{a}+\mathrm{k}\frac{y}{a}+\mathrm{l}\frac{z}{c}=t;$$

t being determined from the condition imposed on the position of the face. But the hexagonal co-ordinates of γ' on X, Y and Z are $(0, a, 0)$.

$$\therefore\ \mathrm{k}=t;$$

and the equation of the face is

$$\mathrm{h}\frac{x}{a}+\mathrm{k}\frac{y}{a}+\mathrm{l}\frac{z}{c}=\mathrm{k}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (86).$$

A rotation of 60° counter-clockwise brings OX to $OU_{,}$, OY to $OX_{,}$ and a length on $OU_{,}$ to OY. Further, the first face intercepts $a \div \mathrm{i}$ on $OU_{,}$, where $\mathrm{i} = -(\mathrm{h}+\mathrm{k})$. Hence the new position of the face is given by

$$-\mathrm{k}\frac{x}{a} - \mathrm{i}\frac{y}{a} + \mathrm{l}\frac{z}{c} = t_{,}.$$

But this is a face of the hexagonal pyramid passing through the point on $OX_{,}$ at distance $-a$ from the origin.

$$\therefore \quad -\mathrm{k} \times -\frac{a}{a} = t_{,}, \text{ and } t_{,} = \mathrm{k}.$$

The equation of the face is therefore

$$-\mathrm{k}\frac{x}{a} - \mathrm{i}\frac{y}{a} + \mathrm{l}\frac{z}{c} = \mathrm{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (87);$$

and the face-symbol is $(\bar{\mathrm{k}}\bar{\mathrm{i}}\mathrm{hl})$.

A second rotation in the same direction brings the original face into the position of one having the symbol (ihkl): its equation is found in the same way and is

$$\mathrm{i}\frac{x}{a} + \mathrm{h}\frac{y}{a} + \mathrm{l}\frac{z}{c} = \mathrm{k}.$$

Repeating the process, the successive faces are

$$\left.\begin{array}{ll} & \mathrm{i}\frac{x}{a} + \mathrm{h}\frac{y}{a} + \mathrm{l}\frac{z}{c} = \mathrm{k} \\ (\bar{\mathrm{h}}\bar{\mathrm{k}}\mathrm{il}), & -\mathrm{h}\frac{x}{a} - \mathrm{k}\frac{y}{b} + \mathrm{l}\frac{z}{c} = \mathrm{k} \\ (\mathrm{kihl}), & \mathrm{k}\frac{x}{a} + \mathrm{i}\frac{y}{b} + \mathrm{l}\frac{z}{c} = \mathrm{k} \\ (\bar{\mathrm{i}}\bar{\mathrm{h}}\mathrm{kl}), & -\mathrm{i}\frac{x}{a} - \mathrm{h}\frac{y}{b} + \mathrm{l}\frac{z}{c} = \mathrm{k} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (88).$$

The equations of the six faces parallel to the above six faces are obtained by changing the sign of the constant term k. The equations of the six faces of the dihexagonal pyramid having the first three indices in the opposite cyclical order are found in the same way: they also have the same constant term k.

The problems solved in Arts. 22–24 by the aid of Millerian axes can now be all solved in hexagonal notation.

Grassmann's method of axial representation.

28. Grassmann suggested a different method of axial representation, by which the difficulty of following in the imagination the various combinations of planes could be to a considerable extent overcome. With this object he took for axes three normals not lying in one plane; and for the parametral ratios he took the edges along the axial normals of a parallelepiped which has for diagonal a fourth normal not lying in a plane with any two of the

axial normals. The system of planes can then be replaced by a system of normals or *rays*, as he termed them, all emanating from the origin. The direction of each of these rays can, like the parametral-ray, be given by the diagonal of a parallelepiped which has its edges in the axes; the edges being rational multiples of those of the parametral parallelepiped. This system of rays served in Chap. VII as the basis of the representation of a crystal by the stereographic projection. The analytical method to which this axial representation gives rise has never been seriously employed in the solution of problems. Miller gave a short account of it in 1868 (*Proc. Camb. Phil. Soc.* II, p. 75, 1868); and it affords an elegant method for establishing many of the general relations discussed in Arts. 2–20 of this Chapter.

29. To find the equations of a ray.

Take in Fig. 544 for Grassmann's axes the normals OA, OB, OC to the faces which, in the ordinary representation, are the axial planes YOZ, ZOX and XOY; and let the current co-ordinates on OA, OB, OC be denoted by x, y, z. Further, let the face (111) be perpendicular to the parametral-ray OG, and let the edges of the parallelepiped which has for diagonal a definite length on this ray be represented by a, b, c.

Then, from the statement of the method, any other ray is given by

$$\frac{\mathsf{x}}{h\mathsf{a}} = \frac{\mathsf{y}}{k\mathsf{b}} = \frac{\mathsf{z}}{l\mathsf{c}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (89);$$

where h, k and l are any integers, one or two but not all three of which may be zero.

These equations are of the form given in Art. 3 for a zone-axis. But in Art. 16 it was shown that a normal can only in very special cases coincide with a zone-axis. The ray-parameters a, b, c are not therefore commensurable multiples of the face-parameters a, b, c measured on zone-axes OX, OY, OZ.

30. To find the relations which connect the ray-parameters and the face-parameters, when the same faces are used in the determination of the axial representation and the parametral ratios.

Let Grassmann's axes OA, OB, OC, Fig. 544, and the corresponding zone-axes OX, OY, OZ meet a sphere described about O as centre in the points A, B, C and X, Y, Z. Then the triangles ABC and XYZ are polar triangles (Chap. XI, Art. 6); and the

system of lines is fully determined when the sides of either triangle are known. If the axial faces (100), (010), (001), of which A, B, and C are the respective normals, are present, and if the angles over their edges are measurable, then the sides of ABC are given by direct observation.

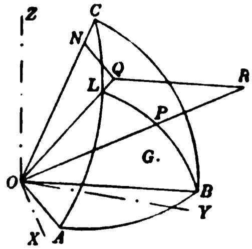


FIG. 544.

Let P be the pole of a face (hkl), so that the radius OP is given by equations (89). On OP take any point R, having when referred to Grassmann's axes the coordinates (x, y, z); and draw the ordinates RQ, QN parallel to the normals OB and OA. Let OQ meet the circle AC in L, and describe the great circle BPL. Since the great circles $[AC]$ and $[BP]$ pass each through two poles, their point of intersection L is a possible pole which has the symbol $(h0l)$. G is the pole (111) corresponding to parameters $\mathbf{a}, \mathbf{b}, \mathbf{c}$ on the zone-axes OX, OY, OZ, and to a, b, c on the normals OA, OB and OC.

From the geometry of the figure,

$$\frac{z}{x} = \frac{ON}{NQ} = \frac{\sin NQO}{\sin QON} = \frac{\sin LOA}{\sin LOC},$$

since NQ is parallel to OA.

And from the spherical triangles LBA, LBC,

$$\frac{\sin LA}{\sin LC} = \frac{\sin PBA \sin AB}{\sin PBC \sin BC} \quad \text{..............} \quad (90).$$

In Chap. XI, Art. 22 it was shown that

$$\frac{l}{c} \sin PBC = \frac{h}{a} \sin PBA.$$

Hence
$$\frac{z}{x} = \frac{\sin PBA \sin AB}{\sin PBC \sin BC} = \frac{al \sin AB}{ch \sin BC}.$$

$$\therefore \quad \frac{x}{\dfrac{h \sin BC}{a}} = \frac{z}{\dfrac{l \sin AB}{c}} = (\text{by symmetry}) \ \frac{y}{\dfrac{k \sin CA}{b}} \ldots (91).$$

These equations are identical with (89), if

$$\frac{a\mathbf{a}}{\sin BC} = \frac{b\mathbf{b}}{\sin CA} = \frac{c\mathbf{c}}{\sin AB} \quad \text{.................} (92).$$

By the relations of the polar triangles, the trigonometrical ratio

can be replaced by their equivalents in terms of the sides or angles of the triangle XYZ; thus

$$\frac{a\mathfrak{a}}{\sin YZ}=\frac{b\mathfrak{b}}{\sin ZX}=\frac{c\mathfrak{c}}{\sin XY}\text{ .... }(92^*).$$

Hence if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ satisfy the above equations, the normal OR is the diagonal of a parallelepiped which has edges in OA, OB, OC respectively proportional to $h\mathfrak{a}$, $k\mathfrak{b}$ and $l\mathfrak{c}$.

31. To find the origin-plane containing two rays.

From Art. 30 it follows that the origin-plane

$$h\frac{x}{a}+k\frac{y}{b}+l\frac{z}{c}=0\text{ .... }(93),$$

and the ray

$$\frac{X}{h\mathfrak{a}}=\frac{Y}{k\mathfrak{b}}=\frac{Z}{l\mathfrak{c}}\text{ .... }(94)$$

are for all possible values of h, k and l at right angles to one another. Since equations (92) and (92*), which connect the parameters in the two systems of axes, involve these parameters in an exactly similar manner, it follows that the plane (referred to Grassmann's axes)

$$u\frac{X}{\mathfrak{a}}+v\frac{Y}{\mathfrak{b}}+w\frac{Z}{\mathfrak{c}}=0\text{ .... }(95)$$

is perpendicular to the zone-axis (referred to zone-axes)

$$\frac{x}{au}=\frac{y}{bv}=\frac{z}{cw}\text{ .... }(96).$$

This can be easily proved from (89).

For let two faces $(h_1k_1l_1)$, $(h_2k_2l_2)$ have for zone-axis the line which has the symbol $[uvw]$ given in Art. 3. Then (96) are the equations of the zone-axis.

Now the ray perpendicular to the face $(h_1k_1l_1)$ is at right angles to $[uvw]$, and so is the ray perpendicular to $(h_2k_2l_2)$. The plane containing these rays is therefore perpendicular to the zone-axis $[uvw]$.

The ray $(h_1k_1l_1)$ is given by

$$\frac{X}{\mathfrak{a}h_1}=\frac{Y}{\mathfrak{b}k_1}=\frac{Z}{\mathfrak{c}l_1},$$

and the ray $(h_2k_2l_2)$ by

$$\frac{X}{\mathfrak{a}h_2}=\frac{Y}{\mathfrak{b}k_2}=\frac{Z}{\mathfrak{c}l_2}.$$

The origin-plane containing these two rays is

$$(k_1l_2 - l_1k_2)\frac{x}{a} + (l_1h_2 - h_1l_2)\frac{y}{b} + (h_1k_2 - k_1h_2)\frac{z}{c} = 0 \;\ldots\ldots\; (97).$$

This latter plane is that of the zone-circle containing the poles of tautozonal faces: as was shown in Art. 16, it is not generally a possible face.

Weiss's, Naumann's and Lévy's notations.

32. Crystallographers have represented the forms of crystals in several ways, and the Millerian notation is not even yet universally adopted; but, with hardly an exception, crystallographers who still use other symbols now give the Millerian equivalents also. We shall briefly indicate the principles of three of the most important notations.

33. Weiss represented a face by its intercepts on the axes. Assuming the axes to be the same, the Millerian symbol is found by dividing by the common multiple, and writing the denominators (expressed as integers) in the order of the axes X, Y, Z. In the hexagonal system four axes were used, and the further transformation given in Chap. XVII, Art. 10, has to be used if Millerian axes are adopted. Weiss's notation is used in Rammelsberg's *Handbuch d. Kryst.-Phys. Chemie*, 1881—2.

34. Naumann sought to make the symbol indicate the shape of the form, and made it depend on the characteristic holohedral form of the system. Thus in the prismatic system he takes as *fundamental* pyramid P that which has its apices A, B and C at distances a, b and c on the axes of X, Y and Z. Every other pyramid is supposed to be drawn through A or B on the axes of X and Y, when its intercepts are given (i) by $a : nb : mc$, or (ii) by $na : b : mc$. The pyramid (i) he represents by $m\bar{P}n$, (ii) by $m\breve{P}n$. The first index m gives the intercept on the vertical axis OZ. The letter P is affected with the long or short sign according as the second index n refers to the makro-axis OY or the brachy-axis OX. By giving to m and n the special values ∞ and 0, all the special forms of the system can be represented. In the tetragonal and hexagonal systems P carries no sign, for the horizontal axes are interchangeable and the parameters equal.

In the cubic system P is replaced by O, for all the parameters are equal, and the fundamental pyramid is an octahedron.

In the oblique system the same method is employed, but when the index n gives the intercept on the inclined axis OX a stroke is drawn slantwise across P, and when it refers to OY a horizontal stroke is drawn across it; it is left unmodified when the indices h and k are equal. To represent pinakoids $(h0l)$ a minus sign must be placed before the symbol,

if the face, meeting OX on the positive side of the origin, meets OZ on the negative side; and a similar artifice has to be adopted in representing the different four-faced forms which we may call plinthoids.

In the anorthic system P is affected with the long and short signs and with dashes placed on various sides of it which serve to indicate the directions in which the intercepts on the several axes are measured. The perplexing arrangement of these various modifications in the oblique and anorthic systems renders the symbols ill-adapted for the purpose they were intended to serve.

Naumann's notation in the rhombohedral system has been already explained in Chap. XVI; and its utility for comprehending the relations of, and for drawing, scalenohedra and rhombohedra was there demonstrated.

35. Lévy's notation, still used by French crystallographers, is based on the modifications of the edges and coigns of a parallelepiped or of a hexagonal prism which may be regarded as Haüy's primitive form. According to the system the primitive forms are a cube, a tetragonal prism terminated by the base, a rhombic prism terminated by a pinakoid, a rhombohedron, a hexagonal prism and pinakoid, an oblique prism terminated by a pinakoid, and an anorthic parallelepiped.

The anorthic parallelepiped, from which all the others (save the hexagonal prism) are derived by making certain of the angles and edges equal, has four different coigns denoted by the vowels a, e, i, o, a being that at the back of the upper face; three different faces p, m, t (from the word *primitive*); and six pairs of different edges, four b, c, d, f in the base, and two g and h vertical. The back edges of the upper face meeting at a are b and c, b being to the left; the front edges are d and f. Again, b and d meet at e on the left, c and f at i on the right. The vertical edges are measured downwards, h passing through a and o, g through e and i. Any face is denoted by the intercepts measured on the edges meeting at the coign which it modifies, these lengths being indicated by indices attached to the corresponding edges. Thus the face $v(\bar{2}\bar{4}1)$ of anorthite modifies the coign e at which b, d and g meet; and the intercepts are in the ratios $b \div 2 : d \div 6 : g \div 1$: the face is given by $b^{1/2}d^{1/6}g^{1}$.

The parametral length on a vertical edge g or h is that intercepted by some definite face (arbitrarily chosen) which is parallel to one of the diagonals of the base. Thus $x(\bar{1}01)$ of Fig. 121 may be taken to give the parameter in anorthite: it modifies the coign a and may be drawn through the diagonal ei, when it has the intercepts b, c and h; it is represented by a^{1}. The face $y(\bar{2}01)$ may be drawn through the same diagonal to modify the same coign a and to meet h at distance $2h$. Its intercepts are b, c and $2h$, or $b \div 2$, $c \div 2$, h; and it is represented by $a^{1/2}$. Any other face parallel to ei modifying a may be given by $b \div n$, $c \div n$, $h \div z$: it is represented by $a^{z/n}$. If a face in the same zone modifies the coign o, it is represented by $o^{z/n}$; the intercepts being $d \div n$, $f \div n$, $h \div z$. Thus $o^{1/2}$ is the face $t(201)$ of

Fig. 120. In the same way a face through the other diagonal ao intercepting on g a length $ng \div z$ is represented by $e^{z/n}$ or $i^{z/n}$, according as it modifies the coign on the left or on the right; for instance, $e\,(021)$ of Figs. 120 and 121 is $i^{1/2}$ and $n\,(0\bar{2}1)$ is $e^{1/2}$.

The face is also represented by a single letter and index z/n when it is parallel to and modifies the edge to which the letter is attached. If the edge is in the base, z gives the intercept on the vertical, and n on a basal edge. Thus $m\,(111)$ of Fig. 120 is $f^{1/2}$, $p\,(\bar{1}11)$ is $c^{1/2}$, and generally $f^{z/n}$ has the intercepts $d \div n$ and $h \div z$ on the edges meeting f at o. Similarly for all the other basal edges. When the modified edge is one of the vertical edges, the symbol is $h^{y/x}$ or $g^{y/x}$ if the face is to the right of the observer; it is ${}^{y/x}h$ or ${}^{y/x}g$ if to the left; y/x being always greater than unity. Thus $f\,(130)$ of Figs. 120 and 121 is g^2; for its intercepts are $c \div 2$, $f \div 1$ and $g \div 0$, since a line through o bisecting the edge c meets the diagonal ei at the centre of gravity of the triangle aio. Similarly $z\,(1\bar{3}0)$ is ${}^2g$.

With the introduction of planes and axes of symmetry some of the angles and edges become equivalent, and fewer different symbols are needed. Since the symbols denote forms, it is always necessary to know the system and class of the crystal to perceive the character of the form.

Although the symbol, such as $b^{1/x}\,d^{1/y}\,g^{1/z}$, resembles Millerian intercepts $a \div h$, $b \div k$, $c \div l$, the indices in the two notations are only in exceptional cases the same. For Miller's axes of X and Y are, as a rule, parallel to the diagonals of the base of Lévy's parallelepiped, XX' being parallel to ao and YY' to ei. The conversion of Lévy's symbols to Millerian involves therefore a transformation of axes in all systems except the cubic and rhombohedral.

In the rhombohedral system the parallelepiped becomes the fundamental rhombohedron p; the two apices are denoted by a, the median coigns by e, the polar edges by b, and the median edges by d. Assuming the same fundamental rhombohedron, p is $\{100\}$; and a face of any form is given by intercepts which are (i) $b^{1/h}b^{1/k}b^{1/l}$, or (ii) $b^{1/h}d^{1/k}d^{1/l}$, according as the face modifies an apex or a median coign.

i. In these forms the intercepts are the same as Miller's, for we may suppose the origin to be shifted to the apex. Hence $b^{1/h}b^{1/k}b^{1/l}$ is $\{hkl\}$, all the indices being positive. An obtuse rhombohedron $\{hll\}$ is $a^{h/l}$, and it is direct or inverse according as $h \gtrless l$: a^1 is $\{111\}$. A face $(h0l)$ parallel to a polar edge is $b^{h/l}$; the form is a scalenohedron, except when $h = l$ or $h = 2l$: in the former case b^1 is the rhombohedron $\{101\}$, in the latter b^2 is the hexagonal bipyramid $\{201\}$.

ii. In $b^{1/h}d^{1/k}d^{1/l}$ the Millerian indices are numerically the same, but the signs of one or two of the indices have to be changed according to the coign modified by the face. If the face is that of a direct scalenohedron modifying the coign μ, Fig. 545, it meets the edges $V_{,}\mu$, $\mu\mu_{,,}$, $\mu\mu_{,}$ at distances $b \div h$, $d \div k$, $d \div l$ respectively. Suppose the origin to be at $V_{,}$,

and the face to be shifted to a point between $V_{,}$ and μ, where the intercept measured from $V_{,}$ is $b \div h$. In its new position the face meets $V_{,}\mu'$ and $V_{,}\mu''$ produced beyond $V_{,}$, *i.e.* on the negative sides of the origin; and its symbol is $(h\bar{k}\bar{l})$, where $h-k-l>0$. The pole lies in the spherical triangle $ra'a_{,,}$ of Fig. 546. When therefore the scalenohedra are direct and the superior faces modify inferior median coigns of $\{100\}$, $b^{1/h}d^{1/k}d^{1/l}$ is $\{h\bar{k}\bar{l}\}$.

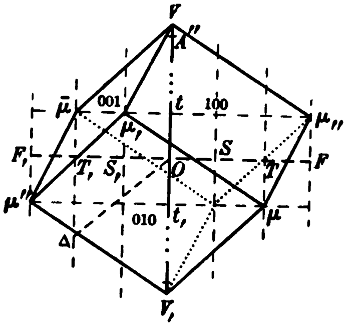


FIG. 545.

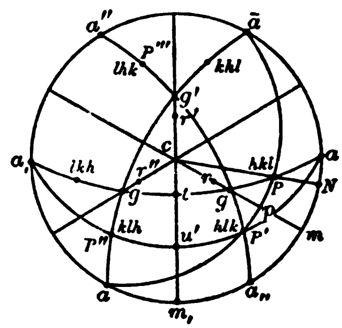


FIG. 546.

Let now a superior face $d^{1/h}b^{1/k}d^{1/l}$ meet a superior median coign μ' (say): it belongs to a direct or inverse scalenohedron according as its pole lies in the spherical triangle $ra_{,,}c$ or in $a_{,,}cm_{,}$ of Fig. 546. When the face is shifted so as to meet the axes at distances from the origin $V_{,}$ equal to the intercepts on the parallel edges meeting at $\mu_{,}$, the Millerian symbol is $(h\bar{k}l)$, where $h-k+l>0$. The direct form is distinguished from the inverse by writing the intercept $b^{1/k}$ first or last. A particular case is given when the face passes through a polar face-diagonal, when the symbol $d^{k/h}b^{1}d^{1}$ is abbreviated to $e_{k/h}$ equivalent to $\{h\bar{k}k\}$. When the intercepts on the median edges are equal, the form is a rhombohedron $e^{h/l}=\{h\bar{l}\bar{l}\}$: it is direct or inverse according as $h-2l \gtrless 0$. When the face is parallel to a median edge, the form is given by $d^{h/l}$ and is the same as $\{h0\bar{l}\}$. The particular cases e^{2} and d^{1} are the hexagonal prisms $\{2\bar{1}\bar{1}\}$ and $\{10\bar{1}\}$ respectively.

CHAPTER XX.

ON GONIOMETERS.

1. THE angles of crystals are measured by instruments called goniometers, of which there are several kinds in use.

The *contact-goniometer* invented by Carangeot, and used by Romé de l'Isle and Haüy, is still employed for the approximate measurement of the angles of large crystals, and more especially when the faces are rough or deficient in lustre, or the crystal is attached to a large piece of matrix. The essential part consists of two flat straight bars—often called limbs (l, l' of Fig. 547)—of steel

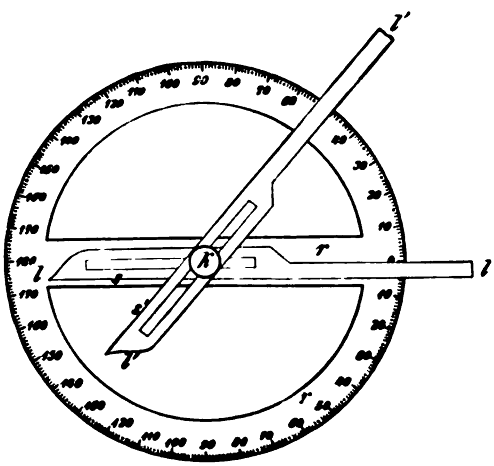


FIG. 547.

or brass connected by a pin and screw k. One edge s of each bar is made accurately straight; and these edges are placed in contact with the two faces the angle between which is required. If the limbs are also perpendicular to the edge in which the faces intersect, the angle between them is the Euclidean angle and its

supplement that between the normals (Chap. II, Art. 2). To secure accurate contact the crystal and limbs are held between the eye and the light; and when the limbs are correctly adjusted, the screw is tightened and the limbs carefully removed from the faces. The limbs are now placed on a graduated circle or semicircle in the centre of which the pin fits, and the included angle is read off. The circle may be divided to degrees, half- or quarter-degrees. But to give correct readings on the circle the continuations of the bars must be cut down, as shown in Fig. 547, to straight edges which are radii of the circle and necessarily parallel to the edges *s* placed in contact with the crystal-faces. Or one limb may be so adjusted

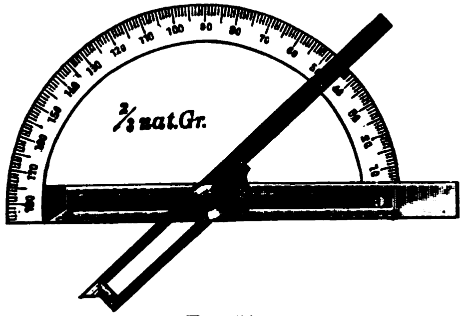


FIG. 548.

that its central line passes through the zero of the scale as shown in Fig. 548. The edges at which readings are made are usually bevelled to a thin edge.

To be able to measure the angles between faces partly enveloped by matrix or other crystals, slots are made in the bars as shown in the figures, and the bars are sloped to points at the ends placed in contact with the crystal. The bars being shortened, their ends can be introduced into shallow cavities, and their tips brought into contact with the faces.

2. *The reflexion-goniometer.* In 1809 Wollaston (*Phil. Trans.* 1809, p. 253) invented this goniometer by which the angle between the normals of two faces is determined by the reflexion of a well-defined signal S (say) from each face in succession; and all modern goniometers suitable for accurate measurement depend on the laws of reflexion. Thus, suppose Fig. 549 to represent a section of a

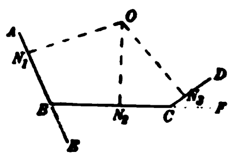


FIG. 549.

crystal by a plane perpendicular to the edges of tautozonal faces; and let the signal S, lying in the plane of section to the left of the face AB, be reflected from AB in a fixed direction. The incident and reflected rays both lie in the plane ABC, and the normal ON_1 bisects the angle between them. If ON_2 is supposed to be equal to ON_1, a rotation about O through $\wedge N_1ON_2 = \wedge CBE$ brings the face BC exactly into the position of AB, and S will be reflected from BC in the fixed direction. A second rotation through N_2ON_3 brings the face CD into the position of AB, and S is reflected from CD in the fixed direction; and so on for all the faces in the zone. If then O is in the axis, perpendicular to a graduated circular disc, which can be turned about this axis, the angles N_1ON_2, N_2ON_3, &c. between the face-normals can be determined. The graduated circle may be either vertical or horizontal, and the instrument is accordingly known as a *vertical-circle* or *horizontal-circle* goniometer. We shall describe one of each kind now in common use.

The axis O has been supposed to be equally distant from the faces, and then the angles turned through are accurately equal to those between the normals. But if, as in the figure, O is unequally distant from AB and BC, a rotation about O through N_1ON_2 brings BC parallel to AB but not into coincidence. If then S is at a short distance from the crystal, the angle of incidence on BC in its new position is less than that on AB, and the reflected ray is not in the fixed direction. If ON_2 is less than ON_1, the crystal must not be turned quite so far, and we get an error in the measured angle due to defective centering. If ON_3 is equal to ON_1, the total angle N_1ON_3 is determined with accuracy, for CD is brought into coincidence with AB. Hence in the measurement of N_2ON_3, where we have an increase in the radius, an error is made of the opposite kind to that made in measuring N_1ON_2. These errors disappear if S is at a very great distance, or, as it is shortly expressed, at infinity: the rays falling on the crystal from S are then all parallel, and the angles of incidence on the successive faces are equal when the latter are brought into parallel positions. The error of centering is therefore eliminated by the use of a very distant signal. It can also be eliminated by bringing each edge into the axis of the instrument. Thus, if the crystal is turned about B through the angle $CBE = N_1ON_2$, BC is brought into the continuation of AB, and the portions of the faces used in reflecting the signal are those near B. To get N_2ON_3, the crystal must be displaced so as to put C in the centre, and so on for all the angles in the zone. The use of signals at a short distance from the crystal has therefore serious disadvantages. Consequently, when considerable distances cannot be secured, collimators giving parallel rays are used.

The vertical-circle goniometer.

3. The instrument shown in Fig. 550 was designed by Professor Miers (*Min. Mag.* IX, p. 214, 1891). The circular disc L, graduated on the edge to half-degrees, is screwed to the axle which is accurately fitted in a bush in the vertical portion of the stand. The stand rests on levelling screws, by means of which the axle can be placed horizontally. The vernier, reading to minutes, is engraved on the edge of a disc V, which is fixed to the stand; and the two discs L and V are kept in contact by a circular steel spring. C is a screw which clamps L to V, and D is a slow motion screw for the purpose of fine adjustment. The axle and disc L are rotated by the milled head B.

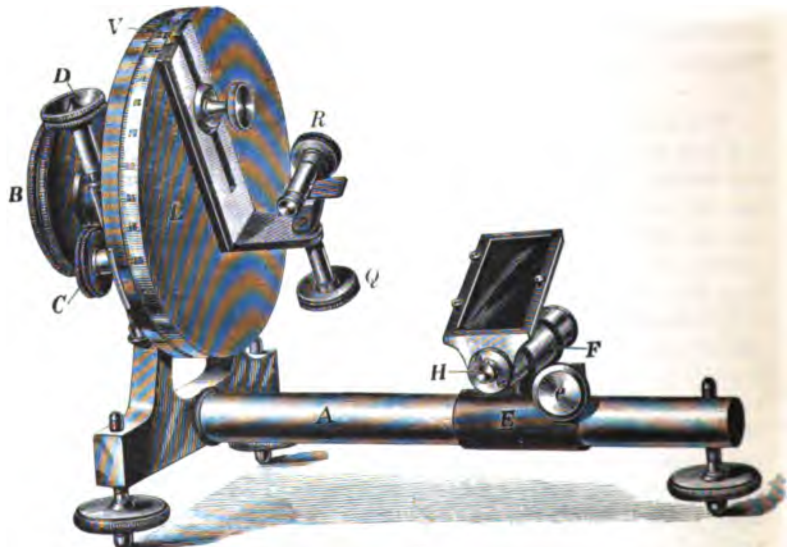


FIG. 550.

The *crystal-holder* consists of (a) a bar of metal bent at right angles, (b) a quadrantal arc turned by an axis Q, and (c) a second axis of rotation R at right angles to Q. (a) One arm of the bent bar is forked and slides on the disc L about a screw which, with the aid of a washer, clamps it firmly to L when the crystal has been adjusted: this movement is used to centre the edge or, approximately, the middle of the crystal when the latter is small. (b) In a bush at the end of the bent bar remote from L the axis Q works,

and it is rigidly attached to the quadrantal arc. Rotation about this axis brings the image of S from a crystal-face into coincidence with that of S seen in the mirror (Art. 4). (*c*) The second axis R (a stout pin) works in a bush at the other end of the quadrantal bar, and gives rotation about an axis at right angles to Q. Rotation about R brings the image of S in a second crystal-face (one as nearly as possible at 90° to the first is the best to take) into coincidence with that of S in the mirror. The crystal is attached by stiff wax to a small plate, which rests in a slit in R; or when very small, the crystal is stuck on a pin by a solution of shellac in alcohol and the pin fixed to R in any convenient way. After the edge has been brought by the axes Q and R into parallelism with the axis, the crystal is centered accurately by sliding the holder in the slot on the disc L.

To test the centering the following simple apparatus may be used. A vertical knitting-needle is supported on a stand, and carries a cork which moves stiffly on it. Through the cork a second needle passes horizontally, the end of which is brought close to the crystal edge. The disc and crystal are then turned through 180°. If the edge is not in the centre, it will describe a semicircle, and in the second position will be separated from the needle by double its distance from the centre. The cork is then moved so as to bring the needle half-way to the edge, and the crystal-edge is moved by a translation of the holder along the slot until it is close to the needle. The operation is repeated until the edge remains close to the needle during the whole of a revolution.

4. *The signals.* To give the *bright* signal S a diamond-shaped hole is cut in a board, which is placed, with a diagonal of the diamond horizontal, across the lower part of a window at the opposite side of the room to that occupied by the goniometer-table. On a stand outside the window a mirror is placed, which reflects the brightest portion of the sky and directs the light through the hole on the crystal: a heliostat and screens for regulating the intensity of the light may be used instead.

For the *faint* signal Σ a white horizontal line on the wall below the bright signal may be used, but it is inconvenient. A small mirror of blackened glass is therefore attached to the stand of the goniometer; and the faint signal is the image in this of a narrow slit cut in a board, which is placed across the upper part of the window with the slit horizontal.

The mirror. On the cylindrical axis A of the stand a tube E slides which can be clamped to A in any desired position by the

screw-head *e*. To *E* another tube *F* is fixed at right angles, in which a cylinder carrying the mirror can slide without rotation. It is clamped by a screw-head hidden behind the mirror. The mirror inclined at a convenient angle is attached to the end of the cylinder by a screw having a capstan-head, and can be turned into any position round the cylinder *F*. It should, by rotation round the axis *A*, be first placed so that the image of *S* is seen near the bottom and that of Σ near the top by an observer looking across the crystal. The mirror is adjusted so as to reflect the bright signal *S* in a plane parallel to *L*, and is then securely clamped by the capstan-head *H*. When once adjusted the mirror should as far as possible be left in the same place. The crystal is easily brought into the plane of reflexion by moving the plate carrying it, away from or towards the disc *L*. To be able to adjust the crystal accurately, the image of *S* should be visible in the mirror when the observer looks across the crystal. The image of *S* is sometimes used as second signal Σ, but such a use has many grave disadvantages and is quite unnecessary.

5. *Adjustments*. The plane of reflexion through the crystal must be parallel to the disc *L* and vertical. It is desirable, though not essential, that the plane of reflexion should be perpendicular to the faint signal and therefore to the window.

i. A line in a vertical plane through *S* perpendicular to the window is traced as accurately as possible on the table. The goniometer is then placed on the table so that the crystal is in the vertical plane through this trace, and the disc *L* very nearly parallel to it. Let the average distance of the crystal from *L* be *n* mm. A cardboard is now attached to the window having two narrow parallel slits in it $2n$ mm. apart. One slit is placed accurately in the position of the vertical diagonal of the rhombus forming the bright signal *S*, the other $2n$ mm. to its right, with regard to the observer. If the surface of *L* is a bright smooth plane the image of the slit covering *S* is seen by reflexion in it; and when the disc is correctly adjusted the image in *L* should exactly cover the other slit seen by direct vision across the edge of *L*. But since the disc seldom gives a distinct image, a plate of blackened glass is cemented to it by wax at some convenient position near the edge; the wax being sufficiently plastic to enable us to adjust the plate at right angles to the axis by gentle pressure. By slightly moving the goniometer (the crystal being maintained in the vertical plane through the trace) the image

in the glass is brought to cover the second slit seen directly. The disc is then turned through 180° by the milled head *B*, and the coincidence of the image and second slit is tested by looking over the opposite edge of the disc. Any divergence is halved by shifting the goniometer. The glass is then pressed until the images become coincident; and the disc is then turned back to its original position, and the coincidence of the images again tested. The divergence, if any, is halved by a change in the direction of the goniometer-axis, and the remainder corrected by again pressing the glass; the disc is now turned through 180°, and the adjustment tested. The process is repeated until the image always coincides with the second slit as the disc *L* is turned, when the plane of reflexion through the crystal is parallel to that of *L*. A strong rod is now screwed to the table pressing gently against the milled heads of the two levelling screws to the left; so that, when once adjusted, the goniometer can be immediately put in position by bringing these milled heads into contact with the rod.

ii. A porcelain dish full of clean mercury is now placed on the floor so that the signal *S*, still limited to the narrow slit in the card, is seen by reflexion in the mercury by an observer looking across the crystal. If the crystal has bright parallel faces—for instance, a good cleavage-fragment of calcite—we proceed by means of the axes of rotation of the holder to bring the image of the slit from one of the faces into coincidence with that seen in the mercury. The crystal is now turned by means of the milled head *B* through 180°, when the image of the slit from the parallel face should coincide with that in the mercury. If the image is displaced to the right or left the axis is not horizontal, and the levelling screws must be turned until the deviation is halved. The remaining half is corrected by rotating one of the axes of the holder; and the crystal is then turned so as to bring the image from the original face into coincidence with that in the mercury. Any deviation is again corrected in the same way, and the process continued until the adjustment is correct. The first adjustment must now again be tested, and any error produced by the levelling must be corrected by the process described under (i). The two processes have to be carried out alternately until the goniometer is adjusted.

iii. The mirror on *A* is now adjusted so that the images from the crystal-faces (known from testing by reflexion from mercury to be all in a vertical plane parallel to *L*) coincide with

the image of S reflected in the mirror. The instrument is now ready for use; and with careful handling it will remain in good adjustment for a considerable time. The adjustments should however be occasionally tested; and any defects corrected in the manner described under (i), (ii) and (iii).

6. *Measuring a crystal.* A freehand sketch of the crystal is made, and if necessary two or three from different sides. The faces are lettered in any way deemed convenient, and any marks on the faces noted. Such examination will generally enable the observer to perceive the symmetry; and this can be tested, if the crystal is translucent, by examining the directions of extinction between crossed Nicols. The observer then judges of the zones which it is desirable to measure. The crystal is now attached to the plate by stiff wax with one face of the zone to be measured as nearly as possible parallel to the plate. The plate is then attached to the holder in such a way that the crystal is in the plane of reflexion and one edge is nearly in the direction of the axis. The crystal is then turned by the milled head B until the image from the face parallel to the plate is seen near that of S in the mirror. By turning the milled head R the two images are brought into coincidence. The crystal is now turned by means of B until the image of S from another face (preferably one nearly at 90° to the first face) is seen near that from the mirror. The two images are brought into coincidence by rotation of the screw-head Q. This adjustment generally affects the first one, and the first face has to be again adjusted; and the process is repeated until both faces are accurately adjusted. The images of S from all the other faces in the zone can by rotating L be then brought in succession to cover that in the mirror.

The edge to be measured is now to be centered, or if the crystal is very small its middle point is put as nearly as possible in the axis. The adjustment of the zone must be checked, and any slight error caused by the change of position of the holder on the disc L corrected. A screen is then interposed, which cuts off all light from the mirror except that from the faint signal.

The crystal and L are next turned by means of B until the image from an easily recognised face is bisected by the faint signal. The corresponding angle is read off, and recorded against the letter used to denote the particular face: any peculiarity of the image, such as its being double or elongated, being noted at the same time.

Distinctly double images should be separately determined. The disc and crystal are now turned until the image from the next face is bisected by the faint signal, and the corresponding angle entered below the first one. If the crystal is not re-centered for each edge, the process is continued without interruption until the first reading is again obtained. The difference between each pair of successive readings gives the angle between the corresponding faces.

7. The author gave in the *Proceedings of the Camb. Phil. Soc.* IV, p. 243, 1882, an analysis of the error due to defective centering which shows that the error can never exceed half the angle subtended at either signal by the extreme positions of the edge when the readings are made. If the signals are, as at Cambridge, at a distance of about 7·25 metres from the crystal, a displacement of the edge through 4 mm. will not cause a greater error than 1′ in the reading. The error arising from defective adjustment of the zone, or of a slight deviation from zonality, is one of the second order, and is inappreciable within the limits which occur with an instrument which only reads to minutes.

The faces of few crystals admit of measurement to half-minutes; and, in passing from one crystal to another, divergences quite beyond those due to errors of centering are continually met with in the angles between corresponding faces. Considerable experience and judgment is needed in selecting the angles to be used in the computation of elements and the theoretical angles.

A fixed telescope supplied with cross-wires may be used to determine the positions of the images from successive faces, and may replace the mirror and faint signal. Vertical-circle goniometers provided with a telescope, with or without a collimator, are fairly common; but the adjustments of the telescope necessary for its axis to be strictly in the plane of reflexion are somewhat troublesome: they are made with two plates of glass in the way described in Art. 10. The telescope may however be employed in conjunction with the mirror, when it merely serves to focus the two signals and to give the position of the faint one when it is concealed behind the crystal. The necessity for very careful adjustment of the telescope is then avoided.

The horizontal-circle goniometer.

8. Fig. 551 represents a vertical section through the axis of the goniometer, Model II, made by Herr Fuess of Berlin to whose

courtesy I am indebted for this figure and for Figs. 547, 548, 552, and 553. The stand consists of a thick metal-plate *o* supported on three legs which are provided with levelling screws. The axis consists of three concentric conical shells fitted the one within the other, and working in a conical bush in the plate *o*. The outer cone *b* carries a circular disc *d*, to which are fixed the verniers and the

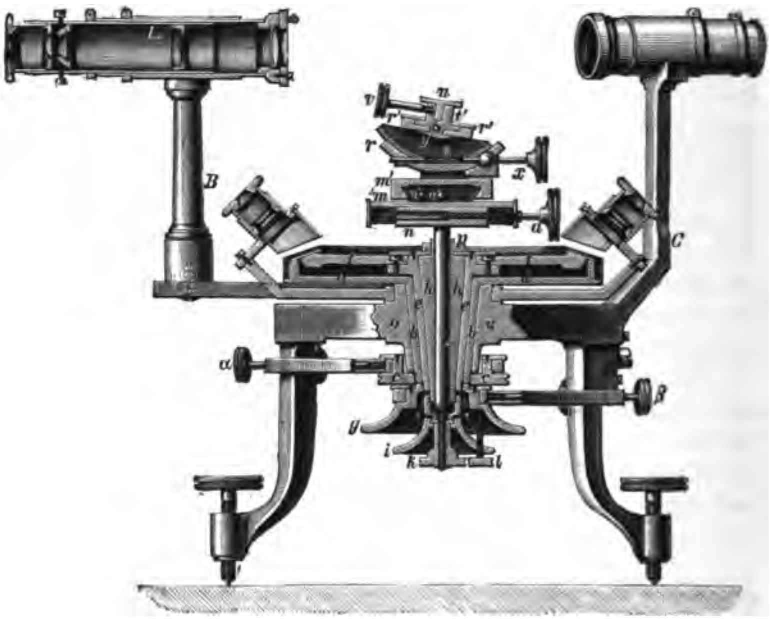


FIG. 551.

standard *B* which carries the observing telescope. The verniers are at opposite extremities of a diameter and read to 30″. The cone *b* is clamped by the screw *a*, so that the telescope and verniers can be fixed in any convenient position. For fine adjustment in determining refractive indices a slow motion screw is attached, the head of which is just visible behind the leg. The second conical shell *e* fits into *b* and carries the graduated disc *f*. This cone terminates in the milled rim *g* for turning the axis. The cone *e* and graduated disc are clamped by the screw *β*, and the corresponding slow motion screw is just visible behind the leg. The third cone *h* works within *e* and is useful for turning the crystal during adjustment without turning the graduated disc: it ends in the milled head *i* and is

clamped to the cone *e* by the screw *l*. Within *h* is a cylindrical steel axis which carries the crystal-holder: it terminates at the lower end in a screw working in *k*, so arranged that the crystal can be raised or lowered for the purpose of bringing it into the centre of the illuminated field.

9. The *crystal-holder* consists of two plane slides *m* and *m′* at right angles to one another and two arrangements for circular motion in planes also at 90° to one another. A movement of translation for centering the edge can be given to the slides *m* and *m′* by screws *a* and *a′*. To *m′* is attached a 'felly' *r* having on its inner (concave) side a groove in which a plate *t* of equal curvature and with toothed edge can move. Motion is given to the plate by the screw *x*, the thread of which works in the toothed edge of *t*. Any line in the plane of *t* can be therefore inclined to the vertical at any desired angle within the range of the felly. On *t* is fixed a similar arrangement of felly and toothed plate, which gives circular motion in a plane at right angles to the first. The centres of the two circular motions are nearly coincident and lie at a short distance above the plate *u* which carries the crystal. This holder can be used with the vertical-circle goniometer described in Art. 3, but here the back-lash in the screws *a*, *x*, &c. which cannot be prevented after the instrument has been used for some time renders the holder untrustworthy (Dauber, *Pogg. Ann.* CIII, p. 107, 1858).

10. The collimator is carried on a fixed standard *C*, and its axis can be adjusted by screws above β perpendicular to the axis of the goniometer. With it different forms of signal can be used, the most useful being one formed by two circular plates which have their centres on opposite sides of the slit and very nearly touch in the centre of the field. For determining refractive indices a straight slit, like that of a spectroscope, should be used. The slits can be illuminated in any convenient way; *e.g.* for measuring a crystal, by an incandescent burner, and for refractive indices, by a Bunsen's burner flame coloured by sodium, lithium, &c.

The observing telescope carried by *B* and attached to the cone *e* is supplied with several eye-pieces for use under different circumstances of definition and brightness of the images. The telescope should be focussed on a very distant object, and the collimator can then be easily adjusted so as to give practically parallel rays. The axes of the telescope and collimator should intersect accurately in

the axis of the goniometer, and for this adjustment the observer has to trust mainly to the skill and care of the maker. A slight deviation of the axes from the same plane is of no consequence provided both pass through, and are perpendicular to, the goniometer-axis. The accurate adjustment of the cross-wires and telescope is one to be made by the observer. For this purpose small screws pass through the eye-piece which allow a limited change of position of the ring holding the cross-wires to be made.

Two plates of glass with parallel sides are used to adjust the telescope. One *a* (say) is adjusted like a crystal on the goniometer, the other *b* is placed in front of the eye-piece of the telescope at 45° to its axis and sends the light of a lamp down the telescope, thus illuminating the cross-wires. The cross-wires, having been first focussed for distinct vision by the eye-piece, and the plate *a* are now adjusted so that the image of the wires reflected successively from the two sides of *a* are superposed on the wires themselves. When this is the case, the axis of the telescope is perpendicular to that of the goniometer; and, if the telescope is adjusted for infinity, the cross-wires are not separated from their image by any displacement of the eye or lamp.

As shown in the figure a small lens can be slipped in front of the objective: it converts the telescope into a microscope which enables the observer to see and center the edge of the crystal[1].

11. To adjust a crystal requires some practice; and it is advisable to attach the crystal to the plate *u* so that the edge to be adjusted is as nearly as the eye can judge perpendicular to the plate. The plate should then be put in its place with one of the crystal-faces parallel to one of the plates *t* or *t′*, and then the screw *v* should be tightened so as to keep the crystal rigidly fixed on the holder. By turning the crystal round until the light falls on the faces nearly at a grazing incidence and looking with the naked eye nearly in the line of the collimator, the observer can see the images near that in the collimator and can, by the circular motions, bring them approximately into the plane of reflexion of the instrument so that they can enter the observing telescope. The crystal is then centered; and the two images correctly adjusted by the aid of the cross-wires in the telescope. When adjusted, the edge is re-centered; the angle is measured and the readings are recorded in the way already described.

[1] For detailed instructions as to the method of adjusting the goniometer, cross-wires and signals, the reader is referred to *Die Optische Instrumente der Firma R. Fuess* by C. Leiss, 1899, and to a memoir by Professor Websky in the *Zeitsch. f. Kryst.* IV, p. 545, 1880.

Theodolite-goniometers.

12. When the crystal to be measured is small and the faces numerous, it is difficult to identify the faces so as to connect accurately the measurements in one zone with those in another. When one good zone has been measured with the vertical-circle goniometer of Art. 3, it is possible, by judicious movements of the plate in the slit in R and of the axes R and Q (the crystal remaining fixed on the plate), to adjust a fresh zone which contains a known face of the first. For the image of the bright signal from a face at some distance out of the first zone can be readily found by looking across the crystal when the light falls on the face nearly at grazing incidence. When this zone has been measured, a similar manipulation of the plate and of the axes R and Q will enable the observer to adjust a third zone containing a second known face of each of the two zones. A triangle being thus obtained, the three measured zones can be correctly projected in a stereogram (Chap. VII, Arts. 15, 19 and 20). Proceeding in this manner from zone to zone, the angles can be measured and the poles projected, and the crystal solved. The method is laborious; and errors may be made, especially if the work has to be interrupted. Such small crystals should be attached to a fine pin by a solution of shellac.

To overcome the difficulty of identification, goniometers with two and three graduated circles have been introduced. Miller[1] in 1874 attached a small vertical-circle goniometer to the disc of a horizontal-circle one, in such a way that the crystal (attached to the holder once for all) lay in the intersection of the axes of the two goniometers. Goniometers with similar combinations of horizontal and vertical circles were in 1889 and 1893 designed by Professors von Fedorow[2] and V. Goldschmidt[3], and by Herr Czapski[4].

13. Fig. 552 represents the theodolite-goniometer (Model II, b) made by Herr Fuess. On the axis of the horizontal circle moves the plate D, which on one side carries the standard E for the support of the vertical circle and on the other side a counterpoise G. At the upper end of E is a conical bush in which fits the axis of a vertical graduated disc to which the crystal is attached by the holder described in Art. 9. The axes and graduated discs are similar in construction to those of the horizontal-circle goniometer, Fig. 551. The central steel axis with the crystal-holder can be taken out of the horizontal axis, and can be fixed in the vertical axis at a; the instrument can then be used as a single circle goniometer.

[1] After his death an account of the method and of the unfinished work was published by the author in the *Proceedings of the Camb. Phil. Soc.* IV, p. 236, 1882: an abstract also in *Zeitsch. f. Kryst.* VII, p. 619, 1882.

[2] *Zeitsch. f. Kryst.* XXI, p. 574, 1893; *Proc. of Min. Soc. of St. Petersburg*, 1889.

[3] *Ibid.* XXI, p. 210, 1893.

[4] *Zeitsch. f. Instr.-Kunde* XIII, p. 1, 1893.

The plate D carrying the vertical circle can move independently of the horizontal circle; it can also be clamped to the latter by the screw e when the two move together as one piece. The two circles are provided with verniers reading to minutes, and with clamping and slow motion screws for fine adjustment.

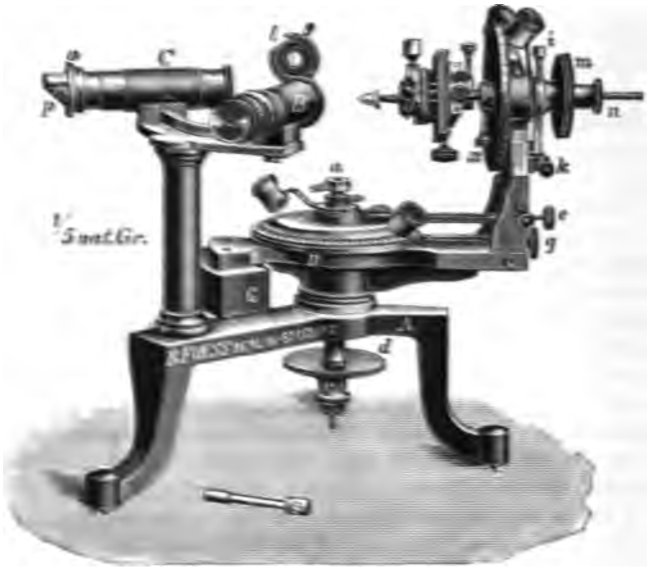


FIG. 552.

The collimator C and observing telescope B are similar to those described in Art. 10. They are both carried on the same standard, and their axes are inclined to one another at an angle of 60°. The best form of signal is one giving lines of light as indicated in Fig. 553; and the most advantageous way of illuminating the signal-disc o is to place before it, as shown in Fig. 552, a total-reflexion prism p which sends the light of an incandescent burner along the axis of the collimator. This burner, if placed above the goniometer, serves also to illuminate the circles and verniers when readings are taken; but its light must be cut off by a screen from the observer and all the instrument, save the prism, whilst the image of a face is adjusted on the cross-wire.

FIG. 553.

The steel axis with the crystal-holder having been placed in the axis of the horizontal circle, the telescope and cross-wires are adjusted by two plates of glass in the way described in Art. 10. By turning the horizontal circle, the signal from the collimator can be reflected from the adjusted glass-plate into the telescope, and the collimator adjusted. The steel axis

is now transferred to the axis of the vertical circle, and the glass-plate again adjusted so as to superpose the image of the cross-wires on the wires themselves. The axis is then turned through 180°, when, if it is perpendicular to that of the horizontal circle, there should be no appreciable displacement of the image of the cross-wires. If there is an appreciable displacement, the screw fastening E to D must be loosened, and a strip or two of tin-foil inserted between E aud D so as to halve the error. When the screw is tightened and the glass re-adjusted, the image of the cross-wires should retain the same position however the plate and vertical circle are turned about the horizontal axis.

The crystal being fixed to the crystal-holder, the only adjustment needed is that of centering; but, to avoid troublesome calculations, it is advisable to adjust some conspicuous face (the *reference-face*) parallel to the vertical circle, for the readings on the horizontal-circle can be then, by Prob. 1 of Chap. VII, directly marked from the centre along diametral circles of a stereogram. The readings of the vertical circle are measured off on the primitive (the equator), starting from any one of the diametral zones. The adjustment of the reference-face is made by placing a plate of glass with parallel faces in front of the telescope eye-piece, and sending down the telescope a beam of light, which illuminates the cross-wires. The observer can then bring the image of the cross-wires reflected from the face into coincidence with the cross-wires themselves. By turning the horizontal axis, the parallelism of the face to the plane of the vertical circle is tested, and any error can be quickly corrected.

The crystal being adjusted, the horizontal axis is turned till one of the edges of the reference-face is vertical. The vertical circle is then clamped and the reading taken. The vertical axis is now turned so as to place the images of all the faces in the vertical zone successively on the cross-wire of the telescope. The corresponding readings of the horizontal circle give the distances between poles on diametral zones of the stereogram. Another edge, possible or actual, of the reference-face is now placed vertically by means of the horizontal axis, and the readings for the new zone taken in the same way as for the first, and so on. The poles are placed on the stereogram in a manner similar to that in which places are marked on a map when their latitude and longitude are known; the horizontal circle giving the co-latitude, and the vertical circle the longitude. The interfacial angles have to be computed from a knowledge of the arcs along meridians and the angles between these meridians, *i.e.* differences of longitude. The spherical triangles are generally oblique-angled, and are frequently disadvantageous for giving accurate values of the third side. Further, to prove that three or more faces, through the poles of which a great circle can be drawn, are truly tautozonal involves troublesome computation of the remaining angles of the oblique-angled triangles; the only zones established by direct observation being the diametral zones to which the reference-face is common.

14. *Three-circle goniometer.* To obviate the necessity for these laborious and somewhat unsatisfactory computations, and also to give the angles between tautozonal faces by direct observation Mr. G. F. Herbert Smith has recently designed a three-circle goniometer (*Min. Mag.* XII, 1899). The circles are a horizontal circle called A, a vertical circle B carried like the vertical circle in Fig. 552, and a third circle C connected to B in the same way as B is to A; *i.e.* the axis of C is always perpendicular to that of B. Each of these circles can be clamped in any desired position, and is provided with a slow motion. The axis of A is perpendicular to the axes of the collimator and telescope, and passes through the point of intersection of these axes (the optic centre), through which the axes of B and C should also pass.

The crystal holder is carried on the axis of C, and is of the type described in Art. 9; the crystal is attached to it in any convenient way.

The graduations are so arranged that A reads 0° 0′ when a plane parallel to B reflects the signal on the cross-wire of the telescope; and that B reads also 0° 0′ when the axis of C is vertical. Any index-errors must be determined, and allowance made for them in taking the readings.

Measurements. The axis of C being vertical and A and B being both clamped at 0° 0′, an edge of the *reference-zone* is adjusted. As far as this zone is concerned the goniometer may be regarded as a single-circle one, having the axis vertical but directed downwards. The interfacial angles in this zone are then given by the readings of C and are recorded in a column labelled C. These measurements having been recorded, C is clamped when the image from a definite face is on the cross-wire, this face being then, from the arrangement of the instrument, parallel to the disc B. By rotating B, the axis of each zone to which this face is common can be brought into the vertical, *i.e.* into parallelism with the axis of A. When the adjustment of one of these zones is perfect, the circle B is clamped, and the crystal with the circles B and C is turned about the axis of A. The differences of the readings of A give the interfacial angles in the adjusted zone, and the difference of the readings of B gives the angle between this zone and that of reference. All the zones to which the face parallel to B is common can be similarly determined. When these have been measured, another face of the reference-zone is brought into parallelism with B; and then all the zones to which it is common are determined in the same way. The interfacial angles of all the faces, and the angles between the several zones can be thus determined, except when one of the circles B and C intervene between the crystal and the telescope.

To obviate the unsteadiness to which the instrument is liable, the circle C may be made very small and the holder described in Art. 3 used. C may be graduated to 1° or 2°, which will suffice for the identification of the faces in the reference-zone. For the angles in this zone can be accurately measured by A, when the zone-axis is truly vertical.

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